International J. of Multidispl. Research & Advcs. in Engg.(IJMRAE),
ISSN 0975-7074, Vol. 9, No. III (December 2017), pp. 25-46

NEW BASIC BELIEF ASSIGNMENTS FOR CONTINUOUS SPACE

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Abstract

Most of the authors are using approaches from belief function to probability and rarely some of the authors are using approaches from probability to belief functions. In this paper we have proposed some indexing of subsets of continuous frame of discernment Ω and basic belief assignments using probability density function of continuous probability distribution. With these basic belief assignments we have obtained lower and upper limits of statistical quantities such as distribution function, raw and central moments and coefficients of skewness and kurtosis. Also we illustrate it by an example (Uniform Continuous Distribution).

1. Introduction

In the world of uncertainty, each and every incidence occurring in our day to day life, always follows some known or unknown probability distribution. Therefore choice of

Key Words : *Frame of discernment, Belief function, Probability density function, Indexing of subsets, Lower and upper limits of statistical quantities.*

appropriate probability distribution plays an important role in decision making. Hence it becomes necessary that we should know common characteristics of all probability distributions.

Here we want to find a new transformations which transform continuous probability function into basic belief assignment hence belief function. While obtaining new transformation, we concentrate on sufficient axioms of basic belief assignment which must be satisfied by our new transformations. Once we have obtained such required new transformation, we are able to find other functions related to belief function.

In this paper, firstly in section 2, we summarize preliminaries of discrete and continuous belief functions and probability functions then in sections 3, 4 and 5, we will explain steps in the development of these new transformations and we will apply to obtain lower and upper bounds of statistical quantities such as distribution function, raw moments, central moments and coefficients of skewness and kurtosis. Finally we will give some applications where these new transformations seems to be applicable.

2. Preliminaries

2.1 Discrete Belief Function Theory

Frame of Discernment : Dictionary meaning of Frame of Discernment is frame of good judgment insight. The word discern means recognize or find out or hear with difficulty. In [10], concepts in discrete frame of discernment are explained in detail. Some of these are as follows: If frame of discernment Θ is

$$\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$$

then every element of Θ is a proposition. The propositions of interest are in one -to -one correspondence with the subsets of Θ. The set of all propositions of interest corresponds to the set of all subsets of Θ, denoted by 2^Θ.

If Θ is *frame of discernment*, then a function $m : 2^\Theta \to [0, 1]$ is called **basic probability assignment** whenever $m(\emptyset) = 0$ and $\sum_{A\subset\Theta} m(A) = 1$. The quantity $m(A)$ is called A's **basic probability number** and it is a measure of the belief committed exactly to A.The *total belief* committed to A is sum of $m(B)$, for all subsets B of A. A function $Bel : 2^\Theta \to [0, 1]$ is called **belief function** over Θ if it satisfies $Bel(A) = \sum_{B\subset A} m(B)$. If Θ is a frame of discernment, then a function $Bel : 2^\Theta \to [0, 1]$ is *belief function* if and only if it satisfies following conditions

1. $Bel(\emptyset) = 0$.
2. $Bel(\Theta) = 1$.
3. For every positive integer n and every collection $A_1, A_2, \ldots, A_n$ of subsets of Θ

$$Bel(A_1 \cup A_2 \cup \ldots \cup A_n) \geq \sum_{I \subset \{1,2,\cdots,n\}} (-1)^{|I|+1} Bel(\bigcap_{i \in I} A_i). \tag{1}$$

A subset of a frame Θ is called a **focal element** of a belief function Bel over Θ if $m(A) > 0$. The union of all the focal elements of a belief function is called its **core**. The quantity $Q(A) = \sum_{B \subset \Theta, A \subset B} m(B)$ is called **commonality number** for A which measures the total probability mass that can move freely to every point of A. A function $Q : 2^\Theta \to [0,1]$ is called **commonality function** for Bel. Also $Bel(A) = \sum_{B \subset \bar{A}}$ and $Q(A) = \sum_{B \subset A} (-1)^{|B|} Bel(\bar{B})$for all $A \subset \Theta$.

Degree of doubt :

$$Dou(A) = Bel(\bar{A})\text{or}Bel(A) = Dou(\bar{A})\text{and } pl(A) = 1 - Dou(A) = \sum_{A \cap B \neq \emptyset} m(B) \tag{2}$$

which expresses the extent to which one finds A credible or plausible. We have relation between belief function, probability function and plausibility function is $Bel(A) \leq p(A) \leq Pl(A), \quad \forall A \subset \Theta$.

A function $P : \Theta \to [0,1]$ is called **probability function** if

1 $\forall A \in \Theta, \quad 0 \leq P(A) \leq 1$.

2 $P(\Theta) = 1$.

A set function μ on a frame of discernment Θ is a **measure** if it satisfies following three conditions:

1. $\mu(A) \in [0, \infty], \quad$ for all $A \in \Theta$.
2. $\mu(\emptyset) = 0$.
3. **Additive Property : For collection** $A_1, A_2, \ldots, A_n, \ldots$,

$$\mu(\cup_{i=1}^{\infty} A_i) = \sum_{\substack{I \subset \{1,2,\ldots,n,\ldots\} \\ I \neq \emptyset}} (-1)^{|A|+1} \mu(\cap_{i=1}^{\infty} A_i). \tag{3}$$

The measure μ is finite or infinite as $\mu(\Theta) < \infty$ or $\mu(\Theta) = \infty$. Also μ is probability measure if $\mu(\Theta) = 1$ [2].
In Shafer's book [10], we have **Bayesian Belief Function** as: If Θ is frame of discernment then a function $Bel : 2^{\Theta} \to [0, 1]$ is called Bayesian belief function if

1 $Bel(\emptyset) = 0$,

2 $Bel(\Theta) = 1$,

3 $Bel(A \cup B) = Bel(A) + Bel(B)$ whenever $A, B \in \Theta$ and $A \cap B = \emptyset$.

Suppose $Bel : 2^{\Theta} \to [0, 1]$ is belief function. Then following statements are equivalent:

1 Bel is Bayesian.

2 All of Bel's focal elements are singletons.

3 Bel awards a zero commonality number to any subset containing more than one element.

4 $Bel(A) = 1 - Bel(\bar{A})$ for all $A \subset \Theta$.

