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L-FUZZY BP-IDEAL

Y. CHRISTOPHER JEFFERSON[1] AND M. CHANDRAMOULEESWARAN[2]
[1] Department of Mathematics,
Spicer Adventist University, Pune, Maharashtra, India
[2] Head, PG & Research Department of Mathematics,
S.B.K. College, Aruppukottai, Tamilnadu, India

Abstract
In this paper, we define the notion of L-Fuzzy BP-ideal. We discuss the properties of L-Fuzzy BP-ideals and prove some results.

1. Introduction

In 1966 Y. Imai and K. Iseki introduced two classes of abstract algebra, BCK algebras and BCI algebras [3,4]. In 2012 Sun Shin Ahn and Jeong Soon Han introduced the notion of BP-Algebras [7]. In 1975 Iseki introduced the concept of implicative ideals [5]. In 1971 A. Rosenfeld initiated the study of fuzzy algebraic structures [6] In 1965 L. A. Zadeh introduced the notion of fuzzy sets [8]. L Goguen extended the notion of fuzzy sets into L-fuzzy sets where L is a complete lattice [2]. In our earlier paper we have introduced the notion of fuzzy structures in BP-algebras [1]. In this paper, we introduce the notion of L-Fuzzy BP-ideals.

Key Words : *BP Algebras, L-BP Ideals, L-Fuzzy BP algebras, L-Fuzzy BP Ideals.*
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2. Preliminaries

In this section we recall some basic definitions that are needed for our work.

Definition 2.1 : A BP algebra $(X, *, 0)$ is a non-empty set X with a constant 0 and a binary operation $*$ satisfying the following conditions: for all $x, y, z \in X$,

(i) $x * x = 0$

(ii) $x * (x * y) = y$

(iii) $(x * z) * (y * z) = x * y$.

Definition 2.2 : A non-empty subset I of BP-algebra $(X, *, 0)$ is said to be an Ideal of X if it satisfies the following conditions: $\forall \;\; x, y \in I$

(i) $0 \in I$

(ii) $x * y \in I$ and $y \in I \Rightarrow x \in I$.

Definition 2.3 : Let S be a non-empty set. A mapping $\mu : S \to L$ is called a fuzzy subset of S.

Definition 2.4 : A lattice is a partially ordered set in which any two elements have a least upper bound and a greatest lower bound.

Definition2.5 : A lattice L is called a complete lattice if every subset $A = \{a_\alpha\}$ has a sup denoted by $\vee a_\alpha$ and inf denoted by $\wedge a_\alpha$ where $0 \equiv \wedge a_\alpha$ is the least element of L and $1 \equiv \wedge a_\alpha$ is the greatest element of $L : 0 \leq a$ and $1 \geq a$ for every $a \in L$.

Definition 2.6 : Let X be a non-empty set and $L : (L, \leq)$ be a complete lattice with least element 0 and greatest element 1. A L-fuzzy subset μ of X is a function $\mu : X \to L$.

Definition 2.7 : A L-fuzzy subset μ of a BP-algebra $(X, *, 0)$ is called a L-fuzzy BP sub algebra if $\mu(x * y) \geq \mu(x) \wedge \mu(y) \;\; \forall \;\; x, y \in X$.

3. L-Fuzzy BP-Ideals

In this section we introduce the notion of L-Fuzzy BP ideals and prove some simple results.

Definition 3.1 : Let X be a BP-algebra. A L-fuzzy subset set μ of X is said to be a L-fuzzy subset BP-ideal of X if it satisfies the following conditions:

(i) $\mu(0) \geq \mu(x) \;\; \forall \;\; x \in X$

(ii) $\mu(x) \geq (x * y) \wedge \mu(y) \;\; \forall \;\; x, y \in X$.

Example 3.2 : Let $(X = \{0, 1, 2, 3\}, *, 0)$ be a BP-algebra with the following Cayley table.

