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UNIQUENESS AND VALUE DISTRIBUTION FOR q-SHIFT DIFFERENCE POLYNOMIALS

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Abstract

We investigate the zero distribution of q-shift difference polynomials of entire and meromorphic functions with zero order and obtain some results that extend previous results of Liu et al. [18]

1. Introduction and Main Results

Let $f(z)$ and $g(z)$ be two non constant meromorphic functions in the complex plane. By $S(r, f)$, we denote any quantity satisfying $S(r, f) = o(T(r, f))$, as $r \longrightarrow \infty$, possibly outside a set of r with finite linear measure. The meromorphic function α is called a small function of $f(z)$, if $T(r, \alpha) = S(r, f)$. If $f(z) - \alpha$ and $g(z) - \alpha$ have same zeros, counting multiplicity (ignoring multiplicity), then we say that $f(z)$ and $g(z)$ share the

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small function α CM(IM).(see, e.g.,[1,2]) The logarithmic density of a set F_n is defined as follows:

$$\limsup_{r\to\infty} \frac{1}{logr} \int_{[1,r]\cap F_n} \frac{1}{t} dt$$

In recent years, there has been an increasing interest in studying the uniqueness problems related to meromorphic functions and their shifts or difference operators(see, e.g.,[3-15]). Our aim in this article is to investigate the value distribution for q-shift polynomials of transcendental meromorphic and entire functions with zero order.

Liu et al.[13] considered uniqueness of difference polynomials of meromorphic functions, corresponding to uniqueness theorems of meromorphic functions sharing values(see, e.g.,[16]) and obtained the following results.

Theorem A : Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions with finite order. Suppose that c is a non-zero complex constant and n is an integer. If $n \geq 14$ and $f^n(z)f(z+c)$ and $g^n(z)g(z+c)$ share 1 CM, then $f(z) \equiv tg(z)$ or $f(z)g(z) = t$, where $t^{n+1} = 1$.

Theorem B : Under the conditions of Theorem A, if $n \geq 26$ and $f^n(z)f(z+c)$ and $g^n(z)g(z+c)$ share 1 IM, then $f(z) \equiv tg(z)$ or $f(z)g(z) = t$, where $t^{n+1} = 1$.

Recently, Liu et al. [18], considered the case of q-shift difference polynomials and extended the Theorem A as follows :

Theorem C : Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions with $\rho(f) = \rho(g) = 0$. Suppose that q and c are two non-zero complex constants and n is an integer . If $n \geq 14$ and $f^n(z)f(qz+c)$ and $g^n(z)g(qz+c)$ share 1 CM, then $f(z) \equiv tg(z)$ or $f(z)g(z) = t$, where $t^{n+1} = 1$.

Theorem D : Under the conditions of Theorem C, if $n \geq 26$ and $f^n(z)f(qz+c)$ and $g^n(z)g(qz+c)$ share 1 IM, then $f(z) \equiv tg(z)$ or $f(z)g(z) = t$, where $t^{n+1} = 1$.

Theorem E : Let $f(z)$ and $g(z)$ be two transcendental entire functions with $\rho(f) = \rho(g) = 0$, let q and c be two non-zero complex constants, let $P(z) = a_n z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0$ be a non-zero polynomial , where $a_n (\neq 0), a_{n-1}, ..., a_0$, are complex constants and k denotes the number of the distinct zero of $P(z)$. If $n > 2k+1$ and $P(f(z))f(qz+c)$ and $P(g(z))g(qz+c)$ share 1 CM, then one of the following results holds:

(1)$f(z) \equiv tg(z)$ for a constant t such that $t^d = 1$, where $d = GCD\{\lambda_0, \lambda_1, ..., \lambda_n\}$ and

$$\lambda_j = \begin{cases} n+1, & a_j = 0, \\ j+1, & a_j \neq 0, \end{cases} \qquad j = 0, 1, ..., n;$$