2.2 Continuous Belief Function Theory

The necessary information about probability mass function, distribution function, raw moments, central moments and coefficients of skewness and kurtosis, is refered from Bansi Lal and Sanjay Arora book [1]. We use following notations : $\Omega = [a, b] =$ Continuous frame of Discernment, $f(x) =$ Probability Density Function of continuous probability Distribution, $P[p, q] =$ Probability of interval $[p, q], F(x) =$ Distribution Function at $x = f[a, x], \mu_r^{"} = r^{th}$ raw moment and $\mu_r = r^{th}$ central moment.
Note that $\Omega = [a, b]$ can be replaced by any interval of real line $(-\infty, \infty)$ hence $(-\infty, \infty) \cup \{-\infty, \infty\}$. For continuous frame of discernment, we replace summation by integration as number of subsets are uncountably infinite. Also integration over point interval is zero.
In Smets article [11], Smets proposed theory about continuous frame of discernment as: Let $\Omega = [a, b]$ be continuous frame of discernment and $\mathcal{A} = \{A_i | A_i \subseteq [a, b]$ with $\sum_{i-1}^{n} m(A_i) = 1$. Focal elements A_i of m have $m(A_i) > 0$. For every $(x, y) \in \mathcal{T} =$

$\{(x,y)|x \leq y, \quad x,y \in [a,b]\} \subset \mathcal{R}^2$, the probability density function $f^{\mathcal{T}} : [a,b] \to \mathcal{R}^+$ defined by

$$f^{\mathcal{T}}(x,y) = \sum_{i=1}^{n} m(A_i)\delta(x-x_i)\delta(y-y_i), \quad x_i, y_i \in A_i \tag{4}$$

where δ is Dirac's delta function: $\delta(x-x_i)$ is a function which is 0 everywhere except at its centre x_0 where it is infinite with $\in_{x=-\infty}^{\infty} f(x)\delta(x-x_0)dx = f(x_0)$. Also $\int_{x=c}^{b}(x-b)dx = 1$ and $\int_{x=a}^{b-\epsilon}\delta(x-b)dx = 0, \quad \forall\epsilon > 0$. It gives link between belief functions on $\mathcal{R}$ and probability distribution functions on $\mathcal{R}^2$.

In [3, 4, 11, 12], continuous frame of discernment Ω is $(-\infty,\infty)$ and in [13], continuous frame of discernment Ω is $[a,b]$, a finite closed interval. Some quantity may be assigned to $\emptyset$ by assigning $INT = \in_{x=-\infty}^{\infty}\int_{y=x}^{\infty} m([x,y])dydx \leq 1$. Hence $m(\emptyset) = 1 - INT$. For normalized basic belief density, $f^{\mathcal{T}} = m([a,b])$, for $a \leq b$ and $f^{\mathcal{T}} = 0$ for $a > b$. Some related functions are as follows :

$$Bel([a,b]) = \int_{x=a}^{b}\int_{y=x}^{b} m([x,y])dxdy, \qquad Bel(\emptyset) = 0 \tag{5}$$

$$Pl([a,b]) = \int_{x=-\infty}^{b}\int_{y=max.\{a,x\}}^{\infty} m([x,y])dxdy, \qquad Pl(\emptyset) = 0 \tag{6}$$

$$q([a,b]) = \int_{x=-\infty}^{a}\int_{y=b}^{\infty} m([x,y])dxdy, \qquad q(\emptyset) = 1 \tag{7}$$

$$\text{and } b([a,b]) = Bel([a,b]) + m(\emptyset), \qquad b(\emptyset) = m(\emptyset). \tag{8}$$

In Dempster's articles [5, 6], he has discovered important result as:

$$\text{For each subset } A \subset \Omega, Bel(A) \leq P(A) \leq Pl(A). \tag{9}$$

Many of the authors, done research on consonant basic belief density and discovered important results [3, 4, 7, 8, 12, 14]. Dore and Martin [7, 8] modeled belief functions induced by probabilities with indexing from set of all focal elements to $\mathcal{T} \subset \mathcal{R}^2$ hence modeled belief functions on Borel sigma algebra $\mathcal{B}(\mathcal{R}^n)$. Here belief functions and plausibility functions of A are obtained by integrating over set of image sets contained in A and over set image set having non-empty intersection with A with respect to credal measure μ^{Ω} of subset of index set.

The useful formulae to calculate distribution function $F(x)$, raw moments $\mu_r^{'}$, central moments μ_r and coefficients of skewness and kurtosis based on central moments are referred from Billingsley [2] and Bansi Lal and Sanjay Arora book [1]. Interval arithmetic operations [9] are useful to obtain lower and upper limits of statistical quantities by using result in Dempster's article [5, 6].

3. First Basic Belief Assignment For Continuous Space

In Smets's article [10], he has proposed basic belief assignment consisting based on only finite number of focal elements which are prominent. With this idea, we propose our first new basic belief assignment with slight modification. Let $\mathcal{A} = \{A_i = [a_i, b_i] | i = 1, 2, \ldots, n\}$ be set of prominent focal subsets of Ω, which are not necessarily distinct. Now define $m : \mathcal{A} \to [0, 1]$ by $m([x_i, y_i]) = \dfrac{f([x_i, y_i])}{\sum_{i=1}^{n} f([x_i, y_i])}$ with $\sum_{i=1}^{n} m([x_i, y_i]) = 1$. With above these conditions, m is a basic belief assignment. With this new basic belief assignment m and by using (5), (6) and (7), we have belief function Bel, commonality function q and plausibility function Pl from $\mathcal{A}$ to $[0, 1]$ as :

$$\begin{aligned} Bel([c,d]) &= \sum_{[x,y]\subseteq[c,d]} m([x,y]) \\ &= \sum_{[x,y]\subseteq[c,d]} \frac{f([x,y])}{\sum_{i=1}^{n} f([x,y])} \\ &= \frac{1}{\sum_{i=1}^{n} f([x,y])} \sum_{[x,y]\subseteq[c,d]} \int_{Z=x}^{y} f(z)dz. \end{aligned} \tag{10}$$

$$\begin{aligned} q([c,d]) &= \sum_{[x,y]\supseteq[c,d]} m([x,y]) \\ &= \sum_{[x,y]\supseteq[c,d]} \frac{f([x,y])}{\sum_{i=1}^{n} f([x,y])} \\ &= \frac{1}{\sum_{i=1}^{n} f([x,y])} \sum_{[x,y]\supseteq[c,d]} \int_{Z=x}^{y} f(z)dz. \end{aligned} \tag{11}$$

$$\begin{aligned} Pl([c,d]) &= \sum_{[x,y]\cap[c,d]\neq\emptyset} m([x,y]) \\ &= \sum_{[x,y]\cap[c,d]\neq\emptyset} \frac{f([x,y])}{\sum_{i=1}^{n} f([x,y])} \\ &= \frac{1}{\sum_{i=1}^{n} f([x,y])} \sum_{[x,y]\cap[c,d]\neq\emptyset} \int_{Z=x}^{y} f(z)dz. \end{aligned} \tag{12}$$

