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ON PERMANENTAL POLYNOMIAL IN GRAPHS

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Abstract

Let G be a simple graph of order n with adjacency matrix $A(G)$ and $P(G,x)$ the permanental polynomial of G. Let I_n denote the $n \times n$ identity matrix. Then $P(G,x) = per(xI_n + A(G))$ is called the permanental polynomials of the graph G. In this paper, we discussed the permanental polynomial of the graph such as path, cycle, star, triangular book graph, bistar graph and the new root graph.

1. Introduction

By a simple graph $G = (V(G), E(G))$ we mean a finite undirected graph with the vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ and the edge set $E(G) = \{e_1, e_2, \cdots, e_n\}$, if not specified [5]. The adjacency matrix of a graph G, here denoted by $A(G) = (a_{ij})_{n\times n}$, is a matrix of order n whose entries $_{ij} = 1$ if vertex v_i is adjacent to vertex v_j and $a_{ij} = 0$ otherwise.

Key Words : *Permanental polynomial, Path, Cycle, Triangular book graph.*

Definition 1.1 [6] : Let S_n be the set of all permutation of $(1, 2, \cdots, n)$. The *permanent* of the matrix A is denoted by $per(A)$ is defined as $per(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i,\sigma(i)}$.

Definition 1.2 [4] : Let G be a graph of order n and let $A(G)$ be the adjacency matrix of G. Let I_n denote the $n \times n$ identity matrix. The **permanental polynomial** of G, denoted by $P(G, x)$ is defined as $P(G, x) = per(xI_n + A(G))$.

Theorem 1.3 : The permanental polynomial of the path graph P_n is given by $P(G, x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \begin{pmatrix} n-k \\ k \end{pmatrix} x^{n-2k}$ for $n \geq 2$.

Proof : Let $G = P_n$ be the path graph with n vertices and $n - 1$ edges.

We know that the permanental polynomial of G is $P(G, x) = per(xI_n + A(G))$ where $A(G)$ is the adjacency matrix of the graph P_n.

$$
\begin{aligned}
P(P_n, x) &= per(xI_n + A(P_n)) \\
&= per\left(\begin{pmatrix} x & 0 & 0 & 0 & \cdots & 0 \\ 0 & x & 0 & 0 & \cdots & 0 \\ 0 & 0 & x & 0 & \cdots & 0 \\ & & & & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & x \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 \\ & & & & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & 10 \end{pmatrix}\right) \\
&= per\left(\begin{pmatrix} x & 1 & 0 & 0 & \cdots & 0 \\ 1 & x & 1 & 0 & \cdots & 0 \\ 0 & 1 & x & 1 & \cdots & 0 \\ & & & & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & 1x \end{pmatrix}\right) \\
&= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \begin{pmatrix} n-k \\ k \end{pmatrix} x^{n-2k}.
\end{aligned}
$$

That is,

$$P(P_n, x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \begin{pmatrix} n-k \\ k \end{pmatrix} x^{n-2k}.$$

This is true for all $n \geq 2$.

Illustration 1.4 :

Figure 1 : P_4

Let

$$A(P_4) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

Then

$$\begin{aligned} P(P_x, x) &= per(xI_4 + A(P_4)) \\ &= per\left(\begin{pmatrix} x & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \right) \\ &= per \begin{pmatrix} x & 1 & 0 & 0 \\ 1 & x & 1 & 0 \\ 0 & 1 & x & 1 \\ 0 & 0 & 1 & x \end{pmatrix} \\ &= xper \begin{pmatrix} x & 1 & 0 \\ 1 & x & 1 \\ 0 & 1 & x \end{pmatrix} + per \begin{pmatrix} 1 & 0 & 0 \\ 1 & x & 1 \\ 0 & 1 & x \end{pmatrix} \\ &= x(x^3 + 0 + 0 + 0 + x + x) + (x^2 + 0 + 0 + 1) \\ &= x(x^3 + 2x) + x^2 + 1 \\ &= x^4 + 3x^2 + 1. \end{aligned}$$

Hence $P(P_4, x) = x^4 + 3x^2 + 1$.

Note 1.5 : The first few permanental polynomial of the path graph P_n is given below:

1. $P(P_2, x) = x^2 + 1$
2. $P(P_3, x) = x^3 + 2x$
3. $P(P_4, x) = x^4 + 3x^2 + 1$
4. $P(P_5, x) = x^5 + 4x^3 + 3x$
5. $P(P_6, x) = x^6 + 5x^4 + 6x^2 + 1$.