(2)$f(z)$ and $g(z)$ satisfy the algebraic equation $R(f(z), g(z)) = 0$, where

$$R(w_1, w_2) = P(w_1)w_1(qz+c) - P(w_2)w_2(qz+c)$$

In this paper, we deal with value distribution for q-shift difference polynomials of transcendental meromorphic and entire functions of the form $f^n(z)P_m(f(qz+c))f'(z)$ in Theorems C,D,E and prove the following theorems:

Theorem 1 : Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions with $\rho(f) = \rho(g) = 0$. Let q and c be two non-zero complex constants, n an integer and $P_m(z) = a_m z^m + a_{m-1}z^{m-1} + ... + a_1 z + a_0$. If $n \geq 5m+19$ and $f^n(z)P_m(f(qz+c))f'(z)$ and $g^n(z)P_m(g(qz+c))g'(z)$ share 1 CM, then $f(z) \equiv tg(z)$ or $f(z)g(z) = t$, where $t^d = 1, d = GCD(n+m+1, n+m, ..., n+m+1-i, ..., n+1)$,$a_{m-i} \neq 0$, for some $i = 0, 1, ..., m$.

Theorem 2 : Under the conditions of Theorem 1, if $n \geq 11m+31$, $f^n(z)P_m(f(qz+c))f'(z)$ and $g^n(z)P_m(g(qz+c))g'(z)$ share 1 IM, then conclusion of Theorem 1 still holds.

Theorem 3 : Let $f(z)$ and $g(z)$ be two transcendental entire functions with $\rho(f) = \rho(g) = 0$, q and c are two non-zero complex constants and k denote the number of distinct zeros of $P_m(z)$. If $m > n+2k+4$, $f^n(z)P_m(f(qz+c))f'(z)$ and $g^n(z)P_m(g(qz+c))g'(z)$ share 1 CM , then one of the following results holds:

(1)$f(z) \equiv tg(z)$ for a constant t such that $t^d = 1$, where

$$d = GCD(n+m+1, n+m, ..., n+m+1-i, ..., n+1).$$

$a_{m-i} \neq 0$, for some $i = 0, 1, ..., m$.

(2)$f(z)$ and $g(z)$ satisfy the algebraic equation $R(f, g) \equiv 0$ where

$$\begin{aligned} R(w_1, w_2) &= w_1^{n+1}\left[\frac{a_m w_1^m}{n+m+1} + \frac{a_{m-1}w_1^{m-1}}{n+m} + ... + \frac{a_0}{n+1}\right] \\ &\quad - w_2^{n+1}\left[\frac{a_m w_2^m}{n+m+1} + \frac{a_{m-1}w_2^{m-1}}{n+m} + ... + \frac{a_0}{n+1}\right] \end{aligned}$$

2. Preliminary Lemmas

The following lemma is a q-difference analogue of the logarithmic derivative lemma.

Lemma 2.1 (see [14]) : Let $f(z)$ be a meromorphic function of zero order, let c and q be two non-zero complex numbers, then

$$m\left(r, \frac{f(qz+c)}{f(z)}\right) = S(r,f)$$

on a set of logarithmic density 1.

Lemma 2.2 (see [7]) : If $T : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing function such that,

$$\limsup_{r\to\infty} \frac{logT(r)}{logr} = 0$$

then, the set

$E := \{r : T(C_1 r) \geq C_2 T(r)\}$ has the logarithmic density 0 for all $C_1 > 1$ and $C_2 > 1$.

