

A CRANK NICHOLSON METHOD WITH FIVE POINTS TO SOLVE FISHER'S EQUATION

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Abstract

In Fisher's equation, the mechanism of logistic growth and linear diffusion are combined in order to model the spreading and proliferation of population, e.g., in ecological contexts. We present an approximate solution of nonlinear Fisher's reaction diffusion equation using high order finite difference method. The proposed method has the advantage of reducing the problem to a nonlinear system, which will be derived and solved using Newton method. FTCS and Crank Nicholson methods will be introduced, compared and tested.

1. Introduction

In 1937, Fisher proposed a nonlinear reaction diffusion equation to describe the propagation of a viral mutant in an infinitely long habitat, see [6]. The Fisher equation

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arises in heat and mass transfer, biology, and ecology. It's introduced to describe the spreading of genes [2] has found applications in different fields of research ranging from ecology [3] to plasma physics [4].

Lie symmetry method is applied to analyze Fisher's equation in cylindrical coordinates. Symmetry algebra is found and symmetry invariance is used to reduce the equation to a first order ODE. The first order ODE is further analyzed to obtain exact solution of Fisher's equation in explicit form. In this paper we will not investigate the theoretical methods how to solve the Fisher's equation. The problem is reduced to:

$$\text{PDE : } \frac{\partial}{\partial t} u(t, x) = \beta(t, x) \frac{\partial^2}{\partial x^2} u(t, x) + \mathcal{F}(u(t, x)) \quad (1)$$

$$\text{Boundary condition : } u(t, L) = u(t, -L) = 0 \quad (2)$$

$$\text{Initial condition : } u(0, x) = f(x) \quad (3)$$

where \mathcal{F} is a given (presumably nonlinear) function of $u(t, x)$, $f(x)$ is also a given function, L is a constant and $\beta(t, x)$ is the diffusion function. We take t as a time variable and $x \in [-L, L]$.

In the case where \mathcal{F} is linear the PDE can be solved by the separation of variables methods. Here, β is the diffusion constant. This is a parabolic, nonlinear and nonhomogeneous PDE in two dimensions. In this equation, t plays the role of a time variable and x is a spatial variable; that is, we are going to solve the PDE as an initial value problem. In order to obtain a numeric solution, we will need to supplement the PDE with boundary and initial conditions. However if, as in many of the applications considered in [5], \mathcal{F} is nonlinear function then the problem is much more burdensome. It is not usually possible to obtain general analytical traveling wave solutions and one must analyze such problems numerically. Fisher's equation, which describes a balance between linear diffusion and nonlinear reaction or multiplication, is studied numerically by the Sinc collocation method, [1].

Spectral methods (see, for instance, [8, 9, 10, 11, 12]) are powerful techniques that we use to numerically solve linear and nonlinear partial differential equations either in their strong or weak forms. Legendre Spectral Collocation Method is used to solve the Fisher's equation, see [13]. In [14], the author solve the Fisher's equation using Petrov-Galerkin finite element method. In [15], some numerical study of Fisher's equation by

Adomian's method was proposed. In [16], the author study a nonlinear reaction-diffusion equation for its traveling waves. These methods look very difficult. For this reason, we will focus on a nonlinear PDEs using finite difference methods and we restrict our work in the case where the function $\beta(t, x) = \beta$ constant. This method used in this paper can be applied for any nonlinear function, but here we limit our presentation to the case of Fisher's equation which corresponding to the choice of $\mathcal{F}(u) = u(1 - u)$. It was suggested by Fisher as a deterministic version of a stochastic model for the spatial spread of a favored gene in a population. The rest of this paper is organized as follows: In section 2, we present the method of finite difference and Newton technique to solve a system of nonlinear equations. In section 3, we present the two method FTCS and CN. In 4, we solve the Fisher's equation using FTSC and we look to it's stability and some remarks about maximum principle will be presented. Section 5, give a comparison between the FTCS and CN methods and their stability. Finally, we conclude.

2. Preliminary

2.1 Finite Difference Method

It is often helpful to know the "order of accuracy" of a computational method. For one thing, computational methods with a higher order of accuracy tend to give more accurate results. Also, verifying that a code produces answers with the theoretical order of accuracy is a use check of the correctness of the code. For example, if boundary conditions are accidentally applied at grid point one instead of grid point zero, the order of accuracy of the forward Euler method will go from 2 to 1. This buggy code would still converge to the right answer as $\Delta x \rightarrow 0$, but fixing the bug would lead to large improvements in accuracy for a given Δx .

