International J. of Engg. and Research of Indu. Appls. (IJERIA)
ISSN 0974-1518, Vol. 10, No. I (August 2017), p. 1-14

SOME BASIC PROPERTIES OF ANALYTIC AND MULTIVALENT FUNCTION WITH NEGATIVE COEFFICIENTS DEFINED BY DIFFERENTIAL OPERATOR

PRAVIN GANPAT JADHAV
Assistant Professor in Mathematics,
Hon. Balasaheb Jadhav Arts, commerce and Science College, Ale, Pune (M.S.), India

Abstract

By means of certain differential operator we introduce and investigate two subclasses $\mathcal{L}^*(p, \lambda, \phi, \delta, \alpha)$ and $Y\mathcal{L}^*(p, \lambda, \phi, \delta, \alpha)$ of p-valently analytic functions. The results obtained here for each of these classes. We have attempted to obtain coefficient estimate, growth and distortion theorem for the classes $\mathcal{L}^*(p, \lambda, \phi, \delta, \alpha)$ and $Y\mathcal{L}^*(p, \lambda, \phi, \delta, \alpha)$.

1. Introduction

This paper is devoted to study of multivalent functions and its various properties. By means of certain differential operator we introduce and investigate two subclasses $\mathcal{L}^*(p, \lambda, \phi, \delta, \alpha)$ and $Y\mathcal{L}^*(p, \lambda, \phi, \delta, \alpha)$ of p-valently analytic functions. The various results obtained here for each of these classes.

Let $A(p)$ denote the class of functions $f(z)$ of the form

—————————————————

Key Words : *Multivalent function, Coefficient estimate, Distortion theorem, Differential operator.*

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \tag{1.1}$$

which are analytic and multivalent in the unit disk $E = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ for $p \in \mathbb{N}$.

Definition 1.1 : A function $f(z) \in A(p)$ is said to be in the subclass $S(\xi)$ of starlike function if

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \xi, \quad z \in E, \quad 0 \leq \xi < 1.$$

Definition 1.2 : A function $f(z) \in A(p)$ is said to be in the subclass $C(\xi)$ of convex function if

$$Re\left(1 + \frac{zf'(z)}{f(z)}\right) > \xi, \quad z \in E, \quad 0 \leq \xi < 1.$$

Definition 1.3 : A function $f(z) \in A(p)$ is said to be in the subclass $K(\xi)$ of close to convex function if

$$Re\left(\frac{f'(z)}{z^{p-1}}\right) > \xi, \quad z \in E, \quad 0 \leq \xi < 1.$$

Definition 1.4 : A function $f(z) \in A(p)$ is said to be in the subclass $\pounds(p, \lambda, \phi, \delta, \alpha)$ if and only if

$$\left|\frac{\delta z(D_z^{q+1}(\Omega_p(r,p)f(z))) + \lambda z^2(D_z^{q+2}(\Omega_p(r,p)f(z)))}{(1-\lambda)(D_z^{q}(\Omega_p(r,p)f(z))) + z(D_z^{q+1}(\Omega_p(r,p)f(z)))} - (\delta - \phi)\right| < \alpha$$

$$z \in E, \quad q \in \mathbb{N}U\{0\}, \quad 0 < \alpha \leq 1, \quad \phi \in \mathbb{R}, \quad \phi < 1, \quad p > q, \quad \gamma, \delta \leq 1.$$

Further more a function $f(z) \in A(p)$ is said to be in the subclass $Y\pounds(p, \lambda, \phi, \delta, \alpha)$ if and only if $zf'(z) \in \pounds(p, \lambda, \phi, \delta, \alpha)$.

Let $T(p)$ denote the subclass of $A(p)$ consisting of functions of the form

$$f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k, \qquad a_k \geq 0.$$

We dente by $\pounds^*(p, \lambda, \phi, \delta, \alpha)$ and $Y\pounds^*(p, \lambda, \phi, \delta, \alpha)$ the classes obtained by taking intersection respectively of the classes $\Gamma(p, \lambda, \phi, \delta, \alpha)$ and $Y\pounds(p, \lambda, \phi, \delta, \alpha)$ with $T(p)$.

We define the operator $\Omega(a, p)$ on $f(z)$ as follows

$$\Omega_p(r,p)f(z) = z^p - \sum_{k=p+1}^{\infty} \left(\frac{k+\gamma}{p+\gamma}\right)^r a_k z^k.$$

The operator $\Omega_p(r,p)$ is closely related to the Salagean derivative operator $D_z^q f(z)$ is the q^{th} order differential operator for $f(z) \in A(p)$ defined in (1.1)

$$D_z^q(\Omega_p(r,p)f(z)) = \frac{p!}{(p-q)!}z^{p-q} - \sum_{k=p+1}^{\infty} \frac{k!}{(k-q)!}\left(\frac{k+\gamma}{p+\gamma}\right)^4 a_k z^{k-q}, \quad p > q.$$

2. Coefficient Estimates

Theorem 2.1 : A function $f(z) = z^p - \sum\limits_{k=p+1}^{\infty} a_k z^k, \quad a_k \geq 0$ is in $\pounds^*(p,\lambda,\phi,\delta,\alpha)$ if and only if

$$\sum_{k=p+1}^{\infty} \frac{k!}{(k-q)!}\left(\frac{k+\gamma}{p+\gamma}\right)^r [\lambda(k-q)(k-q-1)+\phi(k-q)-(\delta-\phi)(1-\lambda)+\alpha(1-\lambda+k-q)]a_k$$

$$\leq \frac{p!}{(p-q)!}[\alpha(1-\lambda+p-q)-\lambda(p-q)(p-q-1)-\phi(p-q)+(\delta-\phi)(1-\lambda)]. \quad (2.1)$$

