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SPACES FOR HANKEL TYPE TRANSFORMATION

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Abstract

In this paper we have defined different spaces for the Hankel type transformation [1] defined by,

$$(\mathcal{H}_\lambda \phi)(y) = \int_0^\infty \left(\frac{y}{x}\right)^{\lambda/2} J_\lambda(2\sqrt{xy})\phi(x)dx$$

where J_λ denotes the Bessel function of the first kind and of order λ.

1. Introduction

The Hankel type transformation defined by

$$(\mathcal{H}_\lambda \phi)(y) = \int_0^\infty \left(\frac{y}{x}\right)^{\lambda/2} J_\lambda(2\sqrt{xy})\phi(x)dx \tag{1}$$

where, J_λ denotes the Bessel function of the first kind and order λ was studied on distribution spaced, $\mathcal{H}'_{a,\lambda}$ by M. S. Chaudhary [1].

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Given $\lambda \in \square, a > 0$ this author introduced the space $\mathcal{H}_{a,\lambda}$ of all infinitely differentiable functions $\phi = \phi(x), x \in (0, \infty)$, k-non-negative integer, such that

$$\gamma_k^{\lambda}(\phi) = \gamma_k^{\lambda,a}(\phi) = \sup |e^{-ax}\Delta_{\lambda,x}^k \phi(x)| < \infty$$

where

$$\Delta_{\lambda,x}^k = [Dx^{-\lambda+1}Dx^{\lambda}]^k; \quad D = \frac{d}{dx}$$

topology is generated by the family of Seminorms $\{\gamma_k^{\lambda,a}\}$, $\mathcal{H}_{a,\lambda}$ becomes a frechet space. $\mathcal{H}'_{a,\lambda}$ - is the the dual space of $\mathcal{H}_{a,\lambda}$.

2. The Space $\mathcal{H}_{\lambda}^p$

We introduce a space $\mathcal{H}_{\lambda}^p, p \in \square$ of all those $\phi \in L^2(I)$ such that the distributions $h_{\lambda,j}\phi(0 \leq j \leq p)$ are regular and satisfy

$$\|\phi\|_{\lambda,p} = \left\{ \sum_{i+j=0}^{p} \int_0^{\infty} |x^i h_{\lambda,j}\phi(x)|^2 dx \right\}^{1/2} < \infty$$

where $h_{\lambda,0}$ is the identity operator and $h_{\lambda,j}$ denotes the operator $N_{\lambda+j-1\cdots N_{\lambda}}$ $(j \in \square, j \geq 1)$, with $x^{\lambda+1}Dx^{-\lambda+1}$.

Now let's denote $\mathcal{H}_{\lambda}^{-p}$ as dual space of $\mathcal{H}_{\lambda}^p$. For $\phi, \psi \in \mathcal{H}_{\lambda}^p$.

We define inner product as,

$$[\phi, \psi]_{\lambda,p} = \sum_{1+j=0}^{p} \int_0^{\infty} x^{zi} h_{\lambda,j}\phi(x) h_{\lambda,j}\overline{\psi}(x)dx, \ (\phi, \psi \in \mathcal{H}_{\lambda}^p).$$

Under this norm $\mathcal{H}_{\lambda}^{-p}$ and $\mathcal{H}_{\lambda}^p$ both are Hilbert spaces.

Now we will discuss about multipliers and Hankel convolution operators of the spaces $\mathcal{H}_{\lambda}^p$.

3. The Space of Multipliers $\beta_{p,q}$

In this section we introduce the space $\beta_{p,q}, p, q \in \square$ of multipliers from $\mathcal{H}_{\lambda}^p$ into $\mathcal{H}_{\lambda}^q$, i.e. of all those functions $\theta : I \to \square$ such that $\theta\phi \in \mathcal{H}_{\lambda}^q$ for each $\phi \in \mathcal{H}_{\lambda}^p$ and the mapping $\phi \to \theta\phi$ is continuous.

Clearly $\beta_{p,q}$ acts as a space of multiplier form $\mathcal{H}_{\lambda}^{-q}$ into $\mathcal{H}_{\lambda}^{-p}$ by transposition.

Now we define the norm

$$\|\theta\|_{p,q} = \sup\{\|\theta\phi\|_{\lambda,y} : \phi \in \mathcal{H}_{|}^{p} la, \|\phi\|_{\lambda,p} \leq 1\}, (\theta \in \beta_{p,q})$$

under this norm $\beta_{p,q}$ is Banach [2]. With the help of [2] we can give following alternate description about $\beta_{p,q}$.

Proposition 3.1 : Let $p, q \in \square, \theta : I \to C$. Then $\theta \in \beta_{p,q}$ iff $\theta\phi \in \mathcal{H}_{\lambda}^{q}$ for each $\phi \in \mathcal{H}_{a,\lambda}$ and $\|\theta\phi\|_{\lambda,q} \leq C\|\phi\|_{\lambda,p}(\phi \in \mathcal{H}_{a,\lambda})$.

Now let $p, q \in \square$ then $x^{-\lambda+1}\beta_{p,q}$ is a subspace of $H'_{a,\lambda}$. See lemma 2.2 [2].