The Taylor series expansion for $g(s + h)$ about $h = 0$, is

$$g(s + h) \approx g(s) + hg'(s) + \frac{h^2}{2}g''(s) + \frac{h^3}{6}g'''(s) + \dots .$$

Substituting this into the difference approximation gives

$$g'(s) - \frac{g(s + h) - g(s)}{h} \approx \frac{h}{2}g''(s) + \frac{h^2}{6}g'''(s) + \dots . \quad (4)$$

When h is small, the right hand side is almost equal to the first term. In particular, it is roughly proportional to h . If we carry the Taylor series to fifth order, we get a similar

analysis of the centered approximation to the first derivative:

$$\frac{g(s+h) - g(s-h)}{2h} - g'(s) \approx \frac{h^2}{3}g'''(s) + \frac{h^4}{60}g^{(5)}(s) + \dots \quad (5)$$

For the approximation to the second derivative we find

$$\frac{g(s+h) - 2g(s) + g(s-h)}{h^2} - g''(s) \approx \frac{h^2}{12}g^{(4)}(s) + \frac{h^4}{360}g^{(6)}(s) + \dots \quad (6)$$

These Taylor series calculations can be tedious, but they are not very tedious and we don't have to do them very often.

The expressions (4), (5) and (6) are examples of “asymptotic error expansions”. In the general theory of Asymptotic error expansions, we suppose that there is a quantity, A , that we wish to calculate (the “answer”). Given a step, h , we compute an approximation, $R(h)$. An asymptotic error expansion is an expression

$$R(h) - R \approx R_1h^{p_1} + R_2h^{p_2} + \dots \quad (7)$$

The first term on the right will dominate if h is small and the powers of h are increasing: $p_1 < p_2 < p_3 < \dots$. The leading power, p_1 determines the order of accuracy of the approximation. In this notation, (4) says that the one sided approximation to the first derivative is first order accurate, the powers being $p_1 = 1$, $p_2 = 2$, and so on. The centered approximation to the first derivative is second order accurate, as is the three point approximation to the second derivative. In both those cases, $p_1 = 2$, $p_2 = 4$, and so on. This discussion of order of accuracy focuses on what is common to a large class of approximations rather than details that could only be known on a case by case basis, or, in practice, not at all. In this case, the powers, p_1 , p_2 , and so on, depend only on the approximation formula, while the “error constants”, A_1, A_2, \dots , depend on the function we are working with. In solving partial differential equations (PDE's from now on), the difference approximations are used to compute the function in the first place, making it unlikely that we would know the error constant in advance. We can determine the order of accuracy in advance.

It is possible that an approximation has a different order of accuracy than we expect on the basis of these considerations. The actual order could be higher if, by some fluke, the first error constant were zero. In this paper we will derive an approximation of the

second derivative using centered spaced method with five point, this give us a more accuracy and leading to small error.

2.2 Newton's Method

In numerical analysis, Newton's method (also known as the Newton Raphson method), named after Isaac Newton and Joseph Raphson, is perhaps the best known method for finding successively better approximations to the zeroes of a real valued function. Newton's method can often converge remarkably quickly, especially if the iteration begins "sufficiently near" the desired root. We consider a system of nonlinear equations we are looking for the solution of the system. For simplicity, we can write a system of nonlinear equations as:

$$\mathbf{G}(\mathbf{x}) = \mathbf{0} \iff \begin{array}{rcl} G_1(x_1, x_2, \dots, x_n) & = & 0 \\ G_2(x_1, x_2, \dots, x_n) & = & 0 \\ & \vdots & \vdots \\ G_n(x_1, x_2, \dots, x_n) & = & 0 \end{array}$$

