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SOME PROPERTIES OF MULTIVALENT FUNCTION DEFINED BY DIFFERENTIAL OPERATOR

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Abstract

By means of certain differential operator we introduce and investigate two subclasses $\pounds^*(p, \lambda, \phi, \delta, \alpha)$ and $Y\pounds^*(p, \lambda, \phi, \delta, \alpha)$ of p-valently analytic functions. The various results obtained here for each of these classes. We have attempted to obtain radius of starlikeness, convexity and closure theorem for the classes $\pounds^*(p, \lambda, \phi, \delta, \alpha)$ and $Y\pounds^*(p, \lambda, \phi, \delta, \alpha)$.

1. Introduction

This paper is devoted to study of multivalent functions and its various properties. By means of certain differential operator we introduce and investigate two subclasses $\pounds^*(p, \lambda, \phi, \delta, \alpha)$ and $Y\pounds^*(p, \lambda, \phi, \delta, \alpha)$ of p-valently analytic functions. The various results obtained here for each of these classes.

Let $A(p)$ denote the class of functions $f(z)$ of the form

Key Words and Phrases : *Multivalent function, Radius of star likeness, Differential operator.*

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \tag{1.1}$$

which are analytic and multivalent in the unit disk $E = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ for $p \in \mathbb{N}$.

Definition 1.1 : A function $f(z) \in A(p)$ is said to be in the subclass $S(\xi)$ of starlike function if

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \xi, \quad z \in E, \quad 0 \leq \xi < 1.$$

Definition 1.2 : A function $f(z) \in A(p)$ is said to be in the subclass $C(\xi)$ of convex function if

$$Re\left(1 + \frac{zf'(z)}{f(z)}\right) > \xi, \quad z \in E, \quad 0 \leq \xi < 1.$$

Definition 1.3 : A function $f(z) \in A(p)$ is said to be in the subclass $K(\xi)$ of close to convex function if

$$Re\left(\frac{f'(z)}{z^{p-1}}\right) > \xi, \quad z \in E, \quad 0 \leq \xi < 1.$$

Definition 1.4 : A function $f(z) \in A(p)$ is said to be in the subclass $\pounds(p, \lambda, \phi, \delta, \alpha)$ if and only if

$$\left|\frac{\delta z(D_z^{q+1}(\Omega_p(r,p)f(z))) + \lambda z^2(D_z^{q+2}(\Omega_p(r,p)f(z)))}{(1-\lambda)(D_z^q(\Omega_p(r,p)f(z))) + z(D_z^{q+1}(\Omega_p(r,p)f(z)))} - (\delta - \phi)\right| < \alpha$$

$$z \in E, \quad q \in \mathbb{N}U\{0\}, \quad 0 < \alpha \leq 1, \quad \phi \in \mathbb{R}, \quad \phi < 1, \quad p > q, \quad \gamma, \delta \leq 1.$$

Further more a function $f(z) \in A(p)$ is said to be in the subclass $Y\pounds(p, \lambda, \phi, \delta, \alpha)$ if and only if $zf'(z) \in \pounds(p, \lambda, \phi, \delta, \alpha)$.

Let $T(p)$ denote the subclass of $A(p)$ consisting of functions of the form

$$f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k, \quad a_k \geq 0.$$

We dente by $\pounds^*(p, \lambda, \phi, \delta, \alpha)$ and $Y\pounds^*(p, \lambda, \phi, \delta, \alpha)$ the classes obtained by taking intersection respectively of the classes $\pounds(p, \lambda, \phi, \delta, \alpha)$ and $Y\pounds(p, \lambda, \phi, \delta, \alpha)$ with $T(p)$.