3.1 Lower and Upper Limits of Distribution Function

We have distribution function $F(x) = f(X \leq x) = f([a, x])$. Also we have result (9) : For any subset A of $\Omega, Bel(A) \leq f(A) \leq Pl(A)$. Using this result for general set for distribution function $[a, x]$, we have

$$Bel([a, x]) \leq f([a, x]) \leq Pl([a, x]) \tag{13}$$

3.2 Lower and Upper Limits of Raw Moments

We have r^{th} raw moments for continuous probability distribution as $E(x^r) = \int_{X=a}^{b} x^r f(x)dx$. Now we have r^{th} raw moment based on probability of set $E(x^r)_p = \mu'_{(r,p)}$, belief of set $E(x^r)_{Bel} = \mu'_{(r,Bel)}$ and plausibility of set $E(x^r)_{Pl} = \mu'_{(r,Pl)}$ as :

$$\begin{aligned}
\mu'_{(r,p)} &= E(x^r)_p \\
&= \sum_{i=1}^{n} i^r p(A_i) \\
&= \sum_{i=1}^{n} i^r p([a_i, b_i]) \\
&= \sum_{i=1}^{n} i^r \int_{z=a_i}^{b_i} f(z)dz
\end{aligned} \tag{14}$$

$$\begin{aligned}
\mu'_{(r,Bel)} &= E(x^r)_{Bel} \\
&= \sum_{i=1}^{n} i^r Bel(A_i) \\
&= \sum_{i=1}^{n} i^r Bel([a_i, b_i]) \\
&= \sum_{i=1}^{n} i^r \frac{1}{\sum_{i=1}^{n} f([x, y])} \sum_{[x,y] \subseteq [a_i, b_i]} \int_{Z=x}^{y} f(z)dz
\end{aligned} \tag{15}$$

$$\begin{aligned}
\mu'_{(r,Pl)} &= E(x^r)_{Pl} \\
&= \sum_{i=1}^{n} i^r Pl(A_i) \\
&= \sum_{i=1}^{n} i^r Pl([a_i, b_i]) \\
&= \sum_{i=1}^{n} i^r \frac{1}{\sum_{i=1}^{n} f([x, y])} \sum_{[x,y] \cap [a_i, b_i] \neq \emptyset} \int_{Z=x}^{y} f(z)dz
\end{aligned} \tag{16}$$

Note that $\mu'_{(r,Bel)} \leq \mu'_{(r,p)} \leq \mu'_{(r,Pl)}$. Therefore we get lower and upper limits of raw moments by using result :

$$\frac{\mu'_{(r,Bel)}}{\mu'_{(r,p)}} \cdot \mu'_r \leq \mu'_r \leq \frac{\mu'_{(r,Pl)}}{\mu'_{(r,p)}} \cdot \mu'_r. \tag{17}$$

3.3 Lower and Upper Limits of Central Moments, Skewness and Kurtosis

Using relations between raw and central moments, lower and upper limits of raw moments and interval arithmetics, we can obtain lower and upper limits of central moments.

$$\mu_r = \sum_{k=0}^{r} \binom{r}{k} \mu'_{r-k} {\mu'_1}^k. \tag{18}$$

Using lower and upper limits of central moments and interval arithmetics, we will obtain lower and upper limits of coefficients of skewness and kurtosis.

$$\gamma_1 = \sqrt{(\frac{\mu_3^2}{\mu_2^3})} \text{ and } \gamma_2 = \frac{\mu_4}{\mu_2^2}. \tag{19}$$

4. Second Basic Belief Assignment For Continuous Space

In above section, we have proposed new basic belief assignment. With slight modification, we propose second new basic belief assignment. Let $\mathcal{A} = \{[a, x] | x \in \Omega = [a, b]\}$ be set of focal elements of Ω, which are not distinct. Now define $m : \mathcal{A} \to [0, 1]$ by $m([a, x]) = \frac{f([a, x])}{\int_{[a,x]\subseteq\Omega} f([a, x])dx}$ with $\int_{[a,x]\subseteq\Omega} m([a, x])dx = 1$. Now we will check that whether m defined above, is basic belief assignment or not.

Here $f(\emptyset) = 0$ hence $\emptyset$ is not focal element Ω. As $[a, x]$ is focal element of Ω, it implies that $f([a, x]) > 0$ hence $m([a, x]) = \frac{f([a, x])}{\int_{[a,x]\subseteq\Omega} f([a, x])dx} > 0$. Also

$$\begin{aligned}\int_{[a,x]\subseteq\Omega} m([a, x]) &= \int_{[a,x]\subseteq\Omega} \frac{f([a, x])}{\int_{[a,x]\subseteq\Omega} f([a, x])dx} \\ &= \frac{\int_{[a,x]\subseteq\Omega} f([a, x])dx}{\int_{[a,x]\subseteq\Omega} f([a, x])dx} \\ &= 1.\end{aligned}$$