For a system of nonlinear equations, Newton's method is written in matrix form as:

$$[\mathbf{J}] (\Delta \mathbf{x}) = -\mathbf{G}(\mathbf{x})$$

or

$$\begin{bmatrix} J_{11} & J_{12} & \cdots & J_{1n} \\ J_{21} & J_{22} & \cdots & J_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ J_{n1} & J_{n2} & \cdots & J_{nn} \end{bmatrix} \begin{pmatrix} \Delta x_1 \\ \Delta x_2 \\ \vdots \\ \Delta x_n \end{pmatrix} = - \begin{pmatrix} G_1(x_1, x_2, \dots, x_n) \\ G_2(x_1, x_2, \dots, x_n) \\ \vdots \\ G_n(x_1, x_2, \dots, x_n) \end{pmatrix}$$

Here, $\Delta \mathbf{x}$ represents the change in all of the variables at this iteration. The $[\mathbf{J}]$ matrix is called the "Jacobian" matrix and is defined as:

$$J_{ij} = \frac{\partial G_i}{\partial x_j}$$

Given a guess for the solution, the algorithm for Newton's method is

- Calculate the elements of the Jacobian matrix, $J_{ij} = \frac{\partial G_i}{\partial x_j}$.
- Calculate the function values at the current value for \mathbf{x} , $\mathbf{G}(\mathbf{x})$
- Solve the system of equations $[\mathbf{J}] (\Delta \mathbf{x}) = -\mathbf{G}(\mathbf{x})$ for $\Delta \mathbf{x}$
- Update the solution by adding $\Delta \mathbf{x}$ to the current value of \mathbf{x} .

- Check for convergence. If not converged then go to step 1.

3. Approximation

The key step in solving our Fisher's equation numerically using finite difference methods is to replace the derivatives with so-called "finite difference method".

$$\begin{aligned} \text{Forward time Method : } \frac{\partial}{\partial t} u(t, x) &\simeq -\frac{u(t, x)}{s} + \frac{u(t+s, x)}{s} \\ \text{Backward time Method : } \frac{\partial}{\partial t} u(t, x) &\simeq -\frac{u(t-s, x)}{s} + \frac{u(t, x)}{s} \\ \text{Centered space : } \frac{\partial^2}{\partial x^2} u(t, x) &\simeq -\frac{1}{12} \frac{u(t, x-2h)}{h^2} + \frac{4}{3} \frac{u(t, x-h)}{h^2} - \frac{5}{2} \frac{u(t, x)}{h^2} \\ &+ \frac{4}{3} \frac{u(t, x+h)}{h^2} - \frac{1}{12} \frac{u(t, x+2h)}{h^2} \end{aligned}$$

Note that the last term on the RHS of previous approximations is just the leading order term in the error; terms with higher powers of h have been omitted. The power of h in the leading error term is called the order of the method. One can give better approximation using more points in previous approximations.

The FTCS (Forward time Centered space) is obtained by putting forward time and centered space approximation into the PDE, we get:

$$\begin{aligned} -\frac{u(t, x)}{s} + \frac{u(t+s, x)}{s} &= \beta \left(-\frac{1}{12} \frac{u(t, x-2h)}{h^2} + \frac{4}{3} \frac{u(t, x-h)}{h^2} - \frac{5}{2} \frac{u(t, x)}{h^2} \right) \\ &+ \beta \left(\frac{4}{3} \frac{u(t, x+h)}{h^2} - \frac{1}{12} \frac{u(t, x+2h)}{h^2} \right) \\ &+ \mathcal{F}(u(t, x)) \end{aligned}$$

The BTCS method is obtained by putting backward time and centered space method into the PDE:

$$\begin{aligned} -\frac{u(t, x)}{s} + \frac{u(t+s, x)}{s} &= \beta \left(-\frac{1}{12} \frac{u(t+s, x-2h)}{h^2} + \frac{4}{3} \frac{u(t+s, x-h)}{h^2} - \frac{5}{2} \frac{u(t+s, x)}{h^2} \right) \\ &+ \beta \left(\frac{4}{3} \frac{u(t+s, x+h)}{h^2} - \frac{1}{12} \frac{u(t+s, x+2h)}{h^2} \right) \\ &+ \mathcal{F}(u(t+s, x)) \end{aligned}$$