Lemma 2.3 (see [12]) : Let $f(z)$ be a meromorphic function of finite order, let $c \neq 0$ be fixed. Then

$$\overline{N}(r, f(z+c)) \leq \overline{N}(r, f(z)) + S(r,f),$$
$$\overline{N}\left(r, \frac{1}{f(z+c)}\right) \leq \overline{N}\left(r, \frac{1}{f(z)}\right) + S(r,f),$$
$$N(r, f(z+c)) \leq N(r, f(z)) + S(r,f),$$
$$N\left(r, \frac{1}{f(z+c)}\right) \leq N\left(r, \frac{1}{f(z)}\right) + S(r,f).$$

Lemma 2.4 (see[18]) : Let $f(z)$ be a meromorphic function with $\rho(f) = 0$, let c and q be two non-zero complex numbers, then

$$\overline{N}(r, f(qz+c)) \leq \overline{N}(r, f(z)) + S(r,f),$$
$$\overline{N}\left(r, \frac{1}{f(qz+c)}\right) \leq \overline{N}\left(r, \frac{1}{f(z)}\right) + S(r,f),$$
$$N(r, f(qz+c)) \leq N(r, f(z)) + S(r,f),$$
$$N\left(r, \frac{1}{f(qz+c)}\right) \leq N\left(r, \frac{1}{f(z)}\right) + S(r,f).$$

Lemma 2.5 (see[18]) : Let f be a non-constant meromorphic function of zero order, let c and q be two non-zero complex numbers, then

$$T(r, f(qz+c)) \leq T(r, f(z)) + S(r,f)$$

on a set of logarithmic density 1.

Lemma 2.6(see[18]). Let $f(z)$ be an entire function with $\rho(f) = 0$, let c and q be two fixed non-zero complex constants, let $P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0$ be a non-zero polynomial , where $a_n(\neq 0), a_{n-1}, ..., a_0$, are complex constants, then

$$T(r, P(f(z))f(qz + c)) = T(r, P(f(z))f(z)) + S(r, f).$$

Lemma 2.7 (see[17]) : Let F and G be two non-constant meromorphic functions. If F and G share 1 CM, then one of the following three cases holds:

$$(i) max\{T(r, F), T(r, G)\} \leq N_2\left(r, \frac{1}{F}\right) + N_2(r, F) + N_2\left(r, \frac{1}{G}\right)$$
$$+ N_2(r, G) + S(r, F) + S(r, G)$$
$$(ii) F = G,$$
$$(iii) FG \equiv 1,$$

where $N_2\left(r, \frac{1}{F}\right)$ denotes the counting function of zero of F, such that simple zero are counted once and multiple zeros are counted twice.

Lemma 2.8 (see[16]) : Let F and G be two non-constant meromorphic functions. Let F and G share 1 IM and

$$H = \frac{F''}{F'} - 2\frac{F'}{F-1} - \frac{G''}{G'} + 2\frac{G'}{G-1}$$

If $H \not\equiv 0$, then

$$T(r, F) + T(r, G) \leq 2\left(N_2\left(r, \frac{1}{F}\right) + N_2(r, F) + N_2\left(r, \frac{1}{G}\right) + N_2(r, G)\right)$$
$$+3\left(\overline{N}(r, F) + \overline{N}(r, G) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{G}\right)\right) + S(r, F) + S(r, G).$$

3. Proof of Theorem 1

Let

$$\begin{aligned} F(z) &= f^n(z) P_m(f(qz + c)) f'(z) \\ G(z) &= g^n(z) P_m(g(qz + c)) g'(z) \end{aligned}$$

Thus, F and G share 1 CM. Combining the first main theorem with Lemma 2.5, we obtain