We define the operator $\Omega_p(a, p)$ on $f(z)$ as follows

$$\Omega_p(r,p)f(z) = z^p - \sum_{k=p+1}^{\infty} \left(\frac{k+\gamma}{p+\gamma}\right)^r a_k z^k.$$

The operator $\Omega_p(r,p)$ is closely related to the Salagean derivative operator $D_z^q f(z)$ is the q^{th} order differential operator for $f(z) \in A(p)$ defined in (1.1)

$$D_z^q(\Omega_p(r,p)f(z)) = \frac{p!}{(p-q)!} z^{p-q} - \sum_{k=p+1}^{\infty} \frac{k!}{(k-q)!} \left(\frac{k+\gamma}{p+\gamma}\right)^r a_k z^{k-q}, \quad p > q.$$

Theorem 1.1 : A function $f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k, \quad a_k \geq 0$ is in $\pounds^*(p,\lambda,\phi,\delta,\alpha)$ if and only if

$$\sum_{k=p+1}^{\infty} \frac{k!}{(k-q)!} \left(\frac{k+\gamma}{p+\gamma}\right)^r [\lambda(k-q)(k-q-1)+\phi(k-q)-(\delta-\phi)(1-\lambda)+\alpha(1-\lambda+k-q)]a_k$$

$$\leq \frac{p!}{(p-q)!}[\alpha(1-\lambda+p-q)-\lambda(p-q)(p-q-1)-\phi(p-q)+(\delta-\phi)(1-\lambda)].$$

Corollary 1.1 : If the function $f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k, a_k \geq 0$ is in $\pounds^*(p,\lambda,\phi,\delta,\alpha)$ then

$$a_k \leq \left(\frac{p+\gamma}{k+\gamma}\right)^r \frac{p!(k-q)![\alpha(1-\lambda+p-q)-\lambda(p-q)(p-q-1)-\phi(p-q)+(\delta-\phi)(1-\lambda)]}{k!(p-q)![\lambda(k-q)(k-q-1)+\phi(k-q)-(\delta-\phi)(1-\lambda)+\alpha(1-\lambda+k-q)]}.$$

For $k = 1+p, 2+p \cdots$.

With the equality for function

$$f_z = z^p - \left(\frac{p+\gamma}{k+\gamma}\right)^r \frac{p!(k-q)![\alpha(1-\lambda+p-q)-\lambda(p-q)(p-q-1)-\phi(p-q)+(\delta-\phi)(1-\lambda)]}{k!(p-q)![\lambda(k-q)(k-q-1)+\phi(k-q)-(\delta-\phi)(1-\lambda)+\alpha(1-\lambda+k-q)]} z^k.$$

For $k = 1+p, 2+p \cdots$.

Theorem 1.2 : A function $f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k,\ a_k \geq 0$ is in $Y\ \pounds^*(p,\lambda,\phi,\delta,\alpha)$ if and only if

$$\sum_{k=p+1}^{\infty} \frac{(k+1)!}{(k-q)!} \left(\frac{k+\gamma}{p+\gamma}\right)^r [\lambda(k-q)(k-q-1)+\phi(k-q)-(\delta-\phi)(1-\lambda)+\alpha(1-\lambda+k-q)]a_k$$

$$\leq \frac{p+1!}{p-q!}[\alpha(1-\lambda+p-q)-\lambda(p-q)(p-q-1)-\phi(p-q)+(\delta-\phi)(1-\lambda)].$$

Corollary 1.2 : If the function $f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k, a_k \geq 0$ is in $Y\pounds^*(p, \lambda, \phi, \delta, \alpha)$ then

$$a_k \leq \left(\frac{p+\gamma}{k+\gamma}\right)^r \frac{(p+1)!(k-q)![\alpha(1-\lambda+p-q) - \lambda(p-q)(p-q-1) - \phi(p-q) + (\delta-\phi)(1-\lambda)]}{[\lambda(k-q)(k-q-1) + \phi(k-q) - (\delta-\phi)(1-\lambda) + \alpha(1-\lambda+k-q)]}.$$

For $k = 1+p, 2+p \cdots$.