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Define $\mu : X \to L$ by

$$\mu = \begin{cases} 1 & \text{if} \quad x = 0 \\ t_1 & \text{if} \quad x = 2 \\ t_2 & \text{if} \quad x = 1 \\ 0 & \text{if} \quad x = 3 \end{cases}$$

$t_1, t_2 \in L$ and $\inf L \leq t_1 \leq t_2 \leq \sup L$.

$\therefore$ μ is a L-fuzzy BP-ideal of the BP-algebra X.

Proposition 3.3 : Intersection of two L-fuzzy BP-ideals of X is again a L-fuzzy BP-ideal of X.

Proof : Let μ and ψ be any two fuzzy BP-ideals of X.

$$\begin{aligned} (\mu \cap \psi)(0) &= (\mu \cap \psi)(x * x) \\ &\geq \mu(x * x) \wedge \psi(x * x) \\ &\geq \{\{\mu(x) \wedge \mu(x)\} \wedge \{\psi(x) \wedge \psi(x)\}\} \\ &= \{\{\mu(x) \wedge \psi(x)\} \wedge \{\mu(x) \wedge \psi(x)\}\} \\ &= \{(\mu \cap \psi)(x) \wedge (\mu \cap \psi)(x)\} \\ &= (\mu \cap \psi)(x) \end{aligned}$$

$$\therefore \quad (\mu \cap \psi)(0) \geq (\mu \cap \psi)(x)$$

$$\begin{aligned} (\mu \cap \psi)(x) &= \mu(x) \wedge \psi(x) \\ &\geq (\mu(x * y) \wedge \mu(y)) \wedge (\psi(x * y) \wedge \psi(y)) \\ &= (\mu(x * y) \wedge \psi(x * y)) \wedge (\mu(y) \wedge \psi(y)) \\ &= (\mu \cap \psi)(x * y) \wedge (\mu \cap \psi)(y), \quad \text{for all} \;\; x, y \in X. \end{aligned}$$

Hence $\mu \cap \psi$ is a L-fuzzy BP-ideal of X.

Proposition 3.4 : If μ is a L fuzzy BP-ideal of a BP-algebra $(X, *, 0)$, then $\forall \;\; x, y \in X$.

1. μ is order reversing; that is, $x \leq y$ implies $\mu(x) \geq \mu(y)$

2. $\mu(x * (x * y)) \geq \mu(y)$.

Proof : Since μ is a fuzzy BP-ideal of X.

Let $x \leq y \Rightarrow x * y = 0$

$\Rightarrow \mu(x * y) = \mu(0)$

$\therefore \quad \mu(x * y) = \mu(0) \geq \mu(x)$.

$\mu(x) \geq \mu(x * y) \wedge \mu(y)$

$\geq \mu(0) \wedge \mu(y)$

$= \mu(y)$

$\therefore \quad \mu(x) \geq \mu(y)$.

By definition 2.1(ii) $x * (x * y) = y$

$$\therefore \quad (x * (x * y)) * y = y * y$$
$$\Rightarrow (x * (x * y)) * y = 0$$
$$\Rightarrow x * (x * y) \leq y.$$

By (1) μ is order reversing, $\mu(x * (x * y)) \geq \mu(y) \;\; \forall \;\; x, y \in X$.

Proposition 3.5 : If μ is a L-fuzzy ideal of a BP-algebra $(X, *, 0)$ and $\mu_\alpha(x) = (\alpha \wedge \mu(x)) \;\; \forall \;\; x \in X$ and $\alpha \in L$, then $\mu_\alpha(x)$ is L fuzzy BP-ideal of X.