The Crank-Nicholson method is the average of the FTCS and BTCS:

$$\begin{aligned} -\frac{u(t, x)}{s} + \frac{u(t + s, x)}{s} &= \frac{1}{2}\beta \left(-\frac{1}{12} \frac{u(t, x - 2h)}{h^2} + \frac{4}{3} \frac{u(t, x - h)}{h^2} - \frac{5}{2} \frac{u(t, x)}{h^2} \right) \\ &+ \frac{1}{2}\beta \left(\frac{4}{3} \frac{u(t, x + h)}{h^2} - \frac{1}{12} \frac{u(t, x + 2h)}{h^2} \right) \\ &+ \frac{1}{2}\beta \left(-\frac{1}{12} \frac{u(t + s, x - 2h)}{h^2} + \frac{4}{3} \frac{u(t + s, x - h)}{h^2} - \frac{5}{2} \frac{u(t + s, x)}{h^2} \right) \\ &+ \frac{1}{2}\beta \left(\frac{4}{3} \frac{u(t + s, x + h)}{h^2} - \frac{1}{12} \frac{u(t + s, x + 2h)}{h^2} \right) \\ &+ \frac{1}{2} (\mathcal{F}(u(t, x)) + \mathcal{F}(u(t + s, x))) \end{aligned}$$

The method used is "centered" because u is evaluated at an equal number of points to the right and left of the point where we want to approximate the derivative. The Crank-Nicholson method relates five future values $u(t + s, x - jh)$, $j = -2, -1, 0, 1, 2$ to five past values $u(t, x - jh)$, $j = -2, -1, 0, 1, 2$. This means that, as for the BTCS method is an undetermined equation for the future field values if the past values are known. Here we ask the following question: why bother with method such as BTCS or Crank-Nicholson (which involve the solution of a potentially large system of equations at each time step) over the simpler FTCS method? The answer is the stability of the various schemes. Let's compare the output from the FTCS and Crank-Nicholson schemes in the next section.

4. Forward Time Centered-space Solution

The time derivative will be replaced with a "forward time" right-oriented method with 2 point. The "forward time" label comes from the fact that we are approximating the time derivative at $u(t, x)$ using a point to the future. Notice that we have denoted the step-size in the time direction by s . The spatial derivative will be approximated with a centered five point with step-size h .

The FTCS can be solved for $u(t + s, x)$ for any form of the potential $\mathcal{F}(u(t, x))$:

$$\begin{aligned} u(t + s, x) &= u(t, x) + \frac{s\beta}{h^2} \left(-\frac{1}{12} \frac{u(t, x - 2h)}{h^2} + \frac{4}{3} \frac{u(t, x - h)}{h^2} - \frac{5}{2} \frac{u(t, x)}{h^2} \right) \\ &+ \frac{s\beta}{h^2} \left(\frac{4}{3} \frac{u(t, x + h)}{h^2} - \frac{1}{12} \frac{u(t, x + 2h)}{h^2} \right) + s\mathcal{F}(u(t, x)) \end{aligned}$$

We can always calculation the solution $u(t, x)$ at a future time step with knowledge of

the field at a previous time step, irregardless of the nature of the nonlinearity in the potential \mathcal{F} . This means that the numerical FTCS algorithm to solve our problem is virtually the same as it is for the linear equation. We introduce, some parameter will be used in our method:

- $\tau = 1, N = 400, M = 20, L = 1, \beta = 1$
- $X := j \mapsto L \left(-1 + 2 \frac{j}{M+1} \right)$
- $T := i \mapsto \frac{\tau i}{N}$
- Time step $s := T(1) - T(0) = 0.0952380952$
- Space step $h := X(1) - X(0) = 0.0025$
- $f(x) = e^{-16x^2}$