With the equality for function

$$f_z = z^p - \left(\frac{p+\gamma}{k+\gamma}\right)^r \frac{(p+1)!(k-q)![\alpha(1-\lambda+p-q) - \lambda(p-q)(p-q-1) - \phi(p-q) + (\delta-\phi)(1-\lambda)]}{(k+1)!(p-q)![\lambda(k-q)(k-q-1) + \phi(k-q) - (\delta-\phi)(1-\lambda) + \alpha(1-\lambda+k-q)]} z^k.$$

For $k = 1+p, 2+p \cdots$.

2. Radii of Close-to-Convexity, Starlikeness and Convexity

Theorem 2.1 : Let $f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k$, $a_k \geq 0$ is in $\pounds^*(p, \lambda, \phi, \delta, \alpha)$ then f is close to convex of order ξ in $|z| < R_1$, where

$$R_1 = \inf_{k \geq 1+p} \left\{ \left\{ \left(\frac{k+\gamma}{p+\gamma}\right)^r \frac{(k-1)!}{(k-q!)} \frac{(p-\xi)(p-q)!}{p!} \frac{[\lambda(k-q)(k-q-1) + \phi(k-q) - (\delta-\phi)(1-\lambda) + \alpha(1-\lambda+k-q)]}{[\alpha(1-\lambda+p-q) - \lambda(p-q)(p-q-1) - \phi(p-q) + (\delta-\phi)(1-\lambda)]} \right\}^{\frac{1}{k+1-p}} \right\}.$$

Proof : It is sufficient to show that $\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq p - \xi$ for $|z| < R_1$. We have

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| = \left|-\sum_{k=p+1}^{\infty} k a_k z^{k-p+1}\right| \leq \sum_{k=p+1}^{\infty} k|a_k||z|^{k-p+1}.$$

Thus $\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq p - \xi$ if

$$\sum_{k=p+1}^{\infty} \frac{k}{(p-\xi)} |a_k||z|^{k-p+1} \leq 1. \tag{2.1}$$

Theorem 1.1 conforms that

$$\sum_{k=p+1}^{\infty}\frac{k!}{(k-q)!}\left(\frac{k+\gamma}{p+\gamma}\right)^{r}[\lambda(k-q)(k-q-1)+\phi(k-q)-(\delta-\phi)(1-\lambda)+\alpha(1-\lambda+k-q)]|a_k|$$

$$\leq \frac{p!}{(p-q)!}[\alpha(1-\lambda+p-q)-\lambda(p-q)(p-q-1)-\phi(p-q)+(\delta-\phi)(1-\lambda)].$$

Therefore

$$\sum_{k=p+1}^{\infty}\frac{k!}{(k-q)!}\left(\frac{k+\gamma}{p+\gamma}\right)^{r}\frac{(p-q)!}{p!}$$
$$\frac{[\lambda(k-q)(k-q-1)+\phi(k-q)-(\delta-\phi)(1-\lambda)+\alpha(1-\lambda+k-q)]}{[\alpha(1-\lambda+p-q)-\lambda(p-q)(p-q-1)-\phi(p-q)+(\delta-\phi)(1-\lambda)]}|a_k|\leq 1 \tag{2.2}$$

Hence (2.1) will be true if

$$\frac{k}{(p-\xi)}|z|^{k-p+1}\leq\left(\frac{k+\gamma}{p+\gamma}\right)^{r}\frac{(p-q)!}{p!}$$
$$\frac{[\lambda(k-q)(k-q-1)+\phi(k-q)-(\delta-\phi)(1-\lambda)+\alpha(1-\lambda+k-q)]}{[\alpha(1-\lambda+p-q)-\lambda(p-q)(p-q-1)-\phi(p-q)+(\delta-\phi)(1-\lambda)]}.$$

We obtain

$$|z|\leq\left\{\left(\frac{k+\gamma}{p+\gamma}\right)^{r}\frac{(p-q)!}{p!}\frac{(k-1)!}{(k-q!)}\frac{(p-\xi)(p-q)!}{p!}\right.$$
$$\left.\frac{[\lambda(k-q)(k-q-1)+\phi(k-q)-(\delta-\phi)(1-\lambda)+\alpha(1-\lambda+k-q)]}{[\alpha(1-\lambda+p-q)-\lambda(p-q)(p-q-1)-\phi(p-q)+(\delta-\phi)(1-\lambda)]}^{\frac{1}{k+1-p}}\right\}$$

as required.