Lemma 3.1 : Let $p, q \in N$ and $\phi \in \mathcal{H}'_{a,\lambda}$ then $T \in x^{-\lambda+1}\beta_{p,q}$ iff $x^{-\lambda+1}\phi(x)T(x) \in \mathcal{H}_{\lambda}^{q}$ for each $\phi \in \mathcal{H}_{a,\lambda}$ with

$$\|x^{-\lambda+1}\phi(x)T(x)\|_{\lambda,q} \leq C\|\phi\|_{\lambda,p} \quad (\phi \in \mathcal{H}_{a,\lambda}). \tag{2}$$

4. Multiplication Distribution Space $\mathcal{MD}(\mathcal{H}_{\lambda}^{p}, \mathcal{H}_{\lambda}^{q})$

Now let's introduce the space called multiplication distribution space from $\mathcal{H}_{\lambda}^{[}$ into $\mathcal{H}_{\lambda}^{q}$ for $p, q \in N$ as

$$\mathcal{MD}(\mathcal{H}_{|}^{p} la, \mathcal{H}_{\lambda}^{q}) = \{F \in \mathcal{H}'_{a,\lambda} | \exists K_F \in L(\mathcal{H}_{\lambda}^{p}, \mathcal{H}_{\lambda}^{q})$$

with

$$x^{-\lambda+1}\phi(x)F(x) = (L_{F\phi})(x)(\phi \in \mathcal{H}_{a,\lambda})$$

the space of multiplication distributions from $\mathcal{H}_{\lambda}^{p}$ into $\mathcal{H}_{\lambda}^{q}$ where $L(\mathcal{H}_{\lambda}^{p}, \mathcal{H}_{\lambda}^{q})$ denotes the space of all continuous linear mappings from $\mathcal{H}_{\lambda}^{p}$ into $\mathcal{H}_{\lambda}^{q}$.

With the help of Lemma 2.1 we can give the relation between two spaces defined above as,

Theorem 4.1 : For each $p, q \in N$ the identity $\mathcal{MD}(\mathcal{H}_{\lambda}^{p}, \mathcal{H}_{\lambda}^{q}) = x^{-\lambda+1}\beta_{p,q}$ holds.

5. The Hankel Translation and Convolution Operator on $\mathcal{H}_{\lambda}^{p}$

In this section we define Hankel Translation and Convolution Operator on $\mathcal{H}_{|}^{p}\lambda$ for this we require the following theorem whose proof is similar to proof of [2].

Theorem 5.1 : For every $p \in \square$and $y \in 1$, the function $g_{\lambda,t}((xt)) = t^{-\lambda+1}(xtg)^{-1} \times J_{\lambda}(xt)(x \in I)$ lies in $\beta_{p,q}$.

With the help of this we define the Hankel translation operator J_t on H_{λ}^{p} by,

$$(J_t\phi)(x) = \mathcal{H}_{\lambda}(g_{\lambda}, H_{\lambda}\phi)(x), \quad (\phi \in H_{\lambda}^{p}, x \in I).$$

Theorem 5.2 : Let $p \in \square$, the operator $J_t(t \in I)$ is well-defined and continuous from $\mathcal{H}_\lambda^p$ into itself. Moreover, if $p \geq 1$ then the identity $(J_t\phi)(x) = (J_x\phi)(t)$ $(x_1 t \in z)$ holds for every $\phi \in \mathcal{H}_|^p la$. For proof see proposition 3.2 [2].
We define the spaces of convolution operators $\beta_{p,q}^*$ as

$$\beta_{p,q}^* = \{T \in H_{a,\lambda}' : t^{-\lambda+1}(H_\lambda' T)(t) \in \beta_{p,q}\}$$

with the norm as

$$|T|_{p,q}^* = \|t^{-\lambda+1}(H_{a,\lambda}' T)(t)\|_{p,q} \quad (T \in \beta_{p,q}^*).$$

Theorem 5.3 : The generalized Hankel Transformation is an isomorphism from $\beta_{p,q}^*$ onto $x^{-\lambda+1}\beta_{p,q}$.
Proof : Proof is similar to proof of [2, proposition 4-1].
Now fix $p, q \in \square$.
By [3, proposition 2.12] $\mathcal{H}_{a,\lambda}$ is dense in $\mathcal{H}_\lambda^p$. For each $T \in \beta_{p,q}^*$ the continuous mapping
$* : (\mathcal{H}_{a,\lambda}, \|\square\|_{\lambda,p}) \to (\mathcal{H}_\lambda^q, \|\square\|_{\lambda,q})$
$\phi \to T * \phi$
admits a unique extension up to $\mathcal{H}_\lambda^p$ preserving the norm, which we denoting by the same symbol $*$. Hence $\beta_{p,q}^*$ may be regarded as a subspace of $L(\mathcal{H}_|^p la, \mathcal{H}_\lambda^q)$ and the norm restricted to $\beta_{p,q}^*$ will be represented by $\|\square\|_{p,q}^*$.
Theorem 5.4 : Let $p, q \in \square$ for each $T \in \beta_{p,q}^*$ and $\phi \in \mathcal{H}_\lambda^p$ the identity

$$\mathcal{H}_\lambda(T * \phi)(t) = t^{-\lambda+1}(\mathcal{H}_\lambda' T)(t)(\mathcal{H}_\lambda\phi)(t) \quad (t \in I) \tag{1}$$