Theorem 1.6 : The permanental polynomial of the cycle graph C_n is given by

$$P(G, x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{k} \begin{pmatrix} n-k-1 \\ k-1 \end{pmatrix} x^{n-2k} + 2 \quad \text{for} \quad n \geq 3.$$

Proof : Let $G = C_n$ be the cycle graph with n vertices and n edges.
We know that the permanental polynomial of G is $P(G, x) = per(xI_n + A(G))$ where $A(G)$ is the adjacency matrix of the graph C_n.

$$\begin{aligned} P(C_n, x) &= per(xI_n + A(C_n)) \\ &= per\left(\begin{pmatrix} x & 0 & 0 & 0 & \cdots & 0 \\ 0 & x & 0 & 0 & \cdots & 0 \\ 0 & 0 & x & 0 & \cdots & 0 \\ & & & & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & x \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 \\ & & & & \vdots & \\ 1 & 0 & 0 & 0 & \cdots & 10 \end{pmatrix}\right) \\ &= per\left(\begin{pmatrix} x & 1 & 0 & 0 & \cdots & 1 \\ 1 & x & 1 & 0 & \cdots & 0 \\ 0 & 1 & x & 1 & \cdots & 0 \\ & & & & \vdots & \\ 1 & 0 & 0 & 0 & \cdots & 1x \end{pmatrix}\right) \\ &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{k} \begin{pmatrix} n-k-1 \\ k-1 \end{pmatrix} x^{n-2k} + 2. \end{aligned}$$

That is,

$$P(C_n, x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n}{k} \begin{pmatrix} n-k-1 \\ k-1 \end{pmatrix} x^{n-2k} + 2.$$

This is true for all $n \geq 3$.

Illustration 1.7 :

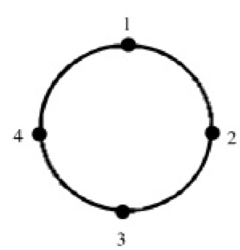


Figure 2 : C_4

$$A(C_4) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

Then

$$\begin{aligned}
P(C_x,x) &= per(xI_4 + A(C_4)) \\
&= per\left(\begin{pmatrix} x & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}\right) \\
&= per\begin{pmatrix} x & 1 & 0 & 1 \\ 1 & x & 1 & 0 \\ 0 & 1 & x & 1 \\ 1 & 0 & 1 & x \end{pmatrix} \\
&= xper\begin{pmatrix} x & 1 & 0 \\ 1 & x & 1 \\ 0 & 1 & x \end{pmatrix} + per\begin{pmatrix} 1 & 0 & 1 \\ 1 & x & 1 \\ 0 & 1 & x \end{pmatrix} + per\begin{pmatrix} 1 & 0 & 1 \\ x & 1 & 0 \\ 1 & x & 1 \end{pmatrix} \\
&= x(x^3 + 0 + 0 + x + x) + (x^2 + 0 + 1 + 1) + (1 + x^2 + 1 + 0) \\
&= x(x^3 + 2x) + (x^2 + 2) + (x^2 + 2) \\
&= x^4 + 4x^2 + 4.
\end{aligned}$$

Hence $P(C_4, x) = x^4 + 4x^2 + 1$.

Note 1.8 : The first few permanental polynomial of the path graph C_n is given below:

1. $P(C_3, x) = x^3 + 3x + 2$
2. $P(C_4, x) = x^4 + 4x^2 + 4$
3. $P(C_5, x) = x^5 + 5x^3 + 5x + 2$
4. $P(C_6, x) = x^6 + 6x^4 + 9x^2 + 4$
5. $P(C_7, x) = x^7 + 7x^5 + 14x^3 + 7x + 2.$

Theorem 1.9 : The permanental polynomial of the star graph is given by

$$P(K_{1,n}, x) = x^{n+1} + nx^{n-1} \quad \text{for} \quad n \geq 1.$$

Proof : Let $G = K_{1,n}$ be the star graph with $n + 1$ vertices and n edges.