Theorem 2.2 : Let $f(z)=z^p-\sum\limits_{k=p+1}^{\infty}a_kz^k$, $a_k\geq 0$ is in $\pounds^*(p,\lambda,\phi,\delta,\alpha)$ then f is starlike of order ξ in $|z|<R_2$, where

$$R_2=\inf_{k\geq 1+p}$$
$$\left\{\left(\frac{k+\gamma}{p+\gamma}\right)^{r}\frac{k!}{(k-q!)}\frac{(p-\xi)(p-q)!}{(k-\xi)p!}\right.$$
$$\left.\frac{[\lambda(k-q)(k-q-1)+\phi(k-q)-(\delta-\phi)(1-\lambda)+\alpha(1-\lambda+k-q)]}{[\alpha(1-\lambda+p-q)-\lambda(p-q)(p-q-1)-\phi(p-q)+(\delta-\phi)(1-\lambda)]}^{\frac{1}{k+1-p}}\right\}.$$

Proof : We must show that

$$\left|\frac{zf'(z)}{f(z)}-p\right|\leq p-\xi$$

$$\begin{aligned} zf'(z)-pf(z) &= pz^p-\sum_{k=p+1}^{\infty}ka_kz^k-pz^p+p\sum_{k=p+1}^{\infty}a_kz^k \\ &= -\sum_{k=p_1}^{\infty}(k-p)a_kz^k. \end{aligned}$$

We have

$$\left|\frac{zf'(z)}{fz}-p\right|=\left|\frac{-\sum\limits_{k=p+1}^{\infty}(k-p)a_kz^k}{z^p-\sum\limits_{k=p+1}^{\infty}a_kz^k}\right|\leq\frac{\sum\limits_{k=p+1}^{\infty}(k-p)|a_k||z|^{k-p}}{1-\sum\limits_{k=p+1}^{\infty}|a_k||z|^{k-p}}\leq p-\xi. \tag{2.3}$$

Hence (2.1) holds true if

$$\sum_{k=p+1}^{\infty}(k-p)|a_k||z|^{k-p}\leq(p-\xi)(1-\sum_{k=p+1}^{\infty}|a_k|\ |z|^{k-p}).$$

Or equivalently

$$\sum_{k=p+1}^{\infty}\frac{(k-\xi)}{(p-\xi)}|a_k||z|^{k-p}\leq 1. \tag{2.4}$$

Theorem 1.1 conforms that

$$\sum_{k=p+1}^{\infty}\frac{k!}{(k-q)!}\left(\frac{k+\gamma}{p+\gamma}\right)^r\frac{(p-q)}{p!} \frac{[\lambda(k-q)(k-q-1)+\phi(k-q)-(\delta-\phi)(1-\lambda)+\alpha(1-\lambda+k-q)]}{[\alpha(1-\lambda+p-q)-\lambda(p-q)(p-q-1)-\phi(p-q)+(\delta-\phi)(1-\lambda)]}|a_k|\leq 1 \tag{2.5}$$

Hence by using (2.4) and (2.5) will be true if

$$\frac{(k-\xi)}{(p-\xi)}|z|^{k-p}\leq\frac{k!}{(k-q)!}\left(\frac{k+\gamma}{p+\gamma}\right)^r\frac{(p-q)!}{p!} \frac{[\lambda(k-q)(k-q-1)+\phi(k-q)-(\delta-\phi)(1-\lambda)+\alpha(1-\lambda+k-q)]}{[\alpha(1-\lambda+p-q)-\lambda(p-q)(p-q-1)-\phi(p-q)+(\delta-\phi)(1-\lambda)]}.$$