With above these conditions, m is a basic belief assignment. With this new basic belief assignment $m$and by using (5), (6) and (7), we have belief function and Bel, commonality function q and plausibility function Pl from 2^{Ω} to $[0,1]$ as :

$$\begin{aligned}
Bel([a,c]) &= \int_{[a,x]\subseteq[a,c]} m([a,x]) \\
&= \int_{[a,x]\subseteq[a,c]} \frac{f([a,x])}{\int_{[a,x]\subseteq\Omega} f([a,x])dx} \\
&= \frac{1}{\int_{[a,x]\subseteq\Omega} f([a,x])dx} \int_{[a,x]\subseteq[a,c]} \int_{Z=a}^{x} f(z)dz
\end{aligned} \tag{20}$$

$$\begin{aligned}
q([a,c]) &= \int_{[a,x]\supseteq[a,c]} m([a,x]) \\
&= \int_{[a,x]\supseteq[a,c]} \frac{f([a,x])}{\int_{[a,x]\subseteq\Omega} f([a,x])dx} \\
&= \frac{1}{\int_{[a,x]\subseteq\Omega} f([a,x])dx} \int_{[a,x]\supseteq[a,c]} \int_{Z=a}^{x} f(z)dz
\end{aligned} \tag{21}$$

$$\begin{aligned}
Pl([a,c]) &= \int_{[a,x]\cap[a,c]\neq\emptyset} m([a,x]) \\
&= \int_{[a,x]\cap[a,c]\neq\emptyset} \frac{f([a,x])}{\int_{[a,x]\subseteq\Omega} f([a,x])dx} \\
&= \frac{1}{\int_{[a,x]\subseteq\Omega} f([a,x])dx} \int_{[a,x]\cap[a,c]\neq\emptyset} \int_{Z=a}^{c} f(z)dz.
\end{aligned} \tag{22}$$

4.1 Lower and Upper Limits of Distribution Function

As proceeding as in above section 3, for :Lower and Upper Limits of Distribution Function, we have $Bel([a,x]) \leq f([a,x]) \leq Pl([a,x])$ where

$$Bel([a,k]) = \int_{x=a}^{k} m([a,x])dx \text{ and } Pl([a,k]) = \int_{x=a}^{b} m([a,x])dx = 1 \tag{23}$$

4.2 Lower and Upper Limits of Raw Moments

Now we have raw moments based on probability, belief and plausibility as proceeding in above section 3 as :

$$\begin{aligned}\mu'_{(r,p)} &= \int_{x=a}^{b} x^r f([a,x])dx \\ &= \int_{x=a}^{b} x^r \int_{z=a}^{x} f(z)dz.\end{aligned} \tag{24}$$

$$\begin{aligned}\mu'_{(r,Bel)} &= \int_{x=a}^{b} x^r Bel([a,x])dx \\ &= \int_{x=a}^{b} x^r \int_{y=a}^{x} m([a,y])dydx \\ &= \int_{x=a}^{b} x^r \int_{y=a}^{x} \frac{f([a,y])}{\int_{[a,y]\subseteq\Omega} f([a,y])dy}dydx \\ &= \int_{x=a}^{b} x^r \int_{y=a}^{x} (\frac{1}{\int_{[a,y]\subseteq\Omega} f([a,y])dy})f([a,y])dydx \\ &= \int_{x=a}^{b} x^r \int_{y=a}^{x} (\frac{1}{\int_{[a,y]\subseteq\Omega} f([a,y])dy}) \int_{z=a}^{y} f(z)dzdydx.\end{aligned} \tag{25}$$

$$\begin{aligned}\mu'_{(r,Pl)} &= \int_{x=a}^{b} x^r Pl([a,x])dx \\ &= \int_{x=a}^{b} x^r \cdot 1dx \\ &= \int_{x=a}^{b} x^r dx \\ &= [\frac{x^{r+1}}{r+1}]_a^b \\ &= \frac{1}{r+1}[b^{r+1} - a^{r+1}].\end{aligned} \tag{26}$$

As proceeding on similar way in section 3, we can obtain lower and upper limits of raw moments, central moments and coefficients of skewness and kurtosis.

5. Third Basic Belief Assignment For Continuous Space

In above section, we have proposed second new basic belief assignment. With slight modification, we propose third new basic belief assignment. Let $\mathcal{A} = \{[x,y]|x,y \in \Omega = [a,b], x \le y\}$ be set of focal elements of Ω, which are not distinct. Now define $m : \mathcal{A} \to [0,1]$ by

$$m([x,y]) = \frac{f([x,y])}{\int_{[x,y]\subseteq\Omega} f([x,y])dxdy} \tag{27}$$

with $\int_{[x,y]\subseteq\Omega} m([x,y])dxdy = 1$. Here $f(\emptyset)=0$ hence $\emptyset$ is not focal element Ω. As $[x,y]$ is focal element of Ω, it implies that $f([x,y])>0$ hence $m([x,y]) = \dfrac{f([x,y])}{\int_{[x,y]\subseteq\Omega} f([x,y])dxdy} > 0$. Also

$$\begin{aligned}\int_{[x,y]\subseteq\Omega} m([x,y])dxdy &= \int_{[x,y]\subseteq\Omega} \frac{f([x,y])}{\int_{[x,y]\subseteq\Omega} f([x,y])dxdy}\\ &= \frac{\int_{[x,y]\subseteq\Omega} f([x,y])dxdy}{\int_{[x,y]\subseteq\Omega} f([x,y])dxdy} = 1.\end{aligned}$$

With above these conditions, m is a basic belief assignment. With this new basic belief assignment m and by using (5), (6) and (7), we have belief function Bel, commonality function q and plausibility function Pl from 2^Ω to $[0,1]$ as :

$$\begin{aligned}Bel([c,d]) &= \int_{[x,y]\subseteq[c,d]} m([x,y])dxdy\\ &= \int_{[x,y]\subseteq[c,d]} \frac{f([x,y])}{\int_{[x,y]\subseteq\Omega} f([x,y])dxdy}\\ &= \frac{1}{\int_{x=a}^{b}\int_{y=x}^{b}\int_{Z=x}^{y} f(z)dzdxdy}\int_{x=c}^{d}\int_{y=x}^{d}\int_{Z=x}^{y} f(z)dzdxdy\end{aligned} \tag{28}$$

$$\begin{aligned}q([c,d]) &= \int_{[x,y]\supseteq[c,d]} m([x,y])dxdy\\ &= \int_{[x,y]\supseteq[c,d]} \frac{f([x,y])}{\int_{[x,y]\subseteq\Omega} f([x,y])dxdy}\\ &= \frac{1}{\int_{x=a}^{b}\int_{y=x}^{b}\int_{Z=x}^{y} f(z)dzdxdy}\int_{x=a}^{c}\int_{y=d}^{b}\int_{Z=x}^{y} f(z)dzdxdy\end{aligned} \tag{29}$$

$$\begin{aligned}Pl([c,d]) &= \int_{[x,y]\cap[c,d]\neq\emptyset} m([x,y])\\ &= \int_{[x,y]\cap[c,d]\neq\emptyset} \frac{f([x,y])}{\int_{[x,y]\subseteq\Omega} f([x,y])dxdy}\\ &= \frac{1}{\int_{x=a}^{b}\int_{y=x}^{b}\int_{Z=x}^{y} f(z)dzdxdy}\{\int_{x=a}^{c}\int_{y=c}^{b}\int_{Z=x}^{y} f(z)dzdxdy\\ &+ \int_{x=c}^{d}\int_{y=x}^{b}\int_{Z=x}^{y} f(z)dzdxdy\}\end{aligned} \tag{30}$$

