International J. of Multidispl. Research & Advcs. in Engg. (IJMRAE),
ISSN 0975-7074, Vol. 8 No. II (August, 2016), pp. 1-6

SOME REMARKS ON HOMOMORPHISMS OF LIE IDEALS IN PRIME RINGS WITH INVOLUTIONS

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Abstract

Let R be a $*$-prime ring of *char* $R \neq 2$ and L be a square closed $*$-Lie ideal of R. Suppose that R admits a generalized left $*$-derivation $F : R \to R$. If F acts as a homomorphism on L, then F is right $*$-centralizer on R.

1. Introduction

Throughout this paper, R will denote an associative ring (may be without unity 1, unless specifically thier use) with center $Z(R)$. The ring R is n-torsion free for any prime integer where $n > 1$ is an integer, if $nx = 0$, $x \in R$ implies $x = 0$. As usual the

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Key Words and Phrases : *Generalized left $*$-derivations, Jordan $*$-centralizer, Prime rings, Lie ideals.*

2000 AMS Subject Classification : 16N60, 16W25, 16W10, 16U80.

commutator $xy - yx$ will be denoted by $[x, y]$. We shall frequently use the commutator identities $[xy, z] = x[y, z] + [x, z]y$ and $[x, yz] = y[x, z] + [x, y]z$ for all $x, y, z \in R$. Recall that a ring R is prime if for any $a, b \in R$, $aRb = \{0\}$ implies that either $a = 0$ or $b = 0$. An additive mapping $x \mapsto x^*$ on a ring R is called an involution on R if $(xy)^* = y^*x^*$ and $(x^*)^* = x$ hold for all $x, y \in R$. A ring equipped with an involution is called a ring with an involution or $*$-ring. A ring with an involution '$*$' is said to be a $*$-prime ring if $aRb = aRb^* = \{0\}$ (or $aRb = a^*Rb = \{0\}$) implies either $a = 0$ or $b = 0$. It is obvious that every prime ring with an involution '$*'$ is a *-prime ring but the converse may not be necessarily true in general. An additive subgroup L of R is said to be Lie ideal of R if $[L, R] \subseteq L$. A Lie ideal L is said to be square closed if $u^2 \in L$ for all $u \in L$ and L is said to be $*$-Lie ideal if $L = L^*$.

An additive mapping δ is called derivation (resp. Jordan derivation) if $\delta(xy) = \delta(x)y + x\delta(y)$ (resp. $\delta(x^2) = \delta(x)x + x\delta(x)$) holds for all $x, y \in R$. An additive mapping $f : R \to R$ is called generalized derivation if there exists a derivation $\delta : R \to R$ such that $f(xy) = f(x)y + x\delta(y)$ holds for all $x, y \in R$. An additive mapping $d : R \to R$ is said to be left derivation (resp. Jordan left derivation) if $d(xy) = xd(y) + yd(x)$ (resp. $d(x^2) = 2xd(x)$) holds for all $x, y \in R$. Clearly, every left derivation on a ring R is a Jordan left derivation but the converse need not be true in general (see for example [16, Example 1.1]). In [2], Ashraf and first author showed that if R is a 2-torsion free prime ring and $d : R \to R$ is an additive mapping such that $d(u^2) = 2ud(u)$ for all u in a square closed Lie ideal L of R, then $d(uv) = ud(v) + vd(u)$ for all $u, v \in L$.

Let S be a nonempty subset of R and $\delta : R \to R$ be a derivation of R. If $\delta(xy) = \delta(x)\delta(y)$ (resp. $\delta(xy) = \delta(y)\delta(x)$) for all $x, y \in S$, then δ is said to acts as homomorphism (resp. anti-homomorphism) on S. In [3], Bell and Kappe proved that if I is a nonzero right ideal of a prime ring R and δ is a derivation of R such that δ acts as homomorphism (resp. anti-homomorphism) on I, then $\delta = 0$. Further, this result was extended by Rehman [10] for generalized derivation. Similar types of results have been proved for generalized left derivation by Rehman and Ansari in [14]. A mapping $B : R \times R \to R$ is said to be symmetric if $B(x, y) = B(y, x)$ for all $x, y \in R$. Following [5], a bi-additive map $B : S \times S \to R$ is called bi-derivation on S if it is a derivation in each argument, the map $y \mapsto B(x, y)$ is a derivation S into R. Typical examples are mappings of the form $(x, y) \mapsto \lambda[x, y]$, where λ is an element of the center of R. The concept of bi-derivation was introduced by Maska [8]. Further, Bresar [5] showed that every bi-derivation B of a non-commutative prime ring R is of the form $B(x, y) = \lambda[x, y]$ for some $\lambda \in C$, the extended centroid of R.