Proof : Let μ be a L-fuzzy ideal of the BP-algebra $(X, *, 0)$ and $\alpha \in L$.

$$\therefore \quad \mu(0) \geq \mu(x) \quad \forall \;\; x \in X.$$

Now,

$$\mu_\alpha(0) = \{\alpha \wedge \mu(0)\} \geq \{\alpha \wedge \mu(x)\} = \mu_\alpha(x) \;\; \forall \;\; x \in X.$$

Also, μ is a L-fuzzy ideal of X shows that

$$\mu(x) \geq \mu(x * y) \wedge \mu(y) \quad \forall \quad x, y \in X.$$

$$\begin{aligned} \mu_\alpha(x) &= (\alpha \wedge \mu(x)) \\ &\geq \{\alpha \wedge (\mu(x * y) \wedge \mu(y))\} \\ &= (\alpha \wedge \mu(x * y)) \wedge (\alpha \wedge \mu(y))\} \\ &= \{\mu_\alpha(x * y) \wedge \mu_\alpha(y)\} \end{aligned}$$

$\Rightarrow \mu_\alpha(x)$ is a L-fuzzy ideal of X. Since this is true for all $\alpha \in L$, μ_α is L-fuzzy BP-ideal of X for all $\alpha \in L$.

Corollary 3.6 : If μ is a L-fuzzy BP-ideal of a BP-algebra X and

$$\mu_{\mu(\alpha)}(x) = \{(\mu(\alpha) \wedge \mu(x)\} \ \ \forall \ \ \alpha, \quad x \in X.$$

Then $\mu_{\mu(\alpha)}$ is a L-fuzzy BP-ideal of $X \ \forall \ \alpha, \quad x \in X$.

Theorem 3.7 : A L-fuzzy subset μ of a BP-algebra $(X, *, 0)$ is a L-fuzzy BP-ideal if and only if for any $\lambda \in L$,

$$U(\mu, \lambda) = \{x : x \in X, \mu(x) \geq \lambda\}$$

is an ideal of X where $U(\mu, \lambda) \neq \emptyset$.

Proof : Suppose μ is a L fuzzy ideal of X and $U(\mu, \lambda) \neq \emptyset$ for $\lambda \in L$.

Let $x \in U(\mu, \lambda)$, then $\mu(x) \geq \lambda$. By definition of L-fuzzy BP-ideal, we have $\mu(0) \geq \mu(x) \geq \lambda$. Thus $0 \in U(\mu, \lambda)$.

Suppose $x * y \in U(\mu, \lambda)$ and $y \in U(\mu, \lambda)$. Therefore, $\mu(x * y) \geq \lambda$ and $\mu(y) \geq \lambda$. By definition, we have $\mu(x) \geq \min\{\mu(x * y) \wedge \mu(x)\} \geq \lambda$. So $x \in U(\mu, \lambda)$.

Hence (μ, λ) is an BP-ideal of X.

Conversely, suppose that for each $\lambda \in L, U(\mu, \lambda)$ is either empty or an ideal of X.

For any $x \in X$, let $\mu(x) = \lambda$. Then $x \in U(\mu, \lambda)$.

Since $U(\mu, \lambda) \neq \emptyset$ is an ideal of X, we have $0 \in U(\mu, \lambda)$ and hence $\mu(0) \geq \lambda = \mu(x)$. Thus $\mu(0) \geq \mu(x) \ \ \forall \ \ x \in X$.

Assume $\mu(x) \geq \{\mu(x * y) \wedge \mu(y)\} \ \ \forall \ \ x, y \in X$ is not true. Then there exists $x_0, y_0 \in X$ such that

$$\begin{aligned} \mu(x_0) &\leq \{\mu(x_0 * y_0) \wedge \mu(y_0)\} \\ &\Rightarrow \mu(x_0) < \lambda_0 < \{\mu(x_0 * y_0) \wedge \mu(y_0)\}. \end{aligned}$$

We have $x_0 * y_0, y_0 \in U(\mu, \lambda_0)$ and $U(\mu, \lambda_0) \neq \emptyset$.

But $U(\mu, \lambda_0)$ is an ideal of X. So $x_0 \in U(\mu, \lambda_0)$ by the definition of BP-ideal. $\mu(x_0) \geq \lambda_0$, contradicting $(\mu(0) \geq \mu(x) \ \ \forall \ \ x \in X)$.

Therefore $\mu(x) \geq \{\mu(x * y) \wedge \mu(y)\}$.