We know that the permanental polynomial of G is where is $P(G,x) = per(xI_n + A(G))$ where $A(G)$ the adjacency matrix of the graph $K_{1,n}$.

$$\begin{aligned} P(K_{1,n},x) &= per(xI_n + A(K_{1,n})) \\ &= per\left(\begin{pmatrix} x & 0 & 0 & 0 & \cdots & 0 \\ 0 & x & 0 & 0 & \cdots & 0 \\ 0 & 0 & x & 0 & \cdots & 0 \\ & & & & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & x \end{pmatrix} + \begin{pmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 \\ & & & & \vdots & \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}\right) \\ &= per\left(\begin{pmatrix} x & 1 & 1 & & \cdots & 1 \\ 1 & x & 0 & & \cdots & 0 \\ 1 & 0 & x & & \cdots & 0 \\ & & & & \vdots & \\ 1 & 0 & 0 & 0 & \cdots & x \end{pmatrix}\right) \\ &= x^{n+1} + nx^{n-1}. \end{aligned}$$

That is,

$$P(K_{l,n},x) = x^{n+1} + nx^{n-1}.$$

This is true for all $n \geq 3$.

Note 1.9 : The first few permanental polynomial of the path graph is given below:

1. $P(K_{l,1},x) = x^2 + 1$
2. $P(K_{l,2},x) = x^3 + 2x$
3. $P(K_{l,3},x) = x^4 + 3x^2$
4. $P(K_{l,4},x) = x^5 + 4x^3$
5. $P(K_{l,5},x) = x^6 + 5x^5$
6. $P(K_{l,6},x) = x^7 + 6x^6$.

Definition 1.10 : The Triangular book graph is the complete tripartite graph $K_{1,1,n}$ triangles sharing a common edge. A book of this type is a Split graph. Here it is denoted by T_n. It has $(n+2)$ vertices and $(2n+1)$ edges.

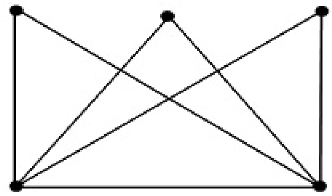


Figure 3 : T_3

Theorem 1.11 : The permanental polynomial of the triangular book graph T_n is given by

$$P(T_n, x) = x^{n+2} + (2n+1)x^n + 2n(n-1)x^{n-2} + 2nx^{n-1} \quad \text{for} \quad n \geq 1.$$

Proof : Let $G = T_n$ be the triangular book graph with $(n+2)$ vertices and $(2n+1)$ edges.

We know that the permanental polynomial of G is $P(G, x) = per(xI_n + A(G))$ where $A(G)$ is the adjacency matrix of the graph T_n.

$$\begin{aligned} P(T_n, x) &= per(xI_n + A(T_n)) \\ &= per\left(\begin{pmatrix} x & 0 & 0 & 0 & \cdots & 0 \\ 0 & x & 0 & 0 & \cdots & 0 \\ 0 & 0 & x & 0 & \cdots & 0 \\ & & & & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & x \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 \\ & & & & \vdots & & \\ 1 & 1 & 1 & 1 & \cdots & 1 & 0 \end{pmatrix}\right) \\ &= per\left(\begin{pmatrix} x & 0 & 0 & & \cdots & 1 & 1 \\ 0 & x & 0 & & \cdots & 1 & 1 \\ 0 & 0 & x & & \cdots & 1 & 1 \\ & & & & \vdots & & \\ 1 & 1 & 1 & 1 & \cdots & 1 & x \end{pmatrix}\right) \\ &= x^{n+2} + (2n+1)x^n + 2n(n-1)x^{n-2} + 2nx^{n-1}. \end{aligned}$$

That is,

$$P(T_n, x) = x^{n+2} + (2n+1)x^n + 2n(n-1)x^{n-2} + 2nx^{n-1}.$$

This is true for all $n \geq 3$.

Note 1.12 : The first few permanental polynomials of the triangular book graph T_n is given below :

1. $P(T_1, x) = x^2 + 3x + 2$

2. $P(T_2, x) = x^4 + 5x^2 + 4x + 4$
3. $P(T_3, x) = x^5 + 7x^3 + 6x^2 + 12x$.
4. $P(T_4, x) = x^6 + 9x^4 + 8x^3 + 24x^2$
5. $P(T_5, x) = x^7 + 11x^5 + 10x^4 + 40x^3$.

Definition 1.13 : The Bistar $B(n, n)$ is obtained by taking two stars on disjoint two vertex sets and then by making their centres u and v adjacent to each other by introducing a new edge uv. Clearly, $\{u, v\}$ is the centre of $B(n, n)$.