5.1 Lower and Upper Limits of Distribution Function

As proceeding on similar way in section 3, for: Lower and Upper Limits of Distribution Function, we have $Bel([a,x]) \leq f([a,x]) \leq Pl([a,x])$ where $Bel([a,k]) =$

$\int_{x=a}^{k}\int_{y=x}^{k} m([x,y])dxdy$ and $Pl([a,k]) = \int_{x=a}^{k}\int_{y=x}^{k} m([x,y])dxdy$.

5.2 Indexing of Subsets of Ω

The subsets of Ω of our interest are of the type $[x, y]$. Now we define a surjective mapping 2^{Ω} to $\mathcal{T} = \{(x,y)|a \leq x \leq y \leq b\}$ such that for each subset $[x, y]$ of Ω, we associate a number $v = (x, y) \in \mathcal{T}$ such that it's enumeration is sum of x and area of triangle shaded in yellow colour in $\mathcal{T}$, which is equal to sum of x and half of the area of rectangle with sides $x - a$ and $y - a$. Thus enumeration of (x, y) is $x + (1/2)[(x - a)(y - a)]$.

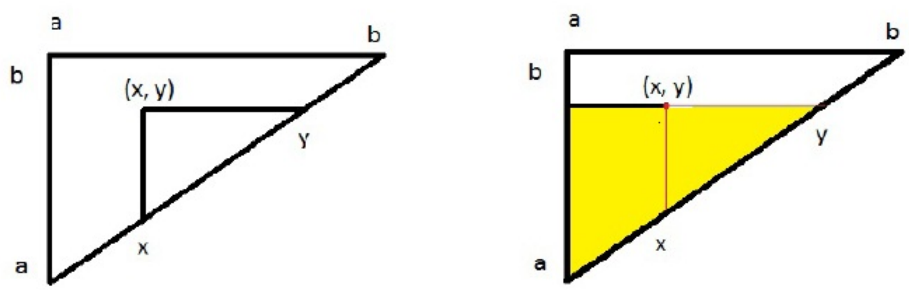

Figure 1: Indexing of subsets $[x, y]$ of Continuous Frame of Discernment Ω.

5.3 Lower and Upper Limits of Raw Moments

Now we have r^{th} raw moment based on probability, belief and plausibility of set, as proceeding in above section 3, as:

$$\begin{aligned} \mu'_{(r,p)} &= \int_{x=a}^{b}\int_{y=x}^{b} (x,y)^r f([x,y])dxdy \\ &= \int_{x=a}^{b}\int_{y=x}^{b} (x + (1/2)[(x-a)(y-a)])^r \int_{z=x}^{y} f(z)dzdydx. \end{aligned} \tag{31}$$

$$
\begin{aligned}
\mu'_{(r,Bel)} &= \int_{x=a}^{b}\int_{y=x}^{b}(x,y)^r Bel([x,y])dxdy \\
&= \int_{x=a}^{b}\int_{y=x}^{b}(x+(1/2)[(x-a)(y-a)])^r \int_{p=x}^{y}\int_{q=p}^{y} m([p,q])dqdpdxdy \\
&= \int_{x=a}^{b}\int_{y=x}^{b}(x+(1/2)[(x-a)(y-a)])^r \int_{p=x}^{y}\int_{q=p}^{y} \frac{f([p,q])}{\int_{[p,q]\subseteq\Omega} f([p,q])dpdq}dqdpdxdy \\
&= \frac{1}{\int_{[x,y]\subseteq\Omega} f([x,y])dxdy}\int_{x=a}^{b}\int_{y=x}^{b}(x+(1/2)[(x-a)(y-a)])^r \\
&\int_{p=x}^{y}\int_{q=p}^{y}\int_{z=p}^{q} f(z)dzdqdpdxdy \\
&= \frac{1}{\int_{x=a}^{b}\int_{y=x}^{b}\int_{z=x}^{y} f(z)dzdxdy}\int_{x=a}^{b}\int_{y=x}^{b}(x+(1/2)[(x-a)(y-a)])^r \\
&\int_{p=x}^{y}\int_{q=p}^{y}\int_{z=p}^{q} f(z)dzdqdpdxdy.
\end{aligned}
\tag{32}
$$

$$
\begin{aligned}
\mu'_{(r,Pl)} &= \int_{x=a}^{b}\int_{y=x}^{b}(x,y)^r Pl([x,y])dxdy \\
&= \int_{x=a}^{b}\int_{y=x}^{b}(x+(1/2)[(x-a)(y-a)])^r \{\int_{p=a}^{x}\int_{q=x}^{b} m([p,q])dqdp \\
&+ \int_{p=x}^{y}\int_{q=p}^{b} m([p,q])dqdp\}dxdy \\
&= \int_{x=a}^{b}\int_{y=x}^{b}(x+(1/2)[(x-a)(y-a)])^r \{\int_{p=a}^{x}\int_{q=x}^{b} \frac{f([p,q])}{\int_{[p,q]\subseteq\Omega} f([p,q])dpdq}dqdp \\
&+ \int_{p=x}^{y}\int_{q=p}^{b} \frac{f([p,q])}{\int_{[p,q]\subseteq\Omega} f([p,q])dpdq}dqdp\}dxdy \\
&= \frac{1}{\int_{[p,q]\subseteq\Omega} f([p,q])dpdq}\int_{x=a}^{b}\int_{y=x}^{b}(x+(1/2)[(x-a)(y-a)])^r \{\int_{p=a}^{x}\int_{q=x}^{b}\int_{z=p}^{q} f(z)dzdqdp \\
&+ \int_{p=x}^{y}\int_{q=p}^{b}\int_{z=p}^{q} f(z)dzdqdp\}dxdy.
\end{aligned}
\tag{33}
$$

As proceeding on similar way in section 3, we can obtain lower and upper limits of raw moments, central moments and coefficients of skewness and kurtosis.