$$(n+2)T(r,f(z)) \leq T(r,f^n(z)f^2(qz+c)) + O(1)$$

Hence, we obtain

$$(n-m-2)T(r,f(z)) \leq T(r,F(z)) + S(r,f) \tag{1}$$

$$\text{Similarly, } (n-m-2)T(r,g(z)) \leq T(r,G(z)) + S(r,g) \tag{2}$$

From Lemma 2.5, we have

$$T(r,F) \leq (n+m+2)T(r,f) + S(r,f) \tag{3}$$

$$T(r,G) \leq (n+m+2)T(r,g) + S(r,g) \tag{4}$$

By Second main theorem, Lemma 2.5 and (4), we obtain

$$\begin{aligned} T(r,F) &\leq \overline{N}(r,F) + \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{F-1}\right) + S(r,F) \\ &\leq \overline{N}(r,f) + \overline{N}(r,P_m(f(qz+c))) + \overline{N}(r,f^{'}) \\ &+ \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{P_m(f(qz+c))}\right) + \overline{N}\left(r,\frac{1}{f'}\right) + \overline{N}\left(r,\frac{1}{G-1}\right) + S(r,f) \\ T(r,F) &\leq (7+m)T(r,f) + (n+m+2)T(r,g) + S(r,f) + S(r,g) \end{aligned} \tag{5}$$

Hence (1) and (5) imply that

$$(n-2m-9)T(r,f) \leq (n+m+2)T(r,g) + S(r,f) + S(r,g) \tag{6}$$

Similarly, we have

$$(n-2m-9)T(r,g) \leq (n+m+2)T(r,f) + S(r,f) + S(r,g) \tag{7}$$

Equations (6) and (7) imply that $S(r,f) = S(r,g)$.

Together the definition of F with Lemma 2.5, we have

$$\begin{aligned} N_2\left(r,\frac{1}{F}\right) &\leq 2\overline{N}\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{P_m(f(qz+c))}\right) + N\left(r,\frac{1}{f'}\right) + S(r,f) \\ &\leq (4+m)T(r,f) + S(r,f) \end{aligned} \tag{8}$$

Similarly,

$$\begin{aligned} N_2\left(r, \frac{1}{G}\right) &\leq (4+m)T(r,g) + S(r,g), \\ N_2(r,F) &\leq (4+m)T(r,f) + S(r,f), \\ N_2(r,G) &\leq (4+m)T(r,g) + S(r,g), \end{aligned} \tag{9}$$

Combining Lemma 2.7 with (8)-(9), we obtain

$$\begin{aligned} T(r,F) + T(r,G) &\leq 2N_2\left(r, \frac{1}{F}\right) + 2N_2(r,F) + 2N_2\left(r, \frac{1}{G}\right) + 2N_2(r,G) + S(r,f) + S(r,g) \\ &\leq (16+4m)\big(T(r,f) + T(r,g)\big) + S(r,f) + S(r,g) \end{aligned} \tag{10}$$

Then, by (1) (2) and (10), we obtain

$$(n-5m-18)[T(r,f) + T(r,g)] \leq S(r,f) + S(r,g) \tag{11}$$

which is a contradiction, since $n \geq 5m+19$.
By Lemma 2.7, we have $F \equiv G$ or $FG \equiv 1$.
If $F \equiv G$,
that is,

$$f^n(z)P_m(f(qz+c))f^{'}(z) \equiv g^n(z)P_m(g(qz+c))g^{'}(z)$$

Set H(z)=f(z)/g(z)
Suppose that $H(z)$ is not a constant. Then, we obtain

$$\begin{aligned} \frac{f^n(z)P_m(f(qz+c))f^{'}(z)}{g^n(z)P_m(g(qz+c))g^{'}(z)} &= 1 \\ H^n(z)P_m(H(qz+c))H^{'}(z) &= 1 \end{aligned} \tag{12}$$

From Lemma 2.5 and (12), we get

$$\begin{aligned} nT(r,H) &= T\left(r, \frac{1}{P_m(H(qz+c))H'(z)}\right) \\ &\leq T(r, P_m(H(qz+c))H^{'}(z)) + S(r,H) \\ &\leq (m+2)T(r,H(z)) + S(r,H) \end{aligned} \tag{13}$$

Hence, $H(z)$ must be non-zero constant, since $n \geq 5m+19$.
Set $H(z) = t$

By (12), we have $t^d = 1$.
Thus $f(z) = tg(z)$, where $d = GCD(n+m+1, n+m, ..., n+m+1-i, ..., n+1)$, $a_{m-i} \neq 0$ for some $i = 0, 1, 2, ...m+n$