Or if

$$|z|\leq\left\{\frac{k!}{(k-q)!}\left(\frac{k+\gamma}{p+\gamma}\right)^r\frac{(p-\xi)(p-q)!}{(k-\xi)p!} \frac{[\lambda(k-q)(k-q-1)+\phi(k-q)-(\delta-\phi)(1-\lambda)+\alpha(1-\lambda+k-q)]}{[\alpha(1-\lambda+p-q)-\lambda(p-q)(p-q-1)-\phi(p-q)+(\delta-\phi)(1-\lambda)]}^{\frac{1}{k-p}}\right\}$$

as required.

Theorem 2.3 : Let $f(z) = z^p - \sum\limits_{k=p+1}^{\infty} a_k z^k$, $a_k \geq 0$ is in $\Gamma^*(p, \lambda, \phi, \delta, \alpha)$ then f is convex of order ξ in $|z| < R_3$, where

$$R_3 = \inf_{k \geq 1+p} \left\{ \left\{ \left(\frac{k+\gamma}{p+\gamma} \right)^r \frac{(k-1)!}{(k-q!)} \frac{(p-\xi)(p-\xi)(p-q)!}{(k-\xi)(p-1)!} \frac{[\lambda(k-q)(k-q-1) + \phi(k-q) - (\delta-\phi)(1-\lambda) + \alpha(1-\lambda+k-q)]}{[\alpha(1-\lambda+p-q) - \lambda(p-q)(p-q-1) - \phi(p-q) + (\delta-\phi)(1-\lambda)]} \right\}^{\frac{1}{k-p}} \right\}.$$

Proof : We know that f is convex if and only if zf' is starlike. We must show that

$$\left| \frac{zg'(z)}{g(z)} - p \right| \leq p - \xi$$

where $g(z) = zf'(z)$

$$g(z) = pz^p - \sum_{k=p+1}^{\infty} k a_k z^k$$

$$zg'(z) = p^2 z^p - \sum_{k=p+1}^{\infty} k^2 a_k z^k$$

$$\begin{aligned} zg'(z) - pg(z) &= p^2 z^p - \sum_{k=p+1}^{\infty} k^2 a_k z^k - p^2 z^p + p \sum_{k=p+1}^{\infty} k a_k z^k \\ &= -\sum_{k=p_1}^{\infty} k(k-p) a_k z^k. \end{aligned}$$

$$\left| \frac{zg'(z)}{gz} - p \right| = \left| \frac{-\sum\limits_{k=p+1}^{\infty} k(k-p) a_k z^k}{pz^p - \sum\limits_{k=p+1}^{\infty} k a_k z^k} \right| \leq \frac{\sum\limits_{k=p+1}^{\infty} k(k-p)|a_k||z|^{k-p}}{p - \sum\limits_{k=p+1}^{\infty} k|a_k||z|^{k-p}} \leq p - \xi.$$

Therefore we have

$$\sum_{k=p+1}^{\infty} k(k-p)|a_k||z|^{k-p} \leq (p-\xi)[p - \sum_{k=p+1}^{\infty} k|a_k|\ |z|^{k-p}].$$

$$\sum_{k=p+1}^{\infty} \frac{k(k-\xi)}{p(p-\xi)} |a_k||z|^{k-p} \leq 1. \tag{2.6}$$

Theorem 1.1 conforms that

$$\sum_{k=p+1}^{\infty} \frac{k!}{(k-q)!}\left(\frac{k+\gamma}{p+\gamma}\right)^{r}\frac{(p-q)}{p!} \frac{[\lambda(k-q)(k-q-1)+\phi(k-q)-(\delta-\phi)(1-\lambda)+\alpha(1-\lambda+k-q)]}{[\alpha(1-\lambda+p-q)-\lambda(p-q)(p-q-1)-\phi(p-q)+(\delta-\phi)(1-\lambda)]}|a_k| \leq 1 \tag{2.7}$$