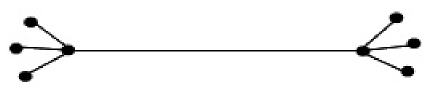


Figure 4 : $B(3, 3)$

Theorem 1.14 : The permanental polynomial of the bistar graph $B(n, n)$is given by

$$P(B(n,n),x) = x^{2n+2} + (2n+1)x^{2n} + n^2x^{2n-2} \quad \text{for‘} \quad n \geq 1.$$

Proof : Let $G = B(n, n)$ be the bistar graph with $(2n+2)$ vertices and $(2n+1)$ edges. We know that the permanental polynomial of G is $P(G, x) = per(xI_n + A(G))$ where $A(G)$ is the adjacency matrix of the graph $B(n, n)$.

$$P(B(n,n),x) = per(xI_n + A(B(n,n))$$

$$= per\left(\begin{pmatrix} x & 0 & 0 & 0 & \cdots & 0 \\ 0 & x & 0 & 0 & \cdots & 0 \\ 0 & 0 & x & 0 & \cdots & 0 \\ & & & & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & x \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & & 1 & 0 & \cdots & 0 \\ 1 & 1 & 1 & 1 & \cdots & & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 1 & \cdots & 1 \\ & & & & \vdots & & & & & \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{pmatrix}\right)$$

$$= per\left(\begin{pmatrix} x & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & x & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & x & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & 1 & \cdots & x & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & x & 1 & \cdots & 1 & 1 \\ & & & & & \vdots & & & & & \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & x \end{pmatrix}\right)$$

$$= x^{2n+2} + (2n+1)x^{2n} + n^2x^{2n-2}.$$

That is,

$$P(B(n,n),x) = x^{2n+2} + (2n+1)x^{2n} + n^2x^{2n-2}.$$

This is true for all $n \geq 3$.

Note 1.15 : The first few permanental polynomials of the bistar graph $B(n,n)$ is given below :

1. $P(B(1,1),x) = x^2 + 3x^2 + 1$
2. $P(B(2,2),x) = x^6 + 5x^4 + 4x^2$
3. $P(B(3,3),x) = x^8 + 7x^6 + 9x^4$
4. $P(B(4,4),x) = x^{10} + 9x^8 + 16x^6$
5. $P(B(5,5),x) = x^{12} + 11x^{10} + 25x^8$.

Theorem 1.16 : The permanental polynomial of the connected graph $C_3\hat{0}K_{1,n}$ is given by

$$P(C_3\hat{0}K_{1,n},x) = x^{n+3} + (n+3)x^{n+1} + nx^{n-1} + 2x^n \quad \text{for} \quad n \geq 1.$$

Proof : Let $G = C_3\hat{0}K_{1,n}$ be the connected graph.

Let the vertices of C_3 be u_0, u_1, u_2. Let the n spokes of $K_{1,n}$ be $v_1, v_2, \cdots, v_n$.

Let v_0 be the centre vertex of the star $K_{1,n}$. Identify u_0 and v_0.

Let the graph so obtained be G. Clearly G has $(n+3)$ vertices and $(n+3)$ edges.

We know that the permanental polynomial of G is $P(G,x) = per(xI_n + A(G))$ where

$A(G)$ is the adjacency matrix of the graph $C_3\hat{0}K_{1,n}$.

$$\begin{aligned} P(C_3\hat{0}K_{1,n},x) &= per(xI_n + A(C_3\hat{0}K_{1,n})) \\ &= per\left(\begin{pmatrix} x & 0 & 0 & 0 & \cdots & 0 \\ 0 & x & 0 & 0 & \cdots & 0 \\ 0 & 0 & x & 0 & \cdots & 0 \\ & & & & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & x \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ & & & & \vdots & & & & \\ 1 & 1 & 1 & 1 & \cdots & 1 & x & 1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & x & 1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & x & 1 \end{pmatrix}\right) \\ &= per\begin{pmatrix} x & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & x & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & x & 0 & \cdots & 1 & 0 & 0 \\ & & & & \vdots & & & \\ 1 & 1 & 1 & 1 & \cdots & x & 1 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & x & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 1 & x \end{pmatrix} \\ &= x^{n+3} + (n+3)x^{n+1} + nx^{n-1}2x^n. \end{aligned}$$

That is,

$$P(C_3, \hat{0}K_{1,n}, x) = x^{n+3} + (n+3)x^{n+1} + nx^{n-1} + 2x^n..$$

This is true for all $n \geq 3$.