If $FG \equiv 1$, that is,

$$f^n(z)P_m(f(qz+c))f^{'}(z).g^n(z)P_m(g(qz+c))g^{'}(z) = 1$$

Let $L(z) = f(z).g(z)$. Using similar method as above, we obtain that $L(z)$ must also be a non-zero constant. Thus we have $fg = t$, where $t^d = 1$, $d = GCD(n+m+1, n+m, ..., n+m+1-i, ..., n+1)$, $a_{m-i} \neq 0$ for some $i = 0, 1, 2, ...m+n$.

4. Proof of Theorem 2

Let

$$\begin{aligned} F(z) &= f^n(z)P_m(f(qz+c))f^{'}(z) \\ G(z) &= g^n(z)P_m(g(qz+c))g^{'}(z) \end{aligned}$$

and H be defined as in Lemma 2.8. Using the same arguments as in Theorem 1, we prove that (1)-(9) holds.
By Lemma 2.5, we obtain

$$\begin{aligned} \overline{N}(r, F(z)) &\leq \overline{N}(r, f(z)) + \overline{N}(r, P_m(f(qz+c))) + \overline{N}(r, f^{'}(z)) + S(r, f) \\ &\leq (m+2)T(r, f) + S(r, f), \\ \text{Similarly}, \overline{N}\left(r, \frac{1}{F(z)}\right) &\leq (m+2)T(r, f) + S(r, f), \\ \overline{N}(r, G(z)) &\leq (m+2)T(r, g) + S(r, g), \\ \overline{N}\left(r, \frac{1}{G(z)}\right) &\leq (m+2)T(r, g) + S(r, g). \end{aligned} \tag{14}$$

Together Lemma 2.8 with (8), (9) and (14), we have

$$T(r, F) + T(r, G) \leq (10m+28)\big(T(r, f) + T(r, g)\big) + S(r, f) + S(r, g) \tag{15}$$

By (1), (2) and (15)

$$(n-m-2)\big(T(r, f) + T(r, g)\big) \leq (10m+28)\big(T(r, f) + T(r, g)\big) + S(r, f) + S(r, g) \tag{16}$$

which is impossible, since $n \geq 11m+31$. Hence, we have $H \equiv 0$.
By integrating H twice, we have

$$F = \frac{(b+1)G + (a-b-1)}{bG + (a-b)} \tag{17}$$

which yields $T(r, F) = T(r, G) + O(1)$.
From (1)-(4), we obtain

$$(n-m-2)T(r,f) \leq (n+m+2)T(r,g) + S(r,f) + S(r,g) \tag{18}$$

$$(n-m-2)T(r,g) \leq (n+m+2)T(r,f) + S(r,f) + S(r,g) \tag{19}$$

Next, we will prove that $F \equiv G$ or $FG \equiv 1$

Case 1 : $(b \neq 0, -1)$. If $a-b-1 \neq 0$, by (17), we obtain

$$\overline{N}\left(r, \frac{1}{F}\right) = \overline{N}\left(r, \frac{1}{G - (a-b-1)/(b+1)}\right) \tag{20}$$

Combining the Nevanlinna second main theorem with Lemma 2.5, (1),(4) and (19), we obtain

$$\begin{aligned}
(n-m-2)T(r,g) &\leq T(r,G) + S(r,g) \\
&\leq \overline{N}\left(r, \frac{1}{G}\right) + \overline{N}(r,G) + \overline{N}\left(r, \frac{1}{G - (a-b-1)/(b+1)}\right) + S(r,g) \\
&\leq \overline{N}\left(r, \frac{1}{G}\right) + \overline{N}(r,G) + \overline{N}\left(r, \frac{1}{F}\right) + S(r,g) \\
&\leq \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{P_m(g(qz+c))}\right) + \overline{N}\left(r, \frac{1}{g'}\right) \\
&+ \overline{N}(r,g) + \overline{N}(r, P_m(g(qz+c))) + \overline{N}(r, g') \\
&+ \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{P_m(f(qz+c))}\right) + \overline{N}\left(r, \frac{1}{f'}\right) + S(r,g) \\
&\leq (7+m)T(r,g) + (m+4)T(r,f) + S(r,g)
\end{aligned} \tag{21}$$