Proof : Suppose a function $f(z) = z^p - \sum\limits_{k=p+1}^{\infty} a_k z^k, a_k \geq 0$ is in $\pounds^*(p,\lambda,\phi,\delta,\alpha)$. Thus

$$\left|\frac{\delta, z(D_z^{q+1}(\Omega_p(r,p)f(z))) + \lambda z^2(D_z^{q+2}(\Omega_p(r,p)f(z)))}{(1-\lambda)(D_z^q(\Omega_p(r,p)f(z))) + z(D_z^{q+1}(\Omega_p(r,p)f(z)))} - (\delta-\phi)\right| < \alpha. \qquad (2.2)$$

Consider

$$\delta z(D_z^{q+1}(\Omega_p(r,p)f(z)))) + \lambda z^2(D_z^{q+2}(\Omega_p(r,p)f(z))) - (\delta-\phi)$$
$$[(1-\lambda)(D_z^1(\Omega_p(r,p)f(z))) + z(D_z^{q+1}(\Omega_p(r,p)f(z)))]$$
$$= \delta\left[\frac{p!}{(p-q)!}(p-q)z^{p-q} - \sum_{k=p+1}^{\infty} \frac{k!}{(k-q)!}\left(\frac{k+\gamma}{p+\gamma}\right)^r (k-q)a_k z^{k-q}\right]$$
$$+\lambda\left[\frac{p!}{(p-q)!}(p-q)(p-q-1)z^{p-q} - \sum_{k=p+1}^{\infty} \frac{k!}{(k-q)!}\left(\frac{k+\gamma}{p+\gamma}\right)^r (k-q)(k-q-1)a_k z^{k-q}\right]$$

$$-(\delta-\phi)(1-\lambda)\left[\frac{p!}{(p-q)!}z^{p-q}-\sum_{k=p+1}^{\infty}\frac{k!}{(k-q)!}\left(\frac{k+\gamma}{p+\gamma}\right)^{r}a_kz^{k-q}\right]$$
$$-(\delta-\phi)\left[\frac{p!}{(p-q)!}(p-q)z^{p-q}-\sum_{k=p+1}^{\infty}\frac{k!}{(k-q)!}\left(\frac{k+\gamma}{p+\gamma}\right)^{r}(k-q)a_kz^{k-q}\right]$$
$$=\frac{p!}{(p-q)!}[\lambda(p-q)(p-q-1)+\phi(p-q)-(\delta-\phi)(1-\lambda)]z^{p-q}$$
$$-\sum_{k=p+1}^{\infty}\frac{k!}{(k-q)!}\left(\frac{k+\gamma}{p+\gamma}\right)^{r}[\lambda(k-q)(k-q-1)+\phi(k-q)-(\delta-\phi)(1-\lambda)]a_kz^{k-q}.$$

Now consider

$$(1-\lambda)(D_z^q(\Omega_p(r,p)f(z)))+z(D_z^{q+1}(\Omega_p(r,p)f(z)))$$
$$=(1-\lambda)\left[\frac{p!}{(p-q)!}z^{p-q}-\sum_{k=p+1}^{\infty}\frac{k!}{(k-q)!}\left(\frac{k+\gamma}{p+\gamma}\right)^{r}a_kz^{k-q}\right]$$
$$+\left[\frac{p!}{(p-q)!}(p-q)z^{p-q}-\sum_{k=p+1}^{\infty}\frac{k!}{(k-q)!}\left(\frac{k+\gamma}{p+\gamma}\right)^{r}(k-q)a_kz^{k-q}\right]$$
$$=\frac{p!}{(p-q)!}\left[1-\lambda+p-q\right]z^{p-q}-\sum_{k=p+1}^{\infty}\frac{k!}{(k-q)!}\left(\frac{k+\gamma}{p+\gamma}\right)^{r}[1-\lambda+k-q]a_kz^{k-q}.$$

From (2.2) we get

$$\left|\frac{\begin{array}{c}\frac{p!}{(p-q)!}[\lambda(p-q)(p-q-1)+\phi(p-q)-(\delta-\phi)(1-\lambda)]z^{p-q}\\ -\sum_{k=p+1}^{\infty}\frac{k!}{(k-q)!}\left(\frac{k+\gamma}{p+\gamma}\right)^{r}[\lambda(k-q)(k-q-1)+\phi(k-q)-(\delta-\phi)(1-\lambda)]a_kz^{k-q}\end{array}}{\frac{p!}{(p-q)!}[1-\lambda+p-q]z^{p-q}-\sum_{k=p+1}^{\infty}\frac{k!}{(k-q)!}\left(\frac{k+\gamma}{p+\gamma}\right)^{r}[1-\lambda+k-q]a_kz^{k-q}}\right|\leq\propto \tag{2.3}$$