Illustration 1.17 :

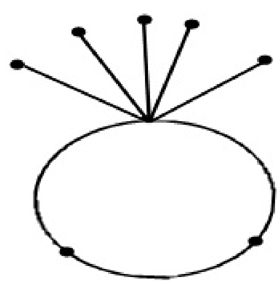


Figure 5 : $C_3\hat{0}K_{1,5}$

Note 1.18 : The first few permanental polynomials of the connected graph $C_3\hat{0}K_{1,n}$ is given below :

1. $P(C_3\hat{0}K_{1,1}, x) = x^4 + 4x^2 + 1 + 2x$

2. $P(C_3\hat{0}K_{1,2}, x) = x^5 + 5x^3 + 2x + 2x^2$
3. $P(C_3\hat{0}K_{1,3}, x) = x^6 + 6x^4 + 3x^2 + 2x^3$
4. $P(C_3\hat{0}K_{1,4}, x) = x^7 + 7x^5 + 4x^3 + 2x^4$.

Definition 1.19 : $K_{1,m} \odot K_{1,n}$ is a tree obtained by adding n pendant edges to each pendant vertices of $K_{1,m}$. It has $1 + m + nm$ vertices and edges.

Theorem 1.20 : The permanental polynomial of the graph $G_n = K_{1,m} \odot K_{1,n}, m = 2$ is given by

$$P(G_n, x) = x^{2n+3} + (2n + 2)x^{2n+1} + n(n + 2)x^{2n-1} \quad \text{for } n \geq 1.$$

Proof : Let $G_n = K_{1,m} \odot K_{1,n}, m = 2$.
$K_{1,2} \odot K_{1,n}$ is a tree obtained by adding n pendant edges to each pendant vertices of $K_{1,2}$.
G_n has $nm + 3$ vertices and $nm + 2$ edges.
We know that the permanental polynomial of G is $P(G, x) = per(xI_n + A(G))$ where $A(G)$ is the adjacency matrix of the graph G_n.

$$\begin{aligned} P(G_n, x) &= per(xI_n + A(G_n)) \\ &= x^{2n+3} + (2n + 2)x^{2n+1} + n(n + 2)x^{2n-1}. \end{aligned}$$

That is,

$$P(G_n, x) = x^{2n+3} + (2n + 2)x^{2n+1} + n(n + 2)x^{2n-1}.$$

This is true for all $n \geq 1$.

Illustration 1.21 :

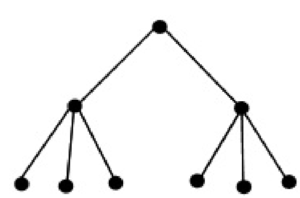


Figure 6 : $K_{1,2} \odot K_{1,2}$

Note 1.22 : The first few permanental polynomial of the graph $G_n = K_{1,2} \odot K_{1,n}$ is given below :

1. $P(G_1, x) = x^2 + 4x^3 + 3x$
2. $P(G_2, x) = x^7 + 6x^5 + 8x^3$
3. $P(G_3, x) = x^9 + 8x^7 + 15x^5$
4. $P(G_4, x) = x^{11} + 10x^9 + 24x^7$
5. $P(G_5, x) = x^{13} + 12x^{11} + 35x^9$.

Definition 1.23 : Consider two stars $K_{1,n}^{(1)}$ and $K_{1,n}^{(2)}$ then $S_n = \langle K_{1,n}^{(1)} \Delta K_{1,n}^{(2)} \rangle$ is the graph obtained by joining apex vertices of stars by an edge as well as to a new vertex x. It has $2n+3$ vertices and edges.

Theorem 1.24 : The permanental polynomial of the graph $S_n = \langle K_{1,2}^{(1)} \Delta K_{1,n}^{(2)} \rangle$ is given by

$$P(S_n, x) = x^{2n+3} + (2n+3)x^{2n+1} + (n^2+2n)x^{2n-1} + 2x^{2n} \quad \text{for} \quad n \geq 1.$$

Proof : Let $S_n = \langle K_{1,2}^{(1)} \Delta K_{1,n}^{(2)} \rangle$.
Consider two stars $K_{1,n}^{(1)}$ and $K_{1,n}^{(2)}$.
Then $S_n = \langle K_{1,2}^{(1)} \Delta K_{1,n}^{(2)} \rangle$ is the graph obtained by joining apex vertices of stars by an edge as well as to a new vertex x.
It has $2n+3$ vertices and $2n+3$ edges.
We know that the permanental polynomial of G is $P(G, x) = per(xI_n + A(G))$ where $A(G)$ is the adjacency matrix of the graph S_n.

$$\begin{aligned} P(S_n, x) &= per(xI_n + A(S_n)) \\ &= x^{2n+3} + (2n+3)x^{2n+1} + (n^2+2n)x^{2n-1} + 2x^{2n} \end{aligned}$$

That is,

$$P(S_n, x) = x^{2n+3} + (2n+3)x^{2n+1} + (n^2+2n)x^{2n-1} + 2x^{2n}.$$

This is true for all $n \geq 1$.