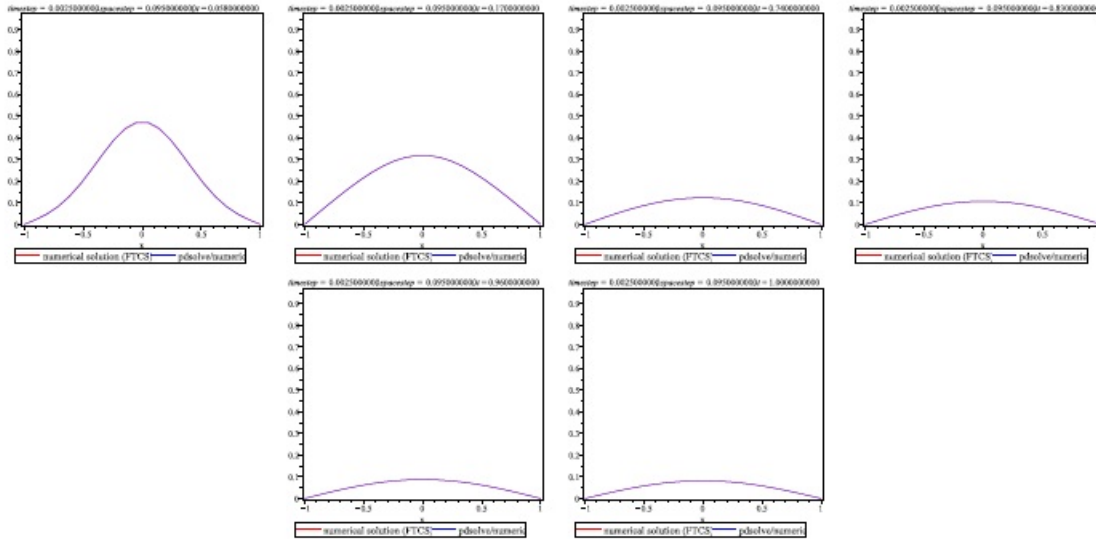


Figure 1: Comparison between FTCS method and the true solution at different time.

Figure 1 compare the numerical and the exact solution obtained using Maple. If the initial data $u(0, x) = f(x)$ is nonnegative then the solution of the Fisher's equation remains nonnegative, this is consistent with its use as a density of probability or density

of population. Figure 1 show that the maximum principle is satisfied. Indeed, the maximum principle is verified because if the solution changes sign from negative to positive or vice versa at (t_0, x_0) and if the solution has a non-degenerate minimum at x_0 then $u_{xx}(t_0, x_0) > 0$. Therefore, the solution cannot involve forward in time into the region $u < 0$. A more careful argument is required to deal with degenerate minimum. So, a similar argument so that if the initial data $f(x) < 1$ then the solution remains less than one, i.e. $u(t, x) < 1$.

5. Crank Nicholson Solution

Here, we denote future field values as: $\phi_j = u(t + s, x_j)$ and past field values as $\psi_j = u(t, x_j)$ Using

$$\begin{aligned} u(t + s, x - 2h) &= \phi_{j-2}, u(t + s, x - h) = \phi_{j-1}, \\ u(t + s, x) &= \phi_j, u(t + s, x + h) = \phi_{j+1}, \\ u(t + s, x + 2h) &= \phi_{j+2}, u(t, x - 2h) = \psi_{j-2}, \\ u(t, x - h) &= \psi_{j-1}, u(t, x) = \psi_j, \\ u(t, x + h) &= \psi_{j+1}, u(t, x + 2h) = \psi_{j+2} \end{aligned}$$

Then we get

$$\begin{aligned} &-24 \phi_j h^2 - \beta s \phi_{j-2} + 16 \beta s \phi_{j-1} - 30 \beta s \phi_j + 16 \beta s \phi_{j+1} - \beta s \phi_{j+2} + 12 \mathcal{F}(\phi_j) sh^2 \\ = &-24 h^2 \psi_j + \beta s \psi_{j-2} - 16 \beta s \psi_{j-1} + 30 \beta s \psi_j - 16 \beta s \psi_{j+1} + \beta s \psi_{j+2} - 12 sh^2 \mathcal{F}(\psi_j) \end{aligned}$$

Here is an example of the nonlinear system of equations we have to solve at each time step for $\{\phi_j\}_{j=1}^M$ When $M = 5$ interior lattice points and the boundary conditions imply