We know that $|Re(z)\leq|z|$, so choosing values of z on real axis. After clearing denominator in (2.3) and allowing $z \to 1_-$ through real values, we get

$$\frac{p!}{(p-q)!}[\lambda(p-q)(p-q-1)+\phi(p-q)-(\delta-\phi)(1-\lambda)]$$
$$+\sum_{k=p+1}^{\infty}\frac{k!}{(k-q)!}\left(\frac{k+\gamma}{p+\gamma}\right)^{r}[\lambda(k-q)(k-q-1)+\phi(k-q)-(\delta-\phi)(1-\lambda)]a_k$$
$$\leq \alpha\left[\frac{p!}{(p-q)!}[1-\lambda+p-q]-\sum_{k=p+1}^{\infty}\frac{k!}{(k-q)!}\left(\frac{k+\gamma}{p+\gamma}\right)^{r}[1-\lambda+k-q]a_k\right].$$

Therefore

$$\sum_{k=p+1}^{\infty}\frac{k!}{(k-q)!}\left(\frac{k+\gamma}{p+\gamma}\right)^{r}[\lambda(k-q)(k-q-1)+\phi(k-q)-(\delta-\phi)(1-\lambda)$$
$$+\alpha(1-\lambda+k-q)]a_k$$
$$\frac{p!}{(p-q)!}[\alpha(1-\lambda+p-q)-\lambda(p-q)(p-q-1)-\phi(p-q)+(\delta-\phi)(1-\lambda)].$$

Now we prove the converse.

Suppose that (2.1) is true, we have

$$|\delta z(D_z^{q+1}(\Omega_p(r,p)f(z)))+\lambda z^2(D_z^{q+2}(\Omega_p(r,p)f(z)))$$
$$-(\delta-\phi)(1-\lambda)(D_z^{q}(\Omega_p(r,p)f(z)))+z(D_z^{q+1}(\Omega_p(r,p)f(z)))$$
$$-\propto|(1-\lambda)(D_z^{q}(\Omega_p(r,p)f(z)))+z(D_z^{q+1}(\Omega_p(r,p)f(z)))|$$
$$=\left|\frac{p!}{(p-q)!}[\lambda(p-q)(p-q-1)+\phi(p-q)-(\delta-\phi)(1-\lambda)]z^{p-q}\right.$$
$$\left.-\sum_{k=p+1}^{\infty}\frac{k!}{(k-q)!}\left(\frac{k+\gamma}{p+\gamma}\right)^{r}[\lambda(k-q)(k-q-1)+\phi(k-q)-(\delta-\phi)(1-\lambda)]a_k z^{k-q}\right|-\propto$$
$$\left|\frac{p!}{(p-q)!}[1-\lambda+p-q]z^{p-q}-\sum_{k=p+1}^{\infty}\frac{k!}{(k-q)!}\left(\frac{k+\gamma}{p+\gamma}\right)^{r}[1-\lambda+k-q]a_k z^{k-q}\right|$$
$$=\sum_{k=p+1}^{\infty}\frac{k!}{(k-q)!}\left(\frac{k+\gamma}{p+\gamma}\right)^{r}$$
$$[\lambda(k-q)(k-q-1)+\phi(k-q)-(\delta-\phi)(1-\lambda)+\alpha(1-\lambda+k-q)]a_k$$
$$-\frac{p!}{(p-q)!}[\alpha(1-\lambda+p-q)-\lambda(p-q)(p-q-1)-\phi(p-q)+(\delta-\phi)(1-\lambda)]$$
$$\leq 0.$$

By using maximum modulus principle and (2.1) $f(z)$ is in $\pounds^*(p,\lambda,\phi,\delta,\alpha)$.

Hence the result follows.

Corollary 2.1 : If the function $f(z) = z^p - \sum\limits_{k=p+1}^{\infty} a_k z^k, a_k \geq 0$ is in $\mathcal{L}^*(p, \lambda, \phi, \delta, \alpha)$ then

$$a_k \leq \left(\frac{p+\gamma}{k+\gamma}\right)^r \frac{p!(k-q)![\alpha(1-\lambda+p-q) - \lambda(p-q)(p-q-1) - \phi(p-q) + (\delta-\phi)(1-\lambda)]}{k!(p-q)![\lambda(k-q)(k-q-1) + \phi(k-q) - (\delta-\phi)(1-\lambda) + \alpha(1-\lambda+k-q)]}.$$

For $k = 1+p, 2+p \cdots$.

With the equality for function

$$f_z = z^p - \left(\frac{p+\gamma}{k+\gamma}\right)^r \frac{p!(k-q)![\alpha(1-\lambda+p-q) - \lambda(p-q)(p-q-1) - \phi(p-q) + (\delta-\phi)(1-\lambda)]}{k!(p-q)![\lambda(k-q)(k-q-1) + \phi(k-q) - (\delta-\phi)(1-\lambda) + \alpha(1-\lambda+k-q)]} a^k.$$

For $k = 1+p, 2+p \cdots$.