2. Generalized Left $*$-derivation

Motivated by the definitions of $*$-derivation and generalized $*$-derivation, Rehman and Ansari [14] recently introduced the notions of left $*$-derivation and generalized left $*$-derivation as follows: Let R be a $*$-ring. An additive mapping $d : R \to R$ is said to be left $*$-derivation if $d(xy) = x^*d(y) + yd(x)$ for all $x, y \in R$. An additive mapping $F : R \to R$ is said to be generalized left $*$-derivation if there exists a left $*$-derivation d such that $F(xy) = x^*F(y) + yd(x)$ for all $x, y \in R$. The concept of generalized left $*$-derivations cover the concept of left $*$-derivations. Moreover, a generalized left $*$-derivation with $d = 0$ includes the concept of right $*$-centralizer i.e., an additive mapping $T : R \to R$ satisfying $T(xy) = x^*T(y)$ for all $x, y \in R$. In [3], Bell and Kappe discussed the derivations acting as a homomorphisms or an anti-homomorphisms on a nonzero right ideal of a prime ring. Recently, Shakir [1], proved some results taking generalized left derivation of a prime ring R which acts either as homomorphisms or as an anti-homomorphisms on a certain well behaved subset of R. Further, this result was extended by Rehman and Ansari in [14] in the setting of generalized left $*$-derivation and generalized left $*$-bi-derivation. In the present section our objective is to extend the results obtained in [15], for Lie ideals.

More precisely, we prove the following:

Theorem 2.1 : Let R be a $*$-prime ring of charR $R \neq 2$ and $L \not\subseteq Z(R)$ is a square closed $*$-Lie ideal of R with involution '$*$'. Suppose that $F : R \to R$ is a generalized left $*$- derivation with associated left $*$-derivation on R. If F acts as a homomorphisms on L, then F is right $*$-centralizer on R.

In order to prove the main result of this section we will make use of the following Lemmas:

Lemma 2.1 ([4, Lemma 4]) : Let R be a $*$-prime ring of characteristic different from two, L be a nonzero $*$-Lie ideal of R and $a, b \in L$. If $aLb^* = \{0\}$ then $a = 0$ or $b = 0$ or $L \subseteq Z(R)$.

Lemma 2.2 ([7, Lemma 3.3]) : Let R be a $*$-prime ring of characteristic different from two, L be a nonzero $*$-Lie ideal of R. If $a \in R$ such that $[a, L] \subseteq Z(R)$, then either $a \in Z(R)$ or $L \subseteq Z(R)$.

The following Lemma is immediate consequence of Lemma 2.2.

Lemma 2.3 : Let R be a $*$-prime ring of characteristic different from two, L be a nonzero $*$-Lie ideal of R. Suppose $[L, L] \subseteq Z(R)$, then $L \subseteq Z(R)$.

Now, we are in a position to prove our main result of this section:

Proof of Theorem 2.1 : If F acts as a homomorphisms on L, then $F(xy) = F(x)F(y)$ for all $x, y \in L$ and also from the definition of generalized left $*$-derivation, we have $F(xy) = x^*F(y) + yd(x)$ for all $x, y \in R$, where d is a left $*$-derivation of R. Now

$$\begin{aligned} F(xyz) &= F(x(yz)) = x^*F(yz) + yzd(x) \\ &= x^*F(y)F(z) + yzd(x) \ \text{ for all } \ x, y, z \in L. \end{aligned} \tag{2.1}$$

On the other hand

$$\begin{aligned} F(xyz) &= F((xy)z) = F(xy)F(z) \\ &= x^*F(y)F(z) + yd(x)F(z) \ \text{ for all } \ x, y, z \in L. \end{aligned} \tag{2.2}$$