Hence by using (2.6) and (2.7) will be true if

$$\frac{(k-\xi)}{(p-\xi)}|z|^{k-p} \leq \frac{k!}{(k-q)!}\left(\frac{k+\gamma}{p+\gamma}\right)^{r}\frac{(p-q)!}{p!} \frac{[\lambda(k-q)(k-q-1)+\phi(k-q)-(\delta-\phi)(1-\lambda)+\alpha(1-\lambda+k-q)]}{[\alpha(1-\lambda+p-q)-\lambda(p-q)(p-q-1)-\phi(p-q)+(\delta-\phi)(1-\lambda)]}.$$

$$|z| \leq \left\{\left(\frac{k+\gamma}{p+\gamma}\right)^{r}\frac{(k-1)!}{(k-q)!}\frac{(p-\xi)(p-q)!}{(k-\xi)(p-1)!} \frac{[\lambda(k-q)(k-q-1)+\phi(k-q)-(\delta-\phi)(1-\lambda)+\alpha(1-\lambda+k-q)]}{[\alpha(1-\lambda+p-q)-\lambda(p-q)(p-q-1)-\phi(p-q)+(\delta-\phi)(1-\lambda)]}^{\frac{1}{k-p}}\right\}$$

as required.

3. Closure Theorem

Theorem 3.1 : Let $f_1(z) = z^p$ and

$$f_k(z) = z^p - \left(\frac{p+\gamma}{k+\gamma}\right)^{r} \frac{p!(k-q)![\alpha(1-\lambda+p-q)-\lambda(p-q)(p-q-1)-\phi(p-q)+(\delta-\phi)(1-\lambda)]}{k!(p-q)![\lambda(k-q)(k-q-1)+\phi(k-q)-(\delta-\phi)(1-\lambda)+\alpha(1-\lambda+k-q)]}z^k$$

for $k \geq 1+p$.Then $f(z) \in \pounds^*(p,\lambda,\phi,\delta,\alpha)$ if anly if $f(z)$ can be expressed in the form $f(z) = \lambda_1 f_1(z) + \sum_{k=p+1}^{\infty} \lambda_k f_k(z)$ where $\lambda_k \geq 0$ and $\lambda_1 + \sum_{k=p+1}^{\infty} \lambda_k = 1$.

Proof : Suppose $f(z)$ can be expressed in the form

$$f(z) = \lambda_1 f_1(z) + \sum_{k=p+1}^{\infty} \lambda_k f_k(z)$$

$$\lambda_1 \left[z^p + \sum_{k=p+1}^{\infty} \lambda_k z^p - \left(\frac{p+\gamma}{k+\gamma}\right)^r \right.$$

$$\left. \frac{p!(k-q)![\alpha(1-\lambda+p-q) - \lambda(p-q)(p-q-1) - \phi(p-q) + (\delta-\phi)(1-\lambda)]}{k!(p-q)![\lambda(k-q)(k-q-1) + \phi(k-q) - (\delta-\phi)(1-\lambda) + \alpha(1-\lambda+k-q)]} z^k \right]$$

$$= \left[\lambda_1 + \sum_{k=p+1}^{\infty} \lambda_k \right] z^p - \sum_{k=p+1}^{\infty} \lambda_k \left(\frac{p+\gamma}{k+\gamma}\right)^r$$

$$\frac{p!(k-q)![\alpha(1-\lambda+p-q) - \lambda(p-q)(p-q-1) - \phi(p-q) + (\delta-\phi)(1-\lambda)]}{k!(p-q)![\lambda(k-q)(k-q-1) + \phi(k-q) - (\delta-\phi)(1-\lambda) + \alpha(1-\lambda+k-q)]} z^k$$