Theorem 3.8 : A fuzzy subset μ of a BP-algebra $(X, *, 0)$ is a L-fuzzy BP-ideal if and only if every nonempty level subset of $U(\mu, s)$, $s \in Im(\mu)$ is a BP-ideal.

Proof : Let μ be a L-fuzzy BP-ideal.

Claim : $U(\mu, s), s \in Im(\mu)$ is a BP-ideal.

Since $U(\mu, s) \neq \emptyset$ there exist $x \in U(\mu, s)$ such that $\mu(x) \geq s$.

Since μ is a fuzzy BP-ideal, $\mu(0) \geq \mu(x) \;\; \forall \;\; x \in X$. Hence for this $x \in U(\mu, s)$, $\mu(0) \geq s$ which shows that $0 \in U(\mu, s)$.

Now, for any $x, y \in X$, assume that $x * y \in U(\mu, s)$ and $y \in U(\mu, s)$.

$$x * y \in U(\mu, s) \Rightarrow \mu(x * y) \geq s.$$

Also

$$y \in U(\mu, s) \Rightarrow \mu(y) \geq s$$

$$\therefore \quad \mu(x * y) \geq s, \quad \mu(y) \geq s.$$

$$\Rightarrow \{\mu(x * y) \wedge \mu(y)\} \geq s.$$

Since μ is a L-fuzzy BP-ideal, $\mu(x) \geq \{\mu(x * y) \wedge \mu(y)\} \geq s$. Thus proving $x \in U(\mu, s)$.

This proves that $U(\mu, s)$ is a BP-ideal of X.

Conversely, let $U(\mu, s), s \in Im(\mu)$ is a BP-ideal of X.

Claim : μ is a L-fuzzy BP-ideal.

Let $x, y \in X$. For any $s \in Im(\mu)$, let $s = \{\mu(x * y) \wedge \mu(y)\}$. Therefore, $\mu(x * y) \geq s$ and $\mu(y) \geq s$.

This shows that $x * y, y \in U(\mu, s)$.

Since $U(\mu, s)$ is a BP-ideal we have $x \in U(\mu, s)$.

This proves that $\mu(x) \geq s = \{\mu(x * y) \wedge \mu(y)\}$.

This shows that μ is a L-fuzzy BP-ideal of X.

Theorem 3.9 : Let μ be a L-fuzzy BP-ideal of BP-algebra X and let $x \in X$. Then $\mu(x) = t$ if and only if $x \in U(\mu, t)$ but $x \notin U(\mu, s) \;\; \forall \;\; s > t$.

Proof : Let μ be a L-fuzzy BP-ideal of X and let $x \in X$. Assume $\mu(x) = t$, so that $x \in U(\mu, t)$.

If possible, let $x \in U(\mu, s)$ for $s > t$. Then $\mu(x) \geq s > t$. This contradicts the fact that $\mu(x) = t$, concludes that $x \notin U(\mu, s) \;\; \forall \;\; s > t$.

Conversely, let $x \in U(\mu, t)$ but $x \notin U(\mu, s) \;\; \forall \;\; s > t$.

$$x \in U(\mu, t) \Rightarrow \mu(x) \geq t.$$

Since $x \notin U(\mu, s)\ \ \forall\ \ s > t, \ \ \mu(x) = t$.

Theorem 3.10 : Let X be a BP-algebra. Let λ and μ be the L-fuzzy BP-ideals of X. Then $\lambda \times \mu$ is a L fuzzy BP-ideal of $X \times X$.