6. Illustrative Example

In this section, we will illustrate above theory by considering continuous uniform distribution. Let continuous frame of discernment is $\Omega = [a, b]$. The probability density function of continuous uniform distribution is

$$f(x) = \frac{1}{b-a} \quad \forall\, x \in [a, b]. \tag{34}$$

Therefore

$$f([c, d]) = \int_{x=c}^{d} f(x)dx = \frac{d-c}{b-a}. \tag{35}$$

Hence distribution function

$$F(x) = f([a, k]) = \frac{k-a}{b-a}. \tag{36}$$

6.1 First Basic Belief Assignment For Continuous frame of Discernment $\Omega = [a, b]$

Let $\mathcal{A} = \{A_i = [x_i, y_i] | i = 1, 2, \ldots, n\}$ where A_i are not necessarily distinct. Here we calculate probability of any subset of $\Omega = [a, b]$ using probability density function $f(x)$ but for calculation of Bel, q and Pl, we consider only subsets $A_i, \quad i = 1, 2, \ldots, n$. Therefore by (10), (11), (12) and (34) we have

$$Bel([c, d]) = \sum_{[x_i, y_i] \subseteq [c, d]} m([x_i, y_i]) \tag{37}$$

$$q([c, d]) = \sum_{[x_i, y_i] \supseteq [c, d]} m([x_i, y_i]) \tag{38}$$

$$Pl([c, d]) = \sum_{[x_i, y_i] \cap [c, d] \neq \emptyset} m([x_i, y_i]). \tag{39}$$

Lower and Upper Bounds of Distribution function :

As $[a, k]$ is general subset required for distribution function. By result (6): For any subset $A, Bel(A) \leq p(A) \leq Pl(A)$, we have by (13),

$$Bel([a, k]) \leq p([a, k]) \leq Pl([a, k])$$

i.e.

$$\sum_{[x_i, y_i] \subseteq [a, k]} m([x_i, y_i]) \leq \frac{k-a}{b-a} \leq \sum_{[x_i, y_i] \cap [a, k] \neq \emptyset} m([x_i, y_i]). \tag{40}$$

Which gives lower and upper limits (bounds) of distribution function.

Lower and Upper Bounds of Raw Moments :
We have r^{th} raw moment

$$\begin{aligned}\mu_r^{'} &= \int_{x=a}^{b} x^r f(x)dx \\ &= \int_{x=a}^{b} x^r \frac{1}{b-a}dx \\ &= \frac{1}{(b-a)(r+1)}[b^{r+1}-a^{r+1}].\end{aligned} \tag{41}$$

Now we calculate raw moments based on probability, belief and plausibility of subsets of $\Omega = [a,b]$ by (14), (15) and (16) as

$$\begin{aligned}\mu_{(r,p)}^{'} &= \sum_{i=1}^{n} i^r p(A_i) \\ &= \sum_{i=1}^{n} i^r p([x_i,y_i]) \\ &= \frac{1}{b-a}\sum_{i=1}^{n} i^r (y_i - x_i)\end{aligned} \tag{42}$$

$$\begin{aligned}\mu_{(r,Bel)}^{'} &= \sum_{i=1}^{n} i^r Bel(A_i) \\ &= \sum_{i=1}^{n} i^r \sum_{\substack{[c,d]\subseteq[x_i,y_i] \\ [c,d]\in\mathcal{A}}} m([c,d])\end{aligned} \tag{43}$$

$$\begin{aligned}\mu_{(r,Pl)}^{'} &= \sum_{i=1}^{n} i^r Pl(A_i) \\ &= \sum_{i=1}^{n} i^r \sum_{\substack{[c,d]\cap[x_i,y_i]\neq\emptyset \\ [c,d]\in\mathcal{A}}} m([c,d])\end{aligned} \tag{44}$$

Therefore the lower and upper bounds for raw moments are given by

$$\frac{\mu_{(r,Bel)}^{'}}{\mu_{(r,p)}^{'}}\mu_r^{'} \leq \mu_r^{'} \leq \frac{\mu_{(r,Pl)}^{'}}{\mu_{(r,p)}^{'}}\mu_r^{'} \tag{45}$$

Using interval arithmetics [9], relation between raw and central moments and coefficients of skewness and kurtosis [1,2], we can obtain lower and upper bounds for central moments and coefficients of skewness and kurtosis for continuous uniform distribution.

6.2 Second Basic Belief Assignment For Continuous frame of Discernment $\Omega = [a,b]$

Let $\mathcal{A} = \{A = [a,k] | k \in [a,b]\}$ subsets of $\Omega = [a,b]$. The second basic belief assignment for continuous uniform distribution is

$$m([a,x]) = \frac{f([a,x])}{\int_{x=a}^{b} f([a,x])dx}. \tag{46}$$

Now we have $\int_{x=a}^{b} f([a,x])dx = \int_{x=a}^{b} \frac{x-a}{b-a}dx = \frac{b-a}{2}$. Therefore second basic belief assignment for continuous uniform distribution becomes

$$m([a,x]) = \frac{f([a,x])}{\int_{x=a}^{b} f([a,x])dx} = \frac{2(x-a)}{(b-a)^2}. \tag{47}$$

Calculation of belief, commonality and plausibility function :-

If $c > a$ then $Bel([c,d]) = 0$ as there is no focal element $[a,x]$ contained ic $[c,d]$. For $a = c$ and by (20), (21) and (22), consider

$$\begin{aligned} Bel([a,d]) &= \int_{[a,x]\subseteq[a,d]} m([a,x])dx \\ &= \int_{x=a}^{d} \frac{2(x-a)}{(b-a)^2}dx \\ &= \frac{(d-a)^2}{(b-a)}. \end{aligned} \tag{48}$$