By simple calculation, we get contradiction, since $n \geq 11m+31$. Hence we obtain, $a-b-1=0$, so

$$F = \frac{(b+1)G}{bG+1} \tag{22}$$

Using the similar method as above, we obtain

$$\begin{aligned}(n-m-2)T(r,g) &\leq T(r,G)+S(r,g)\\ &\leq \overline{N}\left(r,\frac{1}{G}\right)+\overline{N}(r,G)+\overline{N}\left(r,\frac{1}{G+1/b}\right)+S(r,g)\\ &\leq \overline{N}\left(r,\frac{1}{G}\right)+\overline{N}(r,G)+\overline{N}\left(r,\frac{1}{F}\right)+S(r,g)\\ &\leq (7+m)T(r,g)+(m+4)T(r,f)+S(r,g)\end{aligned}$$

which is impossible.

Case 2 : If $b=-1$ and $a=-1$, then $FG\equiv 1$ follows trivially. Therefore, consider $b=-1$ and $a\neq -1$.

By (17), we have

$$F=\frac{a}{a+1-G} \tag{23}$$

Similarly, as above we get contradiction.

Case 3. If $b=0$, $a=1$, then $F\equiv G$ follows trivially. Therefore, consider $b=0$ and $a\neq 1$. By (17), we have

$$F=\frac{G+a-1}{a} \tag{24}$$

Similarly, as above we get contradiction.

5. Proof of Theorem 3

Let $f(z)$ and $g(z)$ be two transcendental entire functions. Since $f^n(z)P_m(f(qz+c))f^{'}(z)$ and $g^n(z)P_m(g(qz+c))g^{'}(z)$ share 1 CM, we have

$$\frac{f^n(z)P_m(f(qz+c))f^{'}(z)-1}{g^n(z)P_m(g(qz+c))g^{'}(z)-1}=e^{l(z)} \tag{25}$$

where $l(z)$ is an entire function, by $\rho(f)=0$ and $\rho(g)=0$, we have $e^{l(z)}\equiv\eta$ a constant. Rewriting (25),

$$\eta g^n(z)P_m(g(qz+c))g^{'}(z)=f^n(z)P_m(f(qz+c))f^{'}(z)+\eta-1 \tag{26}$$

If $\eta \neq 1$, by the first main theorem, the second main theorem and Lemma 2.5, we have

$$\begin{aligned} T(r, f^n(z)P_m(f(qz+c))f'(z)) &\leq \overline{N}(r, f^n(z)P_m(f(qz+c))f'(z)) \\ &+ \overline{N}\left(r, \frac{1}{f^n(z)P_m(f(qz+c))f'(z))}\right) \\ &+ \overline{N}\left(r, \frac{1}{f^n(z)P_m(f(qz+c))f'(z)) - 1}\right) \\ &\leq (n+k+3)T(r,f) + \overline{N}\left(r, \frac{1}{g^n(z)P_m(g(qz+c))g'(z))}\right) \\ &+ S(r,f) + S(r,g) \\ &\leq (n+k+3)T(r,f) + (n+k+3)T(r,g) + S(r,f) + S(r,g) \end{aligned} \tag{27}$$