$$= z^p - \sum_{k=p+1}^{\infty} \lambda_k \left(\frac{p+\gamma}{k+\gamma}\right)^r$$

$$\frac{p!(k-q)![\alpha(1-\lambda+p-q) - \lambda(p-q)(p-q-1) - \phi(p-q) + (\delta-\phi)(1-\lambda)]}{k!(p-q)![\lambda(k-q)(k-q-1) + \phi(k-q) - (\delta-\phi)(1-\lambda) + \alpha(1-\lambda+k-q)]} z^k$$

Then

$$\sum_{k=p+1}^{\infty} \lambda_k \left(\frac{p+\gamma}{k+\gamma}\right)^r$$

$$\frac{p!(k-q)![\alpha(1-\lambda+p-q) - \lambda(p-q)(p-q-1) - \phi(p-q) + (\delta-\phi)(1-\lambda)]}{k!(p-q)![\lambda(k-q)(k-q-1) + \phi(k-q) - (\delta-\phi)(1-\lambda) + \alpha(1-\lambda+k-q)]},$$

$$\frac{k!(p-q)![\lambda(k-q)(k-q-1) + \phi(k-q) - (\delta-\phi)(1-\lambda) + \alpha(1-\lambda+k-q)]}{p!(k-q)![\alpha(1-\lambda+p-q) - \lambda(p-q)(p-q-1) - \phi(p-q) + (\delta-\phi)(1-\lambda)]} z^k$$

$$= \sum_{k=p+1}^{\infty} \lambda_1 = 1 - \lambda_1 \leq 1.$$

Therefore $f(z) \in \pounds^*(p, \lambda, \phi, \delta, \alpha)$.

Conversely, suppose that $f(z) \in \pounds^*(p, \lambda, \phi, \delta, \alpha)$, we have

$$a_k \leq \left(\frac{p+\gamma}{k+\gamma}\right)^r$$

$$\frac{p!(k-q)![\alpha(1-\lambda+p-q) - \lambda(p-q)(p-q-1) - \phi(p-q) + (\delta-\phi)(1-\lambda)]}{k!(p-q)![\lambda(k-q)(k-q-1) + \phi(k-q) - (\delta-\phi)(1-\lambda) + \alpha(1-\lambda+k-q)]}.$$

We take

$$\lambda_k = \left(\frac{p+\gamma}{k+\gamma}\right)^r$$

$$\frac{p!(p-q)![\lambda(k-q)(k-q-1)+\phi(k-q)-(\delta-\phi)(1-\lambda)+\alpha(1-\lambda+k-q)]}{p!(k-q)![\alpha(1-\lambda+p-q)\lambda(p-q)(p-q-1)-\phi(p-q)+(\delta-\phi)(1-\lambda)]}a_k.$$

$k \geq 1+p$ and $\sum\limits_{k=p+1}^{\infty} \lambda_k = 1-\lambda_1$

$$f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k$$

$$= z^p - \sum_{k=p+1}^{\infty} \left(\frac{p+\gamma}{k+\gamma}\right)^r$$

$$\frac{p!(k-q)![\alpha(1-\lambda+p-q)-\lambda(p-q)(p-q-1)-\phi(p-q)+(\delta-\phi)(1-\lambda)]}{k!(p-q)![\lambda(k-q)(k-q-1)+\phi(k-q)-(\delta-\phi)(1-\lambda)+\alpha(1-\lambda+k-q)]}z^k$$

$$= z^p - \sum_{k=p+1}^{\infty} \lambda_k[z^p - f_k(z)] = z^p \left[1 - \sum_{k=p+1}^{\infty} \lambda_k\right] - \sum_{k=p+1}^{\infty} \lambda_k f_k$$

$$\lambda_1 f_1(z) + \sum_{k=p+1}^{\infty} \lambda_k f_k(z).$$

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