(where $\mathcal{H}_\lambda'$ in the generalized Hankel transformation defined on $\mathcal{H}_{a,\lambda}'$).
Further the norms $\|\square\|_{p,q}^*$ and $|\square|_{p,q}^*$ coincide on $\beta_{p,q}^*$.
Proof similar to [2, proposition 4.3].
Let $p, q \in \square$. The Hankel convolution of $T \in \beta_{p,q}^*$ and $u \in H_\lambda^{-q}$ is functional $T^* u \in \mathcal{H}_\lambda^{-p}$, given by,

$$(T * u, \phi) = \langle u, T * \phi\rangle \quad (\phi \in \mathcal{H}_\lambda^p).$$

Note that, for a fixed $T \in \beta_{p,q}^*$, the mapping $u \to T * u$ from H_λ^{-q} into H_λ^{-p} is the transpose of the mapping $\phi \to T * \phi$ from H_λ^p into H_λ^q.
Theorem 5.5 : Let $q \in \square$ and $T \in \mathcal{H}_\lambda'$. The following are equivalents

(i) $T \in \beta_q^*$ where β_q^* is defined in [2].

(ii) The mapping $\phi \to T * \phi$ is continuous from $\mathcal{H}_{a,\lambda}$ into $\mathcal{H}_{\lambda}^{q}$.

(iii) The mapping $u \to T * u$ is continuous from $\mathcal{H}_{\lambda}^{-q}$ into $\mathcal{H}'_{a,\lambda}$ when $\mathcal{H}'_{a,\lambda}$ is enclosed with either its weak or its strong topology.

Proof : Similar to proof proposition 4.7 [2].

6. Multipliers on $\mathcal{H}_{\lambda}^{p}(p \in z)$

Consider the spaces $\beta_{p,q}$ of multipliers from $\mathcal{H}_{\lambda}^{p}$ into $\mathcal{H}_{\lambda}^{q}$ and of $\mathcal{H}_{\lambda}^{-q}$ into $\mathcal{H}_{\lambda}^{-p}$ which is studied above. Now the space β of multiplier of $\mathcal{H}_{a,\lambda}$ and $\mathcal{H}'_{a,\lambda}$ is expressed as a projective-inductive limit of Hilbert spacer as shown below, By [2]

$$\beta_q = ind \lim_{p \to \infty} \beta_{p,q}.$$

Now, the generalized Hankel transformation makes β_q^* and $x^{-\lambda+1}\beta_q$ isomorphic, where the latter space is topological so that the mapping $\phi \to x^{-\lambda+1}\phi(x)$ defined an isomorphism from β_q onto $x^{-\lambda+1}\beta_q$. We may consider,

$$\beta = Proj \lim_{q \to \infty} \beta_q.$$

Here β is the space of multipliers of $\mathcal{H}_{a,\lambda}$ and $\mathcal{H}'_{a,\lambda}$, topolized in such a way that the generalized Hankel transformation is an isomorphism between $\beta'_{\lambda,*}$ and $x^{-\lambda+1/2}\phi(x)$.

Theorem 6.1 : For each $q \in \square, \beta_q$ is the space of all continuous multipliers from $\mathcal{H}_{a,\lambda}$ into $\mathcal{H}_{\lambda}^{q}$ and from $\mathcal{H}_{\lambda}^{-q}$ into $H'_{a,\lambda}$.

Theorem 6.2 : Given $q \in \square$, let $\mathcal{MD}(\mathcal{H}_{a,\lambda}, \mathcal{H}_{\lambda}^{q})$ denote the space all those $F \in H'_{a,\lambda}$ such that $x^{-\lambda+1}\phi(x)F(x) \in \mathcal{H}_{\lambda}^{q}$ for all $\phi \in \mathcal{H}_{a,\lambda}$ and the mapping $\phi \to x^{-\lambda+1}\phi(x)F(x)$ is continuous from $\mathcal{H}_{a,\lambda}$ into $\mathcal{H}_{\lambda}^{q}$. Then $\mathcal{MD}(\mathcal{H}_{a,\lambda}, \mathcal{H}_{\lambda}^{q}) = x^{-\lambda+1}\beta_q$.

Theorem 6.3 : Let $q \in \square$

(i) The identity $x^{-\lambda+1}\beta_q = ind \lim_{k \to \infty}(1+x^2)^k \mathcal{H}_{\lambda}^{q}$ holds. Moreover $x^{-\lambda+1}\beta_q$ is the strong dual of $proj \lim_{k \to \infty}(1+x^2)^{-1}\mathcal{H}_{\lambda}^{-q}$.

(ii) The embedding $x^{-\lambda+1}\beta_q \to H'_{a,\lambda}$ is continuous, when $\mathcal{H}'_{a,\lambda}$ is endowed with either its weak or its strong topology.

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