Theorem 2.2 : A function $f(z) = z^p - \sum\limits_{k=p+1}^{\infty} a_k z^k,\ a_k \geq 0$ is in $Y\ \mathcal{L}^*(p, \lambda, \phi, \delta, \alpha)$ if and only if

$$\sum_{k=p+1}^{\infty} \frac{(k+1)!}{(k-q)!} \left(\frac{k+\gamma}{p+\gamma}\right)^r [\lambda(k-q)(k-q-1) + \phi(k-q) - (\delta-\phi)(1-\lambda) + \alpha(1-\lambda+k-q)] a_k$$

$$\leq \frac{p+1!}{p-q!}[\alpha(1-\lambda+p-q) - \lambda(p-q)(p-q-1) - \phi(p-q) + (\delta-\phi)(1-\lambda)] \qquad (2.4)$$

Proof : Suppose a function $f(z) = z^p - \sum\limits_{k=p+1}^{\infty} a_k z^k, a_k \geq 0$ is in $Y\mathcal{L}^*(p, \lambda, \phi, \delta, \alpha)$. Therefore $zf'(z) \in \mathcal{L}^*(p, \lambda, \phi, \delta, \alpha)$

Let $g(z) = zf'(z)$

$$g(z) = pz^p - \sum_{k=p+1}^{\infty} k a_k z^k.$$

Thus

$$\left| \frac{\delta z(D_z^{q+1}(\Omega_p(r,p)g(z))) + \lambda z^2(D_z^{q+2}(\Omega_p(r,p)f(z)))}{(1-\lambda)(D_z^q(\Omega_p(r,p)g(z))) + z(D_z^{q+1}(\Omega_p(r,p)g(z)))} - (\delta-\phi) \right| < \alpha. \qquad (2.5)$$

Consider

$$\begin{aligned}
&\delta z(D_z^{q+1}(\Omega_p(r,p)g(z))) + \lambda z^2(D_z^{q+2}(\Omega_p(r,p)g(z))) \\
&-(\delta-\phi)[(1-\lambda)(D_z^{q}(\Omega_p(r,p)g(z))) + z(D_z^{q+1}(\Omega_p(r,p)g(z)))] \\
&= \delta\left[\frac{p!}{(p-q)!}p(p-q)z^{p-q} - \sum_{k=p+1}^{\infty}\frac{k!}{(k-q)!}\left(\frac{k+\gamma}{p+\gamma}\right)^r k(k-q)a_k z^{k-q}\right] \\
&+\lambda\left[\frac{p!}{(p-q)!}p(p-q)(p-q-1)z^{p-q} - \sum_{k=p+1}^{\infty}\frac{k!}{(k-q)!}\left(\frac{k+\gamma}{p+\gamma}\right)^r k(k-q)(k-q-1)a_k z^{k-q}\right] \\
&-(\delta-\phi)(1-\lambda)\left[\frac{p!}{(p-q)!}pz^{p-q} - \sum_{k=p+1}^{\infty}\frac{k!}{(k-q)!}\left(\frac{k+\gamma}{p+\gamma}\right)^r k a_k z^{k-q}\right] \\
&-(\delta-\phi)\left[\frac{p!}{(p-q)!}(p-q)z^{p-q} - \sum_{k=p+1}^{\infty}\frac{k!}{(k-q)!}\left(\frac{k+\gamma}{p+\gamma}\right)^r k(k-q)a_k z^{k-q}\right] \\
&= \frac{p!p}{(p-q)!}[\lambda(p-q)(p-q-1)+\phi(p-q)-(\delta-\phi)(1-\lambda)]z^{p-q} \\
&- \sum_{k=p+1}^{\infty}\frac{k!k}{(k-q)!}\left(\frac{k+\gamma}{p+\gamma}\right)^r [\lambda(k-q)(k-q-1)+\phi(k-q)-(\delta-\phi)(1-\lambda)]a_k z^{k-q}.
\end{aligned}$$

Now consider

$$\begin{aligned}
&(1-\lambda)(D_z^{q}(\Omega_p(r,p)g(z))) + z(D_z^{q+1}(\Omega_p(r,p)g(z))) \\
&= (1-\lambda)\left[\frac{p!}{(p-q)!}pz^{p-q} - \sum_{k=p+1}^{\infty}\frac{k!}{(k-q)!}\left(\frac{k+\gamma}{p+\gamma}\right)^r k a_k z^{k-q}\right] \\
&+ \left[\frac{p!}{(p-q)!}p(p-q)z^{p-q} - \sum_{k=p+1}^{\infty}\frac{k!}{(k-q)!}\left(\frac{k+\gamma}{p+\gamma}\right)^r k(k-q)a_k z^{k-q}\right] \\
&= \frac{p!}{(p-q)!}\left[1-\lambda+p-q\right]z^{p-q} - \sum_{k=p+1}^{\infty}\frac{k!k}{(k-q)!}\left(\frac{k+\gamma}{p+\gamma}\right)^r [1-\lambda+k-q]a_k z^{k-q}.
\end{aligned}$$

From (2.5) we get

$$\left| \frac{\frac{p!}{(p-q)!}[\lambda(p-q)(p-q-1)+\phi(p-q)-(\delta-\phi)(1-\lambda)]z^{p-q} - \sum_{k=p+1}^{\infty} \frac{k!k}{(k-q)!}\left(\frac{k+\gamma}{p+\gamma}\right)^r [\lambda(k-q)(k-q-1)+\phi(k-q)-(\delta-\phi)(1-\lambda)]a_k z^{k-q}}{\frac{p!}{(p-q)!}[1-\lambda+p-q]z^{p-q} - \sum_{k=p+1}^{\infty} \frac{k!k}{(k-q)!}\left(\frac{k+\gamma}{p+\gamma}\right)^r [1-\lambda+k-q]a_k z^{k-q}} \right| \leq \propto . \tag{2.6}$$