Illustration 1.25 :

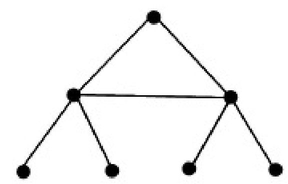


Figure 7 : $\langle K_{1,2}^{(1)} \Delta K_{1,2}^{(2)}$

Note 1.26 : The first few permanental polynomial of the graph $S_n = K_{1,2}^{(1)} \Delta K_{1,2}^{(2)} \rangle$ is given below :

1. $P(S_1, x) = x^5 + 5x^3 + 3x + 2x^2$
2. $P(S_2, x) = x^7 + 7x^5 + 8x^3 + 2x^4$
3. $P(S_3, x) = x^9 + 9x^7 + 15x^5 + 2x^6$
4. $P(S_4, x) = x^{11} + 11x^9 + 24x^7 + 2x^8$
5. $P(S_5, x) = x^{13} + 13x^{11} + 35x^9 + 2x^{10}$.

Construction of New root graph $R_n$and their permanental polynomials

Let $G = C_4$.

R_1 is formed from C_4 by joining the new vertex of K_1 to one of the vertices of degree $\lfloor \frac{4}{2} \rfloor = 2$.

R_2 is formed from R_1 by joining the new vertex of K_1 to one of the vertices of degree $\lfloor \frac{5}{2} \rfloor = 2$.

R_3 is formed from R_2 by joining the new vertex of K_1 to one of the vertices of degree $\lfloor \frac{6}{2} \rfloor = 3$.

R_4 is formed from R_3 by joining the new vertex of K_1 to one of the vertices of degree $\lfloor \frac{7}{2} \rfloor = 3$.

In general R_n is formed from R_{n-1} by joining the new vertex of K_1 to one of the vertices of degree $\lfloor \frac{n+3}{2} \rfloor$.

Theorem 1.27 : The permanental polynomial of the new root graph R_n is given by

$$P(R_n, x) = x^{n+4} + (n + 4_x^{n+2} + 3(n+1)x^n \quad \text{for} \quad n \geq 1.$$

Proof : Let R_n be the new root graph with $(n + 4)$ vertices and edges.

We know that the permanental polynomial of G is $P(G,x) = per(xI_n + A(G))$ where $A(G)$ is the adjacency matrix of the graph R_n.

$$\begin{aligned}
P(R_n,x) &= per(xI_n + A(R_n)) \\
&= per\begin{pmatrix} x & 1 & 0 & 0 & \cdots & 1 & 1 & 1 \\ 1 & x & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & x & 1 & \cdots & 1 & 0 & 0 \\ & & & & \vdots & & & \\ 1 & 0 & 1 & 0 & \cdots & x & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & x & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & x \end{pmatrix} \\
&= x^{n+4} + (n+4)x^{n+2} + 3(n+1)x^n.
\end{aligned}$$

That is,

$$P(R_n,x) = x^{n+4} + (n+4)x^{n+2} + 3(n+1)x^n.$$

This is true for all $n \geq 1$.

Note 1.28 : The first few permanental polynomial of the new root graph R_n is given below :

1. $P(R_1,x) = x^5 + 5x^3 + 6x$
2. $P(R_2,x) = x^6 + 6x^4 + 9x^2$
3. $P(R_3,x) = x^7 + 7x^5 + 12x^3$
4. $P(R_4,x) = x^8 + 8x^6 + 15x^4$
5. $P(R_5,x) = x^9 + 9x^7 + 18x^5$.

Construction of thorn graph TH_n and their permanental polynomials

Let $G = P_4$.

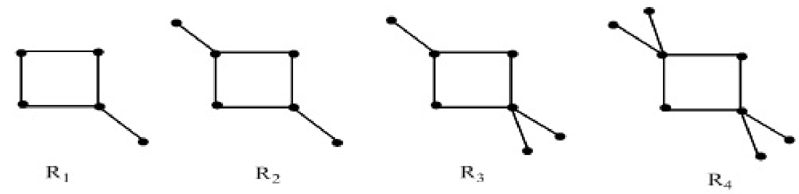
TH_1 is formed from P_4 by joining the new vertex of K_1 to one of the vertices of degree $\lfloor \frac{4}{2} \rfloor = 2$.