Proof : Let X be a BP-algebra and let λ and μ be L-fuzzy BP-ideals of X. For any $(x, y) \in X \times X$.

$$\begin{aligned}(\lambda \times \mu)(0,0) &= \{\lambda(0) \wedge \mu(0)\} \\ &\geq \{\lambda(x) \wedge \mu(x)\} \\ &= (\lambda \times \mu)(x).\end{aligned}$$

Let (x_1, x_2) and $(y_1, y_2) \in X \times X, x = (x_1, x_2)$ and $y = (y_1, y_2)$.

$$\begin{aligned}(\lambda \times \mu)(x) &= (\lambda \times \mu)(x_1, x_2) \\ &= \{\lambda(x_1) \wedge \mu(x_2)\} \\ &\geq \{(\lambda(x_1 * y_1) \wedge \lambda(y_1)) \wedge (\mu(x_2 * y_2) \wedge \mu(y_2))\} \\ &= \{(\lambda(x_1 * y_1) \wedge \mu(x_2 * y_2)) \wedge (\lambda(y_1) \wedge \mu(y_2))\} \\ &= \{(\lambda \times \mu)(x_1 * y_1 \wedge x_2 * y_2) \wedge ((\lambda \times \mu)(y_1, y_2))\} \\ &= \{\lambda \times \mu(x_1, x_2) * (y_1, y_2) \wedge (\lambda \times \mu)(y_1, y_2)\} \\ &= \{(\lambda \times \mu)(x, y) \wedge (\lambda \times \mu)(y)\}.\end{aligned}$$

Thus $(\lambda \times \mu)$ is a fuzzy BP-ideal of $X \times X$.

Theorem 3.11 : For any two L-fuzzy subsets λ and μ of X, if $\lambda \times \mu$ is a L fuzzy BP-ideal of X, then either λ or μ is a L-fuzzy BP-ideal of X.

Proof : Let λ and μ be L-fuzzy subsets of X such that $\lambda \times \mu$ is a L-fuzzy BP-ideal of X.

$\therefore \quad (\lambda \times \mu)(0, 0) \geq (\lambda \times \mu)(x, y)$ for all $(x, y) \in X \times X$.

Assume $\lambda(x) > \lambda(0)$ and $\mu(y) > \mu(0)$ for some $x, y, x \in X$. Then

$$\begin{aligned}(\lambda \times \mu)(x, y) &= \{\lambda(x) \wedge \mu(y)\} \\ &> \{\lambda(0) \wedge \mu(0)\} \\ &= (\lambda \times \mu)(0) \ \text{ for all } \ (x, y) \in X \times X\end{aligned}$$

which is a contradiction. Thus $\lambda(x) \geq \lambda(0)$ or $\mu(0) > \mu(y)\ \ \forall\ \ y \in X$.

Let $x = (x_1, x_2)$ and $y = (y_1, y_2) \in X \times X$

$$\begin{aligned}(\lambda \times \mu)(x) &\geq \{(\lambda \times \mu)(x * y) \wedge (\lambda \times \mu)(y)\} \\ &= \{(\lambda \times \mu)(x_1 * y_1, x_2 * y_2) \wedge (\lambda \times \mu)(y_1, y_2)\} \\ &= \{(\lambda(x_1 * y_1) \wedge \mu(x_1, x_2)) \wedge (\lambda(y_1) \wedge \mu(y_2))\}\end{aligned}$$

$$\begin{aligned}\{(\lambda(x_1) \wedge \mu(x_2)\} &\geq \{(\lambda(x_1 * y_1) \wedge (\lambda(y_1)) \wedge (\mu(x_1, y_2) \wedge \mu(y_2))\} \\ \Rightarrow \text{ either } (\lambda(x_1) &\geq (\lambda(x_1 * y_1) \wedge (\lambda(y_1))\} \text{ or} \\ \mu(x_2) &\geq (\mu(x_1, y_2) \wedge \mu(y_2))\end{aligned}$$

$\Rightarrow \lambda$ or μ is is L-fuzzy ideal of X.

Theorem 3.12 : Let λ and μ be L-fuzzy BP-ideals of X_1 and X_2 respectively. Then $\lambda \times \mu$ is a L-fuzzy BP-ideal of $X_1 \times X_2$.

Proof : Let λ be a fuzzy BP-ideal of X_1.

Let μ be a fuzzy BP-ideal of X_2.