We know that $|Re(z)| \leq |z|$, so choosing values of z on real axis. After clearing denominator in (2.6) and allowing $z \to 1_-$ through real values, we get

$$\begin{aligned}
&\frac{p!}{(p-q)!}[\lambda(p-q)(p-q-1)+\phi(p-q)-(\delta-\phi)(1-\lambda)] \\
&+\sum_{k=p+1}^{\infty} \frac{k!k}{(k-q)!}\left(\frac{k+\gamma}{p+\gamma}\right)^r [\lambda(k-q)(k-q-1)+\phi(k-q)-(\delta-\phi)(1-\lambda)]a_k \\
&\leq \alpha \left[\frac{p!}{(p-q)!}[1-\lambda+p-q] - \sum_{k=p+1}^{\infty} \frac{k!k}{(k-q)!}\left(\frac{k+\gamma}{p+\gamma}\right)^r [1-\lambda+k-q]a_k \right].
\end{aligned}$$

Therefore

$$\begin{aligned}
&\sum_{k=p+1}^{\infty} \frac{(l+1)!}{(k-q)!}\left(\frac{k+\gamma}{p+\gamma}\right)^r [\lambda(k-q)(k-q-1)+\phi(k-q)-(\delta-\phi)(1-\lambda) \\
&+\alpha(1-\lambda+k-q)]a_k \\
&\leq \frac{(p+1)!}{(p-q)!}[\alpha(1-\lambda+p-q)-\lambda(p-q)(p-q-1)-\phi(p-q)+(\delta-\phi)(1-\lambda)].
\end{aligned}$$

On the line of Theorem 2.1 we prove the converse.

Corollary 2.2 : If the function $f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k, a_k \geq 0$ is in $Y\pounds^*(p, \lambda, \phi, \delta, \alpha)$ then

$$\begin{aligned}
&a_k \leq \left(\frac{p+\gamma}{k+\gamma}\right)^r \\
&\frac{(p+1)!(k-q)![\alpha(1-\lambda+p-q)-\lambda(p-q)(p-q-1)-\phi(p-q)+(\delta-\phi)(1-\lambda)]}{[\lambda(k-q)(k-q-1)+\phi(k-q)-(\delta-\phi)(1-\lambda)+\alpha(1-\lambda+k-q)]}.
\end{aligned}$$

For $k = 1+p, 2+p \cdots$.

With the equality for function

$$f_z = z^p - \left(\frac{p+\gamma}{k+\gamma}\right)^r \frac{(p+1)!(k-q)![\alpha(1-\lambda+p-q)-\lambda(p-q)(p-q-1)-\phi(p-q)+(\delta-\phi)(1-\lambda)]}{(k+1)!(p-q)![\lambda(k-q)(k-q-1)+\phi(k-q)-(\delta-\phi)(1-\lambda)+\alpha(1-\lambda+k-q)]} z^k.$$

For $k = 1+p, 2+p\cdots$.

3. Growth and Distortion Theorem

Theorem 3.1 : A function $f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k$, $a_k \geq 0$ is in $\pounds^*(p,\lambda,\phi,\delta,\alpha)$ then

$$|z|^p - \left(\frac{p+\gamma}{p+1+\gamma}\right)^r \frac{p!(1+p-q)![\alpha(1-\lambda+p-q)-\lambda(p-q)(p-q-1)-\phi(p-q)+(\delta-\phi)(1-\lambda)]}{(1+p)!(p-q)![\lambda(1+p-q)(p-q)+\phi(1+p-q)-(\delta-\phi)(1-\lambda)+\alpha(2-\lambda+p-q)]}|z|^{1+p}$$
$$\leq |f(z)| \leq |z|^p \left(\frac{p+\gamma}{p+1+\gamma}\right)^r \frac{p!(1+p-q)![\alpha(1-\lambda+p-q)-\lambda(p-q)(p-q-1)-\phi(p-q)+(\delta-\phi)(1-\lambda)]}{(1+p)!(p-q)![\lambda(1+p-q)(p-q)+\phi(1+p-q)-(\delta-\phi)(1-\lambda)+\alpha(2-\lambda+p-q)]}|z|^{1+p}.$$

Proof : $f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k, a_k \geq 0$ is in $\pounds^*(p,\lambda,\phi,\delta,\alpha)$ if and only if

$$\sum_{k=p+1}^{\infty} \frac{k!}{(k-q)!}\left(\frac{k+\gamma}{p+\gamma}\right)^r [\lambda(k-q)(k-q-1)+\phi(k-q)-(\delta-\phi)(1-\lambda)+\alpha(1-\lambda+k-q)]a_k$$
$$\leq \frac{p!}{(p-q)!}[\alpha(1-\lambda+p-q)-\lambda(p-q)(p-q-1)-\phi(p-q)+(\delta-\phi)(1-\lambda)]$$

$$f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k$$

$$|f(z)| \leq |z^p| + \left|\sum_{k=p+1}^{\infty} a_k z^k\right|$$

$$|f(z)| \leq |z|^p + \left(\frac{p+\gamma}{p+1+\gamma}\right)^r$$
$$\frac{p!(1+p-q)![\alpha(1-\lambda+p-q)-\lambda(p-q)(p-q-1)-\phi(p-q)+(\delta-\phi)(1-\lambda)]}{(1+p)!(p-q)![\lambda(1+p-q)(p-q)+\phi(1+p-q)-(\delta-\phi)(1-\lambda)+\alpha(2-\lambda+p-q)]}|z|^{1+p} \tag{3.1}$$