Now consider

$$\begin{aligned} q([a,d]) &= \int_{[a,x]\supseteq[c,d]} m([a,x])dx \\ &= \int_{x=d}^{b} \frac{2(x-a)}{(b-a)^2}dx \\ &= \frac{[b(b-2a) - d(d-2a)]}{(b-a)^2}. \end{aligned} \tag{49}$$

Now consider

$$\begin{aligned} Pl([a,d]) &= \int_{[a,x]\cap[c,d]\neq\emptyset} m([a,x])dx \\ &= \int_{x=c}^{b} \frac{2(x-a)}{(b-a)^2}dx \\ &= \frac{[b(b-2a) - c(c-2a)]}{(b-a)^2}. \end{aligned} \tag{50}$$

Lower and Upper Bounds of Distribution function :

As $[a,k]$ is general subset required for distribution function. By result (6): For any subset $A, Bel(A) \leq p(A) \leq Pl(A)$, we have by (23),

$$Bel([a,k]) \leq p([a,k]) \leq Pl([a,k])$$

i.e.

$$\frac{(d-a)^2}{(b-a)} \leq \frac{k-a}{b-a} \leq \frac{[b(b-2a)-c(c-2a)]}{(b-a)^2}. \tag{51}$$

Which gives lower and upper limits (bounds) of distribution function.

Calculation of Lower and Upper Bounds of Raw Moments :

Now we calculate raw moments based on probability by using (24), (25) and (26), belief and plausibility of subsets of $\Omega = [a,b]$ as

$$\begin{aligned} \mu'_{(r,p)} &= \int_{x=a}^{b} x^r f([a,x])dx \\ &= \int_{x=a}^{b} x^r \frac{x-a}{b-a}dx \\ &= \frac{1}{b-a}[\frac{b^{r+2}}{r+2} - \frac{ab^{r+1}}{r+1} - \frac{a^{r+2}}{r+2} - \frac{a^{r+2}}{r+1}]. \end{aligned} \tag{52}$$

$$\begin{aligned} \mu'_{(r,Bel)} &= \int_{x=a}^{b} x^r Bel([a,x])dx \\ &= \int_{x=a}^{b} x^r (\frac{x-a}{b-a})^2 dx \\ &= \frac{1}{(b-a)^2}[\frac{b^{r+3}}{r+3} - \frac{2ab^{r+2}}{r+2} + \frac{a^2b^{r+1}}{r+1} \\ &- \frac{a^{r+3}}{r+3} - \frac{2a^{r+3}}{r+2} + \frac{a^{r+3}}{r+1}] \end{aligned} \tag{53}$$

$$\begin{aligned} \mu'_{(r,Pl)} &= \int_{x=a}^{b} x^r Pl([a,x])dx \\ &= \int_{x=a}^{b} x^r 1dx \quad \text{Since } Pl([a,x]) = 1 \quad \forall x \\ &= \frac{1}{r+1}[b^{r+1} - a^{r+1}]. \end{aligned} \tag{54}$$

Therefore the lower and upper bounds for raw moments are given by

$$\frac{\mu'_{(r,Bel)}}{\mu'_{(r,p)}}\mu'_r \leq \mu'_r \leq \frac{\mu'_{(r,Pl)}}{\mu'_{(r,p)}}\mu'_r \tag{55}$$

Using interval arithmetics [9], relation between raw and central moments and coefficients of skewness and kurtosis [1,2], we can obtain lower and upper bounds for central moments and coefficients of skewness and kurtosis for continuous uniform distribution.

6.3 Third Basic Belief Assignment For Continuous frame of Discernment $\Omega = [a, b]$

Let $\mathcal{A} = \{A = [x, y] | x, y \in [a, b]\}$ subsets of $\Omega = [a, b]$. The third basic belief assignment for continuous uniform distribution is

$$m([x, y]) = \frac{f([x, y])}{\int_{x=a}^{b} \int_{y=x}^{b} f([x, y]) dydx}. \tag{56}$$

Now consider

$$\begin{aligned} I &= \int_{x=a}^{b} \int_{y=x}^{b} f([x, y]) dydx \\ &= \int_{x=a}^{b} \int_{y=x}^{b} \frac{y - x}{b - a} dydx \\ &= \frac{[b - a]^2}{6} \end{aligned} \tag{57}$$

Therefore third basic belief assignment becomes

$$m([x, y]) = \frac{f([x, y])}{\int_{x=a}^{b} \int_{y=x}^{b} f([x, y]) dydx} = \frac{6(y - x)}{[b - a]^3}. \tag{58}$$

Now we will calculate belief commonality and plausibility functions based on this third basic belief assignment by using (28), (29) and (30). Therefore

$$\begin{aligned} Bel([c, d]) &= \int_{x=c}^{d} \int_{y=x}^{d} m([x, y]) dydx \\ &= \int_{x=c}^{d} \int_{y=x}^{d} \frac{6(y - x)}{[b - a]^3} dydx \\ &= \frac{[d - c]}{[b - a]}^3. \end{aligned} \tag{59}$$

$$\begin{aligned} q([c, d]) &= \int_{x=a}^{c} \int_{y=d}^{b} m([x, y]) dydx \\ &= \int_{x=a}^{c} \int_{y=d}^{b} \frac{6(y - x)}{[b - a]^3} dydx \\ &= \frac{1}{[b - a]^3} [3bc(b - c) - 3dc(d - c) - 3ab(b - a) + 3ad(d - a)] \end{aligned} \tag{60}$$

We have $Pl([c,d]) = \int_{x=a}^{c}\int_{y=c}^{b} m([x,y])dydx + \int_{x=c}^{d}\int_{y=x}^{b} m([x,y])dydx = I_1 + I_2$. Therefore consider

$$\begin{aligned} I_1 &= \int_{x=a}^{c}\int_{y=c}^{b} m([x,y])dydx \\ &= \int_{x=a}^{c}\int_{y=c}^{b} \frac{6(y-x)}{[b-a]^3}dydx \\ &= \frac{1}{[b-a]^3}[3bc(b-c) - 3ab(b-a) + 3ac(c-a)]. \end{aligned} \tag{61}$$

$$\begin{aligned} I_2 &= \int_{x=c}^{d}\int_{y=x}^{b} m([x,y])dydx \\ &= \int_{x=c}^{d}\int_{y=x}^{b} \frac{6(y-x)}{[b-a]^3}dydx \\ &= \frac{1}{[b-a]^3}[d^3 - c^3 + 3bd(b-d) - 3bc(b-c)]. \end{aligned} \tag{62}$$