Claim : $\lambda \times \mu$ is fuzzy BP-ideals of $X_1 \times X_2$. For any $(x, y) \in X_1 \times X_2$.

$$\begin{aligned}(\lambda \times \mu)(0, 0) &= \{\lambda(0) \wedge \mu(0)\} \\ &\geq \{\lambda(x) \wedge \mu(y)\} \\ &= (\lambda \times \mu)(x, y).\end{aligned}$$

Let (x_1, x_2) and $(y_1, y_2) \in X \times X$.

$$\begin{aligned}(\lambda \times \mu)(x_1, x_2) &= \{(\lambda(x_1) \wedge \mu(x_2))\} \\ &\geq \{(\lambda(x_1 * y_1) \wedge \lambda(y_1)) \wedge (\mu(x_2 * y_2) \wedge \mu(y_2))\} \\ &= \{(\lambda(x_1 * y_1) \wedge \mu(x_2 * y_2)) \wedge (\lambda(y_1) \wedge \mu(y_2))\} \\ &= \{(\lambda \times \mu)(x_1 * y_1 \wedge x_2 * y_2) \wedge (\lambda \times \mu(y_1, y_2))\} \\ &= \{(\lambda \times \mu)((x_1, x_2) * (y_1, y_2)) \wedge (\lambda \times \mu)(y_1, y_2)\}.\end{aligned}$$

Thus $\lambda \times \mu$ is a L-fuzzy BP-ideal of $X_1 \times X_2$.

Theorem 3.13 : Inverse image of fuzzy BP-ideal is again a fuzzy BP-ideal.

Proof : Let $f : X_1 \to X_2$ be an epimorphism. Let σ be fuzzy BP-ideal of X_2.

To prove : $f^{-1}(\sigma)$ is a fuzzy BP-ideal of X_1.

$$\begin{aligned}(f^{-1}(\sigma)(x)) &= \sigma(f(x)) \\ &\geq \{\sigma(f(x) * f(y)) \wedge \sigma(f(y))\} \\ &= \{\sigma(f(x * y) \wedge \sigma(f(y))\} \text{ (since } f \text{ is epimorphismn)} \\ &= (f^{-1}(\sigma)(x * y) \wedge f^{-1}(\sigma)(y) \ \forall \ x, y \in X.\end{aligned}$$

Thus $f^{-1}(\sigma)$ is a L-fuzzy BP-ideal of X_1.

Theorem 3.14 : Let $f : X_1 \to X_2$ be an epimorphism of BP-algebras. Let μ be a L-fuzzy subset of X_2. If $f^{-1}(\mu)$ is a L-fuzzy BP-ideal of X_1, then μ is a L-fuzzy BP-ideal of X_2.

Proof : Let $f : X_1 \to X_2$ be an epimorphism of BP-algebras.
Let μ be a fuzzy subset of X_2. Let $f^{-1}(\mu)$ is a fuzzy BP-ideal of X_1.

Claim : μ is a fuzzy BP-ideal of $X_2.\mu(0_{x_2}) = \mu(f(0_{x_1}) \geq f^{-1}((\mu(x_1) = \mu(f(x_1)) = \mu(x_2)$. Let $x_2, y_2 \in X_2$. Since f is an epimorphism, $x_1, y_1 \in X_1$ such that $f(x_1) = x_2$ and $f(y_1) = y_2$ that is, $x_1 = f^{-1}(x_2)$ and $y_1 = f^{-1}(y_2)$.

$$\begin{aligned}\mu(x_2) &= \mu(f(x_1)) \\ &= f^{-1}(\mu(x_1)) \\ &\geq \{f^{-1}(\mu(x_1 * y_1)) \wedge f^{-1}(\mu(y_1))\} \\ &= \{\mu(f(x_1 * y_1)) \wedge \mu(f(y_1))\} \\ &= \{\mu(f(x_1) * f(y_1)) \wedge \mu(f(y_1))\} \\ &= \{\mu(x_2 * y_2) \wedge \mu(y_2)\}\end{aligned}$$

$\therefore$ μ is a L-fuzzy BP-ideal of X_2.

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