Similarly,

$$|f(z)| \geq |z|^p - \left(\frac{p+\gamma}{p+1+\gamma}\right)^r$$

$$\frac{p!(1+p-q)![\alpha(1-\lambda+p-q)-\lambda(p-q)(p-q-1)-\phi(p-q)+(\delta-\phi)(1-\lambda)]}{(1+p)!(p-q)![\lambda(1+p-q)(p-q)+\phi(1+p-q)-(\delta-\phi)(1-\lambda)+\alpha(2-\lambda+p-q)]}|z|^{1+p} \tag{3.2}$$

From (3.1) and (3.2) we have

$$|z|^p - \left(\frac{p+\gamma}{p+1+\gamma}\right)^r$$

$$\frac{p!(1+p-q)![\alpha(1-\lambda+p-q)-\lambda(p-q)(p-q-1)-\phi(p-q)+(\delta-\phi)(1-\lambda)]}{(1+p)!(p-q)![\lambda(1+p-q)(p-q)+\phi(1+p-q)-(\delta-\phi)(1-\lambda)+\alpha(2-\lambda+p-q)]}|z|^{1+p}$$

$$\leq |f(z)| \leq |z|^p + \left(\frac{p+\gamma}{p+1+\gamma}\right)^r$$

$$\frac{p!(1+p-q)![\alpha(1-\lambda+p-q)-\lambda(p-q)(p-q-1)-\phi(p-q)+(\delta-\phi)(1-\lambda)]}{(1+p)!(p-q)![\lambda(1+p-q)(p-q)+\phi(1+p-q)-(\delta-\phi)(1-\lambda)+\alpha(2-\lambda+p-q)]}|z|^{1+p}$$

Hence the result.

Theorem 3.2 : A function $f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k$, $a_k \geq 0$ is in $Y\pounds^*(p,\lambda,\phi,\delta,\alpha)$ then

$$-\left(\frac{p+\gamma}{p+1+\gamma}\right)^r$$

$$\frac{(p+1)!(1+p-q)![\alpha(1-\lambda+p-q)-\lambda(p-q)(p-q-1)-\phi(p-q)+(\delta-\phi)(1-\lambda)]}{(2+p)!(p-q)![\lambda(1+p-q)(p-q)+\phi(1+p-q)-(\delta-\phi)(1-\lambda)+\alpha(2-\lambda+p-q)]}|z|^{1+p}$$

$$\leq |f(z)| \leq |z|^p + \left(\frac{p+\gamma}{p+1+\gamma}\right)^r$$

$$\frac{(p+1)!(1+p-q)![\alpha(1-\lambda+p-q)-\lambda(p-q)(p-q-1)-\phi(p-q)+(\delta-\phi)(1-\lambda)]}{(2+p)!(p-q)![\lambda(1+p-q)(p-q)+\phi(1+p-q)-(\delta-\phi)(1-\lambda)+\alpha(2-\lambda+p-q)]}|z|^{1+p}.$$

Proof : $f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k, a_k \geq 0$ is in $Y\pounds^*(p,\lambda,\phi,\delta,\alpha)$ if and only if

$$\sum_{k=p+1}^{\infty} \frac{(k+1)!}{(k-q)!}\left(\frac{k+\gamma}{p+\gamma}\right)^r [\lambda(k-q)(k-q-1)+\phi(k-q)-(\delta-\phi)(1-\lambda)+\alpha(1-\lambda+k-q)]a_k$$

$$\leq \frac{(p+1)!}{(p-q)!}[\alpha(1-\lambda+p-q)-\lambda(p-q)(p-q-1)-\phi(p-q)+(\delta-\phi)(1-\lambda)]$$

$$f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k$$

$$|f(z)| \leq |z^p| + \left|\sum_{k=p+1}^{\infty} a_k z^k\right|$$

$$|f(z)| \leq |z|^p + \left(\frac{p+\gamma}{p+1+\gamma}\right)^r$$

$$\frac{(p+1)!(1+p-q)![\alpha(1-\lambda+p-q)-\lambda(p-q)(p-q-1)-\phi(p-q)+(\delta-\phi)(1-\lambda)]}{(2+p)!(p-q)![\lambda(1+p-q)(p-q)+\phi(1+p-q)-(\delta-\phi)(1-\lambda)+\alpha(2-\lambda+p-q)]}|z|^{1+p} \tag{3.3}$$

Similarly,

$$|f(z)| \geq |z|^p - \left(\frac{p+\gamma}{p+1+\gamma}\right)^r$$

$$\frac{(p+1)!(1+p-q)![\alpha(1-\lambda+p-q)-\lambda(p-q)(p-q-1)-\phi(p-q)+(\delta-\phi)(1-\lambda)]}{(2+p)!(p-q)![\lambda(1+p-q)(p-q)+\phi(1+p-q)-(\delta-\phi)(1-\lambda)+\alpha(2-\lambda+p-q)]}|z|^{1+p} \tag{3.4}$$

From (3.3) and (3.4) we have

$$|z|^p - \left(\frac{p+\gamma}{p+1+\gamma}\right)^r$$

$$\frac{(p+1)!(1+p-q)![\alpha(1-\lambda+p-q)-\lambda(p-q)(p-q-1)-\phi(p-q)+(\delta-\phi)(1-\lambda)]}{(2+p)!(p-q)![\lambda(1+p-q)(p-q)+\phi(1+p-q)-(\delta-\phi)(1-\lambda)+\alpha(2-\lambda+p-q)]}|z|^{1+p}$$

$$\leq |f(z)| \leq |z|^p + \left(\frac{p+\gamma}{p+1+\gamma}\right)^r$$

$$\frac{(p+1)!(1+p-q)![\alpha(1-\lambda+p-q)-\lambda(p-q)(p-q-1)-\phi(p-q)+(\delta-\phi)(1-\lambda)]}{(2+p)!(p-q)![\lambda(1+p-q)(p-q)+\phi(1+p-q)-(\delta-\phi)(1-\lambda)+\alpha(2-\lambda+p-q)]}|z|^{1+p}$$

Hence the result.