$$\phi_0 = \phi_{M+1} = \psi_0 = \psi_{M+1} = 0$$

We have the system with $\phi_0 = \psi_0 = \psi_6 = \phi_6 = 0$

$$\left(\begin{array}{c} -24 \phi_1 h^2 - \beta s \phi_{-1} - 30 \beta s \phi_1 + 16 \beta s \phi_2 - \beta s \phi_3 + 12 \mathcal{F}(\phi_1) sh^2 \\ -24 \phi_2 h^2 + 16 \beta s \phi_1 - 30 \beta s \phi_2 + 16 \beta s \phi_3 - \beta s \phi_4 + 12 \mathcal{F}(\phi_2) sh^2 \\ -24 \phi_3 h^2 - \beta s \phi_1 + 16 \beta s \phi_2 - 30 \beta s \phi_3 + 16 \beta s \phi_4 - \beta s \phi_5 + 12 \mathcal{F}(\phi_3) sh^2 \\ -24 \phi_4 h^2 - \beta s \phi_2 + 16 \beta s \phi_3 - 30 \beta s \phi_4 + 16 \beta s \phi_5 + 12 \mathcal{F}(\phi_4) sh^2 \\ -24 \phi_5 h^2 - \beta s \phi_3 + 16 \beta s \phi_4 - 30 \beta s \phi_5 - \beta s \phi_7 + 12 \mathcal{F}(\phi_5) sh^2 \end{array} \right)$$

$$= \begin{pmatrix} -24 h^2 \psi_1 + \beta s \psi_{-1} + 30 \beta s \psi_1 - 16 \beta s \psi_2 + \beta s \psi_3 - 12 s h^2 \mathcal{F}(\psi_1) \\ -24 h^2 \psi_2 - 16 \beta s \psi_1 + 30 \beta s \psi_2 - 16 \beta s \psi_3 + \beta s \psi_4 - 12 s h^2 \mathcal{F}(\psi_2) \\ -24 h^2 \psi_3 + \beta s \psi_1 - 16 \beta s \psi_2 + 30 \beta s \psi_3 - 16 \beta s \psi_4 + \beta s \psi_5 - 12 s h^2 \mathcal{F}(\psi_3) \\ -24 h^2 \psi_4 + \beta s \psi_2 - 16 \beta s \psi_3 + 30 \beta s \psi_4 - 16 \beta s \psi_5 - 12 s h^2 \mathcal{F}(\psi_4) \\ -24 h^2 \psi_5 + \beta s \psi_3 - 16 \beta s \psi_4 + 30 \beta s \psi_5 + \beta s \psi_7 - 12 s h^2 \mathcal{F}(\psi_5) \end{pmatrix}$$

Our system can be transformed to

$$G_i(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5) = 0, i = 1, \dots, 5$$

It is useful to organize the unknowns into a vector $\phi = [\phi_1, \phi_2, \phi_3, \phi_4, \phi_5]^t$. Then, we will linearize the problem by taking $\phi = \phi_{\mathbf{k}} + \delta_{\mathbf{k}}$ where $\phi_{\mathbf{k}}$ is a guess and $\delta_{\mathbf{k}}$ is the error in the guess. Expanding $G_i(\phi) = G_i(\phi_{\mathbf{k}} + \delta_{\mathbf{k}}) = 0$ to linear order in the error, we obtain:

$$G(x_k) + J(x_k) \delta_k = 0 \quad (8)$$

where

$$G(x_k) = \begin{bmatrix} G_1(x_k) \\ G_2(x_k) \\ G_3(x_k) \\ G_4(x_k) \\ G_5(x_k) \end{bmatrix}$$

The matrix $J(x_k)$ is the Jacobian of the system. Then, the linear system (8) can be solved for δ_k which can then be used to update our guess via

$$\mathbf{x}_{\mathbf{k}+1} = \mathbf{x}_{\mathbf{k}} + \delta_{\mathbf{k}} = \mathbf{x}_{\mathbf{k}} - \mathbf{J}^{-1}(\mathbf{x}_{\mathbf{k}}) \mathbf{G}(\mathbf{x}_{\mathbf{k}})$$

One can iterate this process until the distance between the iterates $\frac{|\mathbf{x}_{\mathbf{k}+1} - \mathbf{x}_{\mathbf{k}}|}{n}$ is less than some user defined tolerance. This method is known as a Newton method. Using Newton method to solve nonlinear equations of this type in the process of creating the Crank Nicholson solution. Note that for sufficiently small time steps we expect $\phi_j \sim \psi_j$ so a good initial guess for the future field values are just the corresponding past field values.