TH_2 is formed from TH_1 by joining the new vertex of K_1 to one of the vertices of degree $\lfloor \frac{5}{2} \rfloor = 2$.

TH_3 is formed from TH_2 by joining the new vertex of K_1 to one of the vertices of degree $\lfloor \frac{6}{2} \rfloor = 3$.

TH_4 is formed from TH_3 by joining the new vertex of K_1 to one of the vertices of degree $\lfloor \frac{7}{2} \rfloor = 3$.

In general TH_n is formed from TH_{n-1} by joining the new vertex of K_1 to one of the vertices of degree $\lfloor \frac{n+3}{2} \rfloor$.



Theorem 1.29 : The permanental polynomial of the thorn graph TH_1 is given by

$$P(TH_n, x) = x^{n+4} + (n+3)x^{n+2} + 2nx^n \quad \text{for} \quad n \geq 1.$$

Proof : Let TH_n be the thorn graph with $(n+4)$ vertices and $(n+3)$ edges.

We know that the permanental polynomial of G is $P(G, x) = per(xI_n + A(G))$ where $A(G)$ is the adjacency matrix of the graph TH_n.

$$\begin{aligned} P(TH_n, x) &= per(xI_n + A(TH_n)) \\ &= per \begin{pmatrix} x & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & x & 1 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 1 & x & 1 & \cdots & 1 & 1 & 0 \\ & & & & \vdots & & & \\ 0 & 0 & 1 & 0 & \cdots & x & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & x & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & x \end{pmatrix} \\ &= x^{n+4} + (n+3)x^{n+2} + 2nx^n. \end{aligned}$$

That is,

$$P(TH_n, x) = x^{n+4} + (n+3)x^{n+2} + 2nx^n.$$

This is true for all $n \geq 1$.

Note 1.30 : The first few permanental polynomial of the thorn graph TH_n is given below :

1. $P(TH_1, x) = x^5 + 5x^3 + 2x$
2. $P(TH_2, x) = x^6 + 5x^4 + 4x^2$

3. $P(TH_3, x) = x^7 + 6x^5 + 6x^3$

4. $P(TH_4, x) = x^8 + 7x^6 + 8x^4$

5. $P(TH_5, x) = x^9 + 8x^7 + 10x^5$.

Construction of octupus graph O_n and their permanental polynomials

Let $G = C_4$.

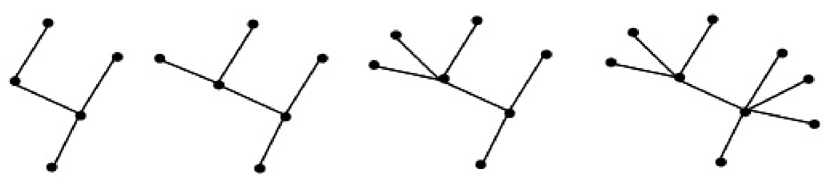
O_1 is formed from C_4 by joining the new vertex of K_1 to one of the vertices of degree $\lfloor \frac{4}{2} \rfloor = 2$.

O_2 is formed from O_1 by joining the new vertex of K_1 to one of the vertices of degree $\lfloor \frac{6}{2} \rfloor = 3$.

O_3 is formed from O_2 by joining the new vertex of K_1 to one of the vertices of degree $\lfloor \frac{8}{2} \rfloor = 4$.

O_4 is formed from O_3 by joining the new vertex of K_1 to one of the vertices of degree $\lfloor \frac{10}{2} \rfloor = 5$.

In general O_n is formed from $)_{n-1}$ by joining the new vertex of K_1 to one of the vertices of degree $\lfloor \frac{2n+2}{2} \rfloor$.



TH_1 TH_2 TH_3 TH_4

Theorem 1.31 : The permanental polynomial of the Octupus graph O_n is given by

$$P(O_n, x) = x^{n+4} + (n+4)x^{n+2} + 2(n+2)x^n \quad \text{for} \quad n \geq 1.$$

Proof : Let O_n be the thorn graph with $(n + 4)$ vertices and edges.

We know that the permanental polynomial of G is $P(G, x) = per(xI_n + A(G))$ where

$A(G)$ is the adjacency matrix of the graph O_n.

$$\begin{aligned} P(O_n, x) &= per(xI_n + A(O_n)) \\ &= per\begin{pmatrix} x & 1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \\ 1 & x & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & \\ 1 & 0 & x & 1 & \cdots & 0 & 0 & 0 & 0 & \\ & & & & \vdots & & & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & x & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & x & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & x \end{pmatrix} \\ &= x^{n+4} + (n+4)x^{n+2} + 2(n+2)x^n. \end{aligned}$$

That is,

$$P(O_n, x) = x^{n+4} + (n+4)x^{n+2} + 2(n+2)x^n.$$

This is true for all $n \geq 1$.