Theorem 3.3 : A function $f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k$, $a_k \geq 0$ is in $\pounds^*(p, \lambda, \phi, \delta, \alpha)$ then

$$p|z|^{p-1} - \left(\frac{p+\gamma}{p+1+\gamma}\right)^r$$
$$\frac{(1+p-q)![\alpha(1-\lambda+p-q)-\lambda(p-q)(p-q-1)-\phi(p-q)+(\delta-\phi)(1-\lambda)]}{(p-q)![\lambda(1+p-q)(p-q)+\phi(1+p-q)-(\delta-\phi)(1-\lambda)+\alpha(2-\lambda+p-q)]}|z|^p$$
$$\leq |f(z)| \leq p|z|^{p-1} + \left(\frac{p+\gamma}{p+1+\gamma}\right)^r$$
$$\frac{(1+p-q)![\alpha(1-\lambda+p-q)-\lambda(p-q)(p-q-1)-\phi(p-q)+(\delta-\phi)(1-\lambda)]}{(p-q)![\lambda(1+p-q)(p-q)+\phi(1+p-q)-(\delta-\phi)(1-\lambda)+\alpha(2-\lambda+p-q)]}|z|^p.$$

Proof : $f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k, a_k \geq 0$ is in $\mathcal{L}^*(p, \lambda, \phi, \delta, \alpha)$ if and only if

$$\sum_{k=p+1}^{\infty} \frac{k!}{(k-q)!} \left(\frac{k+\gamma}{p+\gamma}\right)^r [\lambda(k-q)(k-q-1)+\phi(k-q)-(\delta-\phi)(1-\lambda)+\alpha(1-\lambda+k-q)]a_k$$

$$\leq \frac{p!}{(p-q)!}[\alpha(1-\lambda+p-q)-\lambda(p-q)(p-q-1)-\phi(p-q)+(\delta-\phi)(1-\lambda)]$$

$$f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k$$

$$f'(z) = pz^{p-1} - \sum_{k=p+1}^{\infty} k a_k z^{k-1}$$

$$|f'(z)| \leq |pz^{p-1}| + \left|\sum_{k=p+1}^{\infty} k a_k z^{k-1}\right|$$

$$|f'(z)| \geq p|z|^{p-1} + \frac{p+\gamma)}{p+1+\gamma})^r$$

$$\frac{(1+p-q)![\alpha(1-\lambda+p-q)-\lambda(p-q)(p-q-1)-\phi(p-q)+(\delta-\phi)(1-\lambda)]}{(p-q)![\lambda(1+p-q)(p-q)+\phi(1+p-q)-(\delta-\phi)(1-\lambda)+\alpha(2-\lambda+p-q)]}|z|^p \tag{3.5}$$

Similarly,

$$|f'(z)| \geq p|z|^{p-1} - \left(\frac{p+\gamma}{p+1+\gamma}\right)^r$$

$$\frac{(1+p-q)![\alpha(1-\lambda+p-q)-\lambda(p-q)(p-q-1)-\phi(p-q)+(\delta-\phi)(1-\lambda)]}{(p-q)![\lambda(1+p-q)(p-q)+\phi(1+p-q)-(\delta-\phi)(1-\lambda)+\alpha(2-\lambda+p-q)]}|z|^p \tag{3.6}$$

From (3.5) and (3.6) we have

$$p|z|^{p-1} - \left(\frac{p+\gamma}{p+1+\gamma}\right)^r$$

$$\frac{(1+p-q)![\alpha(1-\lambda+p-q)-\lambda(p-q)(p-q-1)-\phi(p-q)+(\delta-\phi)(1-\lambda)]}{(p-q)![\lambda(1+p-q)(p-q)+\phi(1+p-q)-(\delta-\phi)(1-\lambda)+\alpha(2-\lambda+p-q)]}|z|^p$$

$$\leq |f'(z)| \leq p|z|^{p-1} + \left(\frac{p+\gamma}{p+1+\gamma}\right)^r$$

$$\frac{(1+p-q)![\alpha(1-\lambda+p-q)-\lambda(p-q)(p-q-1)-\phi(p-q)+(\delta-\phi)(1-\lambda)]}{(p-q)![\lambda(1+p-q)(p-q)+\phi(1+p-q)-(\delta-\phi)(1-\lambda)+\alpha(2-\lambda+p-q)]}|z|^p$$

Hence the result.