By Lemma 2.6 and (27), we have

$$\begin{aligned} (n+m+2)T(r,f) &= T(r, f^n(z)P_m(f(qz+c))f'(z)) \\ &\leq (n+k+3)T(r,f) + (n+k+3)T(r,g) + S(r,f) + S(r,g) \\ (m-k-1)T(r,f) &\leq (n+k+3)T(r,g) + S(r,f) + S(r,g) \qquad (28) \end{aligned}$$

$$\text{Similarly, } (m-k-1)T(r,g) \leq (n+k+3)T(r,f) + S(r,f) + S(r,g) \tag{29}$$

Equations (28) and (29) imply that

$$(m-2k-4-n)\big(T(r,f) + T(r,g)\big) \leq S(r,f) + S(r,g) \tag{30}$$

which is impossible , since $m > n + 2k + 4$.
Hence we have $\eta = 1$. Rewriting (25),

$$g^n(z)P_m(g(qz+c))g'(z) = f^n(z)P_m(f(qz+c))f'(z) \tag{31}$$

Set $h(z) = f(z)/g(z)$

Case 1 : Suppose that $h(z)$ is a constant.

Integrating (31), we get

$$\begin{aligned} & f^{n+1}\left[\frac{a_m f^m(qz+c)}{m+n+1} + \frac{a_{m-1}f^{m-1}(qz+c)}{m+n} + ... + \frac{a_0}{n+1}\right] \\ &= g^{n+1}\left[\frac{a_m g^m(qz+c)}{m+n+1} + \frac{a_{m-1}g^{m-1}(qz+c)}{m+n} + ... + \frac{a_0}{n+1}\right] \end{aligned} \tag{32}$$

By substituting $f = gh$ in (32), we obtain

$$g^{n+1}h^{n+1}\left[\frac{a_m g^m(qz+c)h^m}{m+n+1} + \frac{a_{m-1}g^{m-1}(qz+c)h^{m-1}}{m+n} + ... + \frac{a_0}{n+1}\right]$$
$$= g^{n+1}\left[\frac{a_m g^m(qz+c)}{m+n+1} + \frac{a_{m-1}g^{m-1}(qz+c)}{m+n} + ... + \frac{a_0}{n+1}\right]$$
$$\Rightarrow g^{n+1}\left[\frac{a_m g^m(qz+c)}{m+n+1}(h^{m+n+1}-1) + \frac{a_{m-1}g^{m-1}(qz+c)}{m+n+1}(h^{m+n}-1)\right.$$
$$\left. + ... + \frac{a_0}{n+1}(h^{n+1}-1)\right] \equiv 0$$

Since g is a transcendental entire function, we have $g^{n+1}(z) \neq 0$. Hence, we obtain

$$\frac{a_m g^m(qz+c)}{m+n+1}(h^{m+n+1}-1) + \frac{a_{m-1}g^{m-1}(qz+c)}{m+n+1}(h^{m+n}-1) + ... + \frac{a_0}{n+1}(h^{n+1}-1) \equiv 0 \tag{33}$$

Equation (33) implies that $h^d = 1$, where $d = GCD(n+m+1, n+m, ..., n+m+1-i, ..., n+1)$, $a_{m-i} \neq 0$, for some $i = 0, 1, ...m$.
Thus $f = tg$ for a constant t, such that $t^d = 1$, where $d = GCD(n+m+1, n+m, ..., n+m+1-i, ..., n+1)$, $a_{m-i} \neq 0$, for some $i = 0, 1, ...m$.
Case 2 : Suppose that $h(z)$ is not a constant, then by (33) $f(z)$ and $g(z)$ satisfy the algebraic equation $R(f, g) \equiv 0$, where

$$R(w_1, w_2) = w_1^{n+1}\left[\frac{a_m w_1^m}{n+m+1} + \frac{a_{m-1}w_1^{m-1}}{n+m} + ... + \frac{a_0}{n+1}\right]$$
$$- w_2^{n+1}\left[\frac{a_m w_2^m}{n+m+1} + \frac{a_{m-1}w_2^{m-1}}{n+m} + ... + \frac{a_0}{n+1}\right] \tag{34}$$

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