- $\tau = 5, N = 400, M = 100, L = 5, \beta = 1$
- $X := j \mapsto L \left(-1 + 2 \frac{j}{M+1} \right)$
- $T := i \mapsto \frac{\tau i}{N}$

- Time step $s := T(1) - T(0) = 0.0952380952$
- Space step $h := X(1) - X(0) = 0.0025$
- $f(x) = e^{-100x^2}$

For this choice the FTCS is unstable, see Figure 2 however CN method gives reasonable results that are closed to the exact solution, see Figure 3.

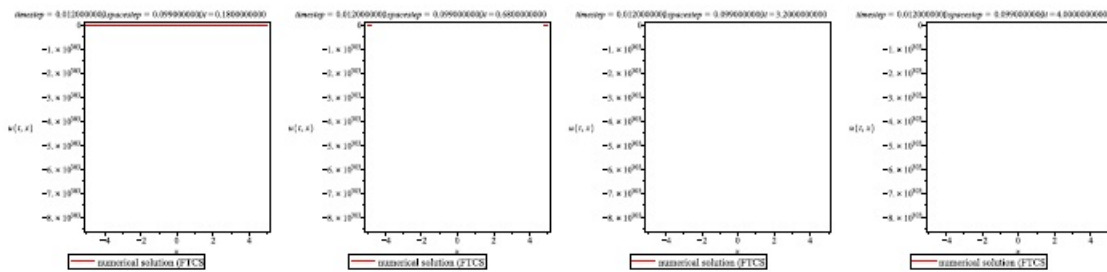


Figure 2: Comparison between FTCS method and the true solution at different time.

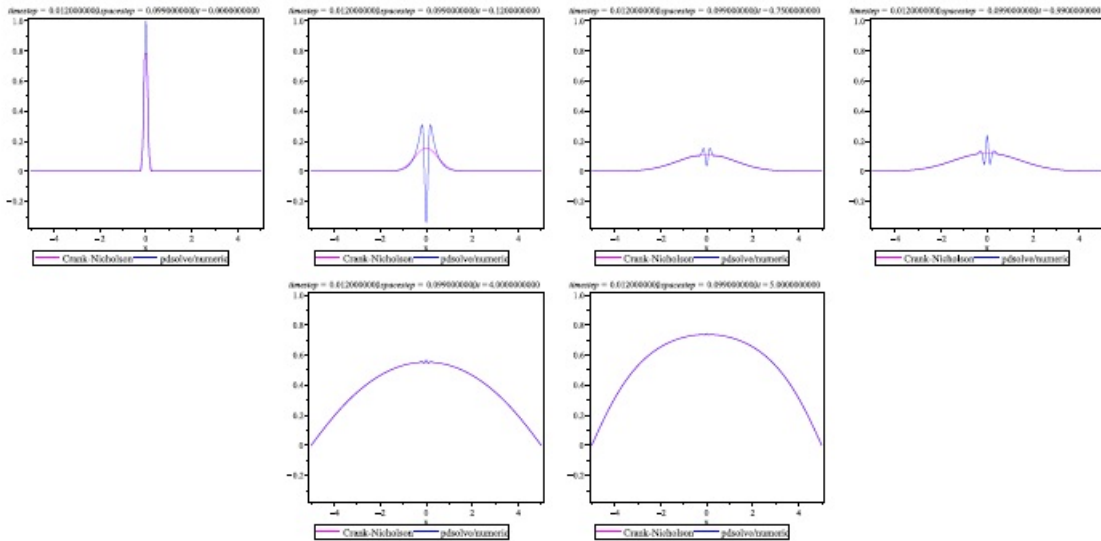


Figure 3: Comparison between CN method and the true solution at different time.

4. Conclusion

Our numerical results can be applied for $\frac{\partial}{\partial t}u(t, x, y) = \Delta u(t, x, y) + \mathcal{F}(u(t, x, y))$. It's a good application of the strategy proposed in this work. The Crank-Nicolson method is more accurate than FTCS or BTCS. Clearly, something is going very wrong with the FTCS method, while the CN one is returning reasonable results. Experimenting with different N and M for the Crank-Nicolson method convince us that its output for $u(t, x)$ is always bounded and not exponentially diverging. Although all three methods have the same spatial truncation error h^2 the better temporal truncation error for the Crank-Nicolson method is big advantage. The Crank-Nicolson scheme is recommended over FTCS and BTCS. One can seek a condition for unstable behavior of FTCS.

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