Theorem 3.4 : A function $f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k$, $a_k \geq 0$ is in $Y\pounds^*(p,\lambda,\phi,\delta,\alpha)$ then

$$p|z|^{p-1} - \left(\frac{p+\gamma}{p+1+\gamma}\right)^r$$
$$\frac{(1+p-q)![\alpha(1-\lambda+p-q)-\lambda(p-q)(p-q-1)-\phi(p-q)+(\delta-\phi)(1-\lambda)]}{(p-q)![(p-q)+\phi(1+p-q)-(\delta-\phi)(1-\lambda)+\alpha(2-\lambda+p-q)]}|z|^p$$
$$\leq |f'(z)| \leq p|z|^{p-1} + \left(\frac{p+\gamma}{p+1+\gamma}\right)^r$$
$$\frac{(1+p-q)![\alpha(1-\lambda+p-q)-\lambda(p-q)(p-q-1)-\phi(p-q)+(\delta-\phi)(1-\lambda)]}{(p-q)![\lambda(1+p-q)(p-q)+\phi(1+p-q)-(\delta-\phi)(1-\lambda)+\alpha(2-\lambda+p-q)]}|z|^p.$$

Proof : $f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k, a_k \geq 0$ is in $Y\pounds^*(p,\lambda,\phi,\delta,\alpha)$ if and only if

$$\sum_{k=p+1}^{\infty} \frac{(k+1)!}{(k-q)!}\left(\frac{k+\gamma}{p+\gamma}\right)^r [\lambda(k-q)(k-q-1)+\phi(k-q)-(\delta-\phi)(1-\lambda)+\alpha(1-\lambda+k-q)]a_k$$

$$\leq \frac{(p+1)!}{(p-q)!}[\alpha(1-\lambda+p-q)-\lambda(p-q)(p-q-1)-\phi(p-q)+(\delta-\phi)(1-\lambda)]$$

$$f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k$$

$$f'(z) = pz^{p-1} - \sum_{k=p+1}^{\infty} k a_k z^{k-1}$$

$$|f'(z)| \leq |pz^{p-1}| + \left|\sum_{k=p+1}^{\infty} k a_k z^{k-1}\right|$$

$$|f'(z)| \leq p|z|^{p-1} + \left(\frac{p+\gamma}{p+1+\gamma}\right)^r$$
$$\frac{(1+p-q)![\alpha(1-\lambda+p-q)-\lambda(p-q)(p-q-1)-\phi(p-q)+(\delta-\phi)(1-\lambda)]}{(p-q)![\lambda(1+p-q)(p-q)+\phi(1+p-q)-(\delta-\phi)(1-\lambda)+\alpha(2-\lambda+p-q)]}|z|^p \tag{3.7}$$

Similarly,

$$|f'(z)| \geq p|z|^{p-1} - \left(\frac{p+\gamma}{p+1+\gamma}\right)^r$$
$$\frac{(1+p-q)![\alpha(1-\lambda+p-q)-\lambda(p-q)(p-q-1)-\phi(p-q)+(\delta-\phi)(1-\lambda)]}{(p-q)![\lambda(1+p-q)(p-q)+\phi(1+p-q)-(\delta-\phi)(1-\lambda)+\alpha(2-\lambda+p-q)]}|z|^p \tag{3.8}$$

From (3.7) and (3.8) we have

$$p|z|^{p-1} - \left(\frac{p+\gamma}{p+1+\gamma}\right)^r$$

$$\frac{(1+p-q)![\alpha(1-\lambda+p-q)-\lambda(p-q)(p-q-1)-\phi(p-q)+(\delta-\phi)(1-\lambda)]}{(p-q)![\lambda(1+p-q)(p-q)+\phi(1+p-q)-(\delta-\phi)(1-\lambda)+\alpha(2-\lambda+p-q)]}|z|^p$$

$$\leq |f'(z)| \leq p|z|^{p-1} + \left(\frac{p+\gamma}{p+1+\gamma}\right)^r$$

$$\frac{(1+p-q)![\alpha(1-\lambda+p-q)-\lambda(p-q)(p-q-1)-\phi(p-q)+(\delta-\phi)(1-\lambda)]}{(p-q)![\lambda(1+p-q)(p-q)+\phi(1+p-q)-(\delta-\phi)(1-\lambda)+\alpha(2-\lambda+p-q)]}|z|^p$$

Hence the result.

References

[1] Goodman A. W., Univalent functions and nonanalytic curves, Proc. Amer. Math. Soc., 8 (1957), 598601.

[2] Murugusundaramoorthy G. and Srivastava H. M., Neighborhoods of certain classes of analytic functions of complex order, J. Inequal. Pure. Appl. Math., 5(2) (2004), Art. 24.

[3] Srivastava H. M., Suchithra K., Adolf Stephen B. and Sivasubramanian S., Inclusion and neighborhood properties of certain subclasses of analytic and multivalent functions of complex order, Journal of inequality in Pure and Applied Mathematics, 7(Issue 5), Article 191, (2006).

[4] Cho N. E. and Srivastava H. M., Argument estimates of certain analytic functions defined by a class of multiplier transformations, Math. Comput. Modelling, 37 (2003), 3949.

[5] Jadhav Pravin Ganpat and Khairnar S. M., Certain subclasses of analytic and multivalent functions in the unit disc International J. of Math. Sci. & Engg. Appls. (IJMSEA), 6(VI) (November, 2012), 175-192.

[6] Raina R. K. and Srivastava H. M., Inclusion and neighborhood properties of some analytic and multivalent functions, J. Inequal. Pure. Appl. Math., 7(1) (2006), Art.5.

[7] Khairnar S. M. and More Meena, On a class of meromorphic multivalent functions with negative coefficients defined by Ruscheweyh derivative, International Mathematical Forum, 3(22) (2008), 10871097/

[8] Ruscheweyh S., Neighborhoods of univalent functions, Proc. Amer. Math. Soc., 81 (1981), 521527.