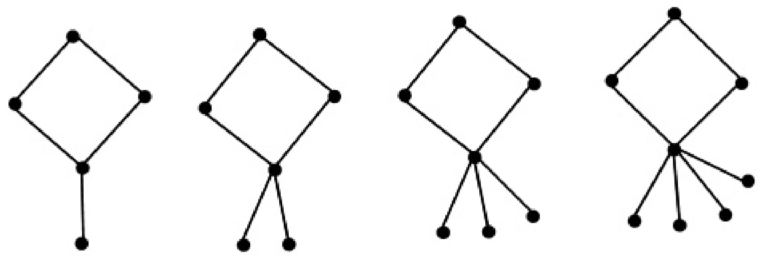
Note 1.32 : The first few permanental polynomial of the thorn graph O_n is given below :

1. $P(O_1, x) = x^5 + 5x^3 + 6x$
2. $P(O_2, x) = x^6 + 6x^4 + 8x^2$
3. $P(O_3, x) = x^7 + 7x^5 + 10x^3$
4. $P(O_4, x) = x^8 + 8x^6 + 12x^4$
5. $P(O_5, x) = x^9 + 9x^7 + 14x^5$.

Construction of octupus graph CL_n and their permanental polynomials

Let $G = F_2$ where F_2 is the two copies of C_3 attached each other with a common vertex.
CL_1 is formed from F_2 by joining the new vertex of K_1 to one of the vertices of degree $\lfloor \frac{4}{2} \rfloor = 2$.
CL_2 is formed from CL_1 by joining the new vertex of K_1 to one of the vertices of degree $\lfloor \frac{5}{2} \rfloor = 2$.
CL_3 is formed from CL_2 by joining the new vertex of K_1 to one of the vertices of degree $\lfloor \frac{6}{2} \rfloor = 3$.
CL_4 is formed from CL_3 by joining the new vertex of K_1 to one of the vertices of degree $\lfloor \frac{7}{2} \rfloor = 3$.

In general CL_n is formed from CL_{n-1} by joining the new vertex of K_1 to one of the vertices of degree $\lfloor \frac{n+3}{2} \rfloor$.



O_1 O_2 O_3 O_4

Theorem 1.33 : The permanental polynomial of the Collar graph is given by

$$P(CL_n, x) = x^{n+5} + (n+6)x^{n+3} + 2(n+2)x^{n+1} + 4x^{n+2} + 4x^n + nx^n + nx^{n-1}$$

for $n \geq 1$.

Proof : Let CL_n be the thorn graph with $(n+5)$ vertices and (n+6) edges.
We know that the permanental polynomial of G is $P(G,x) = per(xI_n + A(G))$ where $A(G)$ is the adjacency matrix of the graph CL_n.

$$\begin{aligned} P(CL_n, x) &= per(xI_n + A(CL_n)) \\ &= per \begin{pmatrix} x & 1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \\ 1 & x & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \\ 1 & 1 & x & 1 & \cdots & 1 & 1 & 1 & 1 & \\ & & & & \vdots & & & & & \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & x & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & x & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & x \end{pmatrix} \\ &= x^{n+5} + (n+6)x^{n+3} + 2(n+2)x^{n+1} + 4x^{n+2} + 4x^n + nx^{n-1}. \end{aligned}$$

That is,

$$P(CL_n, x) = x^{n+5} + (n+6)x^{n+3} + 2(n+2)x^{n+1} + 4x^{n+2} + 4x^n + nx^{n-1}.$$

This is true for all $n \geq 1$.

Note 1.34 : The first few permanental polynomial of the thorn graph CL_n is given below :

1. $P(CL_1, x) = x^6 + 7x^4 + 7x^2 + 4x^3 + 4x + 1$

2. $P(CL_2, x) = x^7 + 8x^5 + 9x^3 + 4x^4 + 4x^2 + 2x$

3. $P(CL_3, x) = x^8 + 9x^6 + 11x^4 + 4x^5 + 4x^3 + 3x^2$

4. $P(CL_4, x) = x^9 + 10x^7 + 13x^5 + 4x^6 + 4x^4 + 4x^3$

5. $P(CL_5, x) = x^{10} + 11x^8 + 15x^6 + 4x^7 + 4x^5 + 5x^4$.

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