Therefore

$$Pl([c,d]) = I_1 + I_2 = \frac{[d^3 - c^3 + 3ac(c-a) - 3ab(b-a) + 3bd(b-d)]}{[b-a]^3}. \tag{63}$$

Lower and Upper Bounds of Distribution Function :-

As proceeding similar way to section 3 for : Lower and Upper Bounds of Distribution Function, and by using (6), (58) and (62), we have $Bel([a,k]) \leq p([a,k]) \leq Pl([a,k])$.i.e

$$\frac{[k-a]^3}{[b-a]} \leq \frac{k-a}{b-a} \leq \frac{[k^3 - a^3 - 3ab(b-a) + 3bk(b-k)]}{[b-a]^3}. \tag{64}$$

Calculation of Lower and Upper Bounds of Raw Moments :

Now we calculate raw moments based on probability, belief and plausibility of subsets of $\Omega = [a, b]$ by using (31), (32) and (33) as

$$\begin{aligned} \mu'_{(r,p)} &= \int_{x=a}^{b}\int_{y=x}^{b} (x,y)^r f([x,y]) dydx \\ &= \int_{x=a}^{b}\int_{y=x}^{b} (x,y)^r \frac{y-x}{b-a} dydx \\ &= \int_{x=a}^{b}\int_{y=x}^{b} x + (1/2)(x-a)(y-a)^r \frac{y-x}{b-a} dydx \end{aligned} \tag{65}$$

$$\begin{aligned} \mu'_{(r,Bel)} &= \int_{x=a}^{b}\int_{y=x}^{b} (x,y)^r Bel([x,y]) dydx \\ &= \int_{x=a}^{b}\int_{y=x}^{b} x + (1/2)(x-a)(y-a)^r \frac{y-x}{b-a}^3 dydx \end{aligned} \tag{66}$$

$$\begin{aligned} \mu'_{(r,Pl)} &= \int_{x=a}^{b}\int_{y=x}^{b} (x,y)^r Pl([x,y]) dydx \\ &= \int_{x=a}^{b}\int_{y=x}^{b} x + (1/2)(x-a)(y-a)^r Pl([x,y]) dydx \end{aligned} \tag{67}$$

Therefore the lower and upper bounds for raw moments are given by

$$\frac{\mu'_{(r,Bel)}}{\mu'_{(r,p)}}\mu'_r \leq \mu'_r \leq \frac{\mu'_{(r,Pl)}}{\mu'_{(r,p)}}\mu'_r. \tag{68}$$

Using interval arithmetics [9], relation between raw and central moments and coefficients of skewness and kurtosis [1,2], we can obtain lower and upper bounds for central moments and coefficients of skewness and kurtosis for continuous uniform distribution.

Remark : All belief functions based on these basic belief assignments are probability functions, measures and Bayesian belief functions as approximately all calculations are based on continuous probability density functions.

7. Applications

In real life, continuous probability distributions have lot of applications. To get lower and upper limits of statistical quantities based on these continuous probability distributions, this approach is helpful. While obtaining these bounds, one should be careful about interval arithmetics as it gives too much wide bounds.

8. Future Scope

This approach of obtaining basic belief assignments can be generalized for focal elements as union of exactly and at most n distinct subsets of continuous frame of discernment.

9. Conclusion

In past, approximately all authors have assumed that subset $[x, y]$ of continuous frame of discernment $[a, b]$, as (x, y) as bivariate continuous probability distribution. This is one of the approach of obtaining beliefs induced by univariate continuous probability distributions. All belief functions based on these basic belief assignments are probability functions, measures and Bayesian belief functions as approximately all calculations are based on continuous probability density functions.

References

[1] Bansi Lal and Sanjay Arora, Mathematical Statistics, Satya Prakashan, New Delhi, (1989).

[2] Billingsley Patrick, Probability and Measures, Wiley India Pvt. Ltd, third edition, (2008).

[3] Branko Ristic, Philippe Smets, Belief function theory on the continuous space with an application to model based classification, Information Processing and Management of Uncertainty, (2004), 1119-1126.

[4] Caron Francois, Branko Ristic, Emmanual Dulfos and Philippe Vanheeghe, Least Committed basic belief density induced by multivariate Guassian pdf: Formulation with Application, International Journal of Approximate Reasoning, 48(2) (2008), 419-436.

[5] Dempster A. P., Lower and upper probabilities induced by a multi-valued mapping, The Annals of Mathematical Statistics,(1966), 325-339.

[6] Dempster A. P., Lower and Upper probabilities generated by a Random closed interval, The Annals of Mathematical Statistics, (1968), 957-966.

[7] Dore P. E. and Matin A., About using beliefs induced by probabilities, Workshop on the Theory of Belief Functions (Breast, France), (2010).

[8] Dore P. E., Martin A. and Khenchaf A., Constructing consonant belief function Induced by a multi-modal probability, COGnitive system with Interactive Sensors (COGIS 2009), Espace Hamelin, Paris, (2009).

[9] Moore Ramon E., Kearfott R. Baker and Cloud Michael J., Introduction to Interval Analysis, Society for Industrial and Applied Mathematics, Philadephia, PA, USA, (2009).

[10] Shafer, Glenn A., A Mathematical Theory of Evidence. Princeton University Press, New Jersey, (1976).

[11] Smets Philippe, Belief functions on real numbers, International Journal of approximate reasoning, 40(3) (2005), 181-223.

[12] Smets Philippe, The combination of evidence in transferable elief model, Pattern Analysis and Machine Intelligence, IEEE Transactions, 12(5) (1990), 447-458. 1990.

[13] Strat Thomas M., Continuous belief functions for evidential reasoning, in Proceedings of national Conference on Artificial Intelligence, (1984), 308-313.

[14] Vannobel Jean-Marc, Continuous belief functions : singletons plausibility function in conjunctive and disjunctive combination operations of consonant bbds, Workshop on the Theory of Belief Functions (Breast, France), (2010).