Combining (2.1) and (2.2), we obtain $x^*F(y)F(z) + yzd(x) = x^*F(y)F(z) + yd(x)F(z)$, for all $x, y, z \in L$. This yields that $y(zd(x) - d(x)F(z)) = 0$ for all $x, y, z \in L$. Multiplying left side by $zd(x) - d(x)F(z)$ to the above relation, we obtain $(zd(x) - d(x)F(z))y(zd(x) - d(x)F(z) = 0$ for all $x, y, z \in L$. Then by Lemma 2.1, we obtain

$$zd(x) - d(x)F(z) = 0 \ \text{ for all } \ x, z \in L. \tag{2.3}$$

Replacing x by $2xy$ and using the fact that *char* $R \neq 2$ in the above relation, we get

$$zx^*d(y) + zyd(x) - x^*d(y)F(z) - yd(x)F(z) = 0 \ \text{ for all } \ x, y, z \in L.$$

Using relation (2.3) in the above relation, we find that

$$zx^*d(y) + zyd(x) - x^*zd(y) - yzd(x) = 0 \ \text{ for all } \ x, y, z \in L.$$

This yields that

$$[z, x^*]d(y) + [z, y]d(x) = 0 \ \text{ for all } \ x, y, z \in L.$$

In particular, replacing y by z in the above relation, we find that

$$[z, x^*]d(z) = 0 \ \text{ for all } \ x, z \in L.$$

Again, replace x by x^* in the above expression, to obtain

$$[x,z]d(z) = 0` \text{ for all } x,z \in L. \tag{2.4}$$

Replacing x by $2xy$ and using the fact that *char* $R \neq 2$, we get $[x,z]yd(z) = 0$ for all $x,y,z \in L$. Now, multiplying left side by $d(z)$ and right side by $[x,z]$, and by Lemma 2.1, we get $d(z)[x,z] = 0$ for all $x,z \in L$. Then, replacing z by y and linearizing the above relation, we obtain $d(x)[z,y] + d(z)[x,y] = 0$ for all $x,y,z \in L$, and hence

$$d(x)[z,y] = -d(z)[x,y] \text{ for all } x,y,z \in L. \tag{2.5}$$

Replacing y by $2uy$ in (2.4) and again using (2.4), we get $2d(x)u[x,y] = 0$ for all $x,y,u \in L$. Since *char* $R \neq 2$, we find that $d(x)u[x,y] = 0$. Now, replace u by $2[z,y]r$ and use the fact that *char* $R \neq 2$, to get $d(x)[z,y]r[x,y] = 0$ for all $x,y,z \in L$ and $r \in R$ and hence application of (2.5), we obtain $d(z)[x,y]r[x,y] = 0$ for all $x,y,z \in L$ and $r \in R$. Again replacing r by $rd(z)$ in the above expression, we get $d(z)[x,y]rd(z)[x,y] = 0$ for all $x,y,z \in L$ and $r \in R$, that is, $d(z)[x,y]Rd(z)[x,y] = \{0\}$ for all $x,y,z \in L$. Thus primeness of R forces that $d(z)[x,y] = 0$ for all $x,y,z \in L$. Again, replacing x by $2tx$ and using the fact that *char* $R \neq 2$, we get $d(z)t[x,y] = 0$ for all $x,y,z,t \in L$. Again by Lemma 2.1, we get $d(z) = 0$ for all $z \in L$. Replacing z by $2r[y,z]$ and using *char* $R \neq 2$, we obtain $[y,z]d(r) = 0$ for all $y,z \in L$ and $r \in R$. Again, replacing y by $2yx$ and using *char* $R \neq 2$, we get $[y,z]xd(r) = 0$ for all $x,y,z \in L$ and $r \in R$. Therefore, by Lemma 2.1, we get $d = 0$ on R. Therefore, $F(xy) = x^*F(y)$ for all $x,y \in R$. Hence, we get the required result. We immediately get the following corollary from the above theorem:

Corollary 2.1 : Let R be a $*$-prime ring of *char* $R \neq 2$ with involution '$*$'. Suppose that $d : R \to R$ is a left $*$-derivation on R. If d acts as a homomorphisms on R, then d is right $*$-centralizer on R.

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