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PATHWAY FRACTIONAL INTEGRAL OPERATOR ASSOCIATED WITH ALEPH - FUNCTION, MULTIVARIABL'S GENERAL CLASS OF POLYNOMIAL, MITTAGE-LEFFLER WITH H - FUNCTION

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Abstract

The following paper is a states the study of a pathway fractional integral operator related to the pathway model and pathway probability density for the Aleph function and Mittag- Leffler function with certain product of H-function and Multivariable's general class of polynomial of R- variables.

1. Introduction

The fractional integral operator which is very important and have applications in

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different subfield of applicable mathematical analysis. Since last four generations many people like Love [7], Saigo [10], etc. have studied in details , the properties, applications and different extensions of various hypergeometric operators of fractional integration. The Pathway Fractional integrals operator S. S. Nair [14] introduced is state as follows: Let $f(x)\,\epsilon L(a,b)\,;\eta\epsilon C, R(\eta)>0; a>0$ and let us take a "pathway parameter" $\alpha<1$. Then the pathway fractional integral operator is defined as follows

$$\left(P_{0+}^{(\eta,\alpha)}f\right)(x)=x^{\eta}\int_{0}^{\left[\frac{x}{a(1-\alpha)}\right]}\left[1-\frac{a\,(1-\alpha)\,t}{x}\right]^{\frac{\eta}{1-\alpha}}f\,(t)\,dt \tag{1}$$

The pathway model is introduced by Mathai [5,2] and discussed further by Mathai and Haubold [3,4].For real scalar α , the pathway model for scalar random variables is represented by the following probability density function (p. d. f.):

$$f\,(x)=c|x|^{\gamma-1}\left[1-a\,(1-\alpha)\,|x|^{\delta}\right]^{\frac{\beta}{1-\alpha}} \tag{2}$$

Provided that $-\infty<x>\infty;\delta>0;\beta\geq0;\left[1-a\,(1-\alpha)\,|x|^{\delta}\right]>0;\gamma>0$ where c is the normalizing constant and α is called the pathway parameter. For real α the normalizating constant is as follows:

$$c=\frac{1}{2}\frac{\delta\,[a\,(1-\alpha)]^{\frac{\gamma}{\delta}}\,\Gamma\left(\frac{\gamma}{\delta}+\frac{\beta}{1-\alpha}+1\right)}{\Gamma\left(\frac{\gamma}{\delta}\right)\Gamma(\frac{\beta}{1-\alpha}+1)};\alpha<1 \tag{3}$$

$$c=\frac{1}{2}\frac{\delta\,[a\,(1-\alpha)]^{\frac{\gamma}{\delta}}\,\Gamma\left(\frac{\beta}{1-\alpha}\right)}{\Gamma\left(\frac{\gamma}{\delta}\right)\Gamma(\frac{\beta}{1-\alpha}-\frac{\gamma}{\delta})};\frac{1}{\alpha-1}-\frac{\gamma}{\delta}>0;\alpha>1 \tag{4}$$

$$c=\frac{1}{2}\frac{\delta\,[a\beta]^{\frac{\gamma}{\delta}}}{\Gamma\left(\frac{\gamma}{\delta}\right)};\alpha\rightarrow1 \tag{5}$$

See that for $\alpha<1$ it is a finite range density with $\left[1-a\,(1-\alpha)\,|x|^{\delta}\right]>0$ and (2) remains in the extended generalized type-1 beta family. The pathway density in (3), for $\alpha<1$, includes the extended type-1 beta density, the triangular density, the uniform density and many other p.d.f.
For $\alpha>1$, writing $1-\alpha=-(\alpha-1)$ we have

$$f\,(x)=c|x|^{\gamma-1}\left[1+a\,(\alpha-1)\,|x|^{\delta}\right]^{\frac{-\beta}{1-\alpha}} \tag{6}$$

Provided that $-\infty<x>\infty;\delta>0;\beta\geq0;\gamma>0$ which is the extended generalized type-2 beta model for real x. It includes the type-2 beta density, the F-density, the

Student-t density, the Cauchy density and many more. Here we consider only the case of pathway parameter $\alpha < 1$. For $\alpha \to 1$ both (2) and (6) take the exponential form. since

$$lim_{\alpha\to 1} c|x|^{\gamma-1}\left[1-a\left(1-\alpha\right)|x|^{\delta}\right]^{\frac{\beta}{1-\alpha}}$$

$$= lim_{\alpha\to 1} c|x|^{\gamma-1}\left[1+a\left(\alpha-1\right)|x|^{\delta}\right]^{\frac{-\beta}{1-\alpha}} = c|x|^{\gamma-1}e^{a\eta|x|^{\delta}} \tag{7}$$

This includes the generalized gamma, the Weibull, the chi-square, the Laplace, Maxwell-Boltzmann and other related densities. For more details on the pathway model, the reader is referred to the recent papers of Mathai and Haubold [3], [4]. The Aleph ($\aleph$) -function introduced by Sudland [7], however the notation and complete definition is presented in the following manner in terms on the Mellin- Barnes type integrals.

$$\aleph\left[z\right] = \aleph^{m,n}_{x_i,y_i,\tau_i,r}\left[z|^{(a_j,A_j)_{1,n},\ldots,[\tau_i(a_j,A_j)]_{n+1,x_i}}_{(b_j,B_j)_{1,m},\ldots,[\tau_i(b_j,B_j)]_{m+1,y_i}}\right]$$

$$= \frac{1}{2\pi\omega}\int_L \Omega^{m,n}_{x_i,y_i,\tau_i,r}\left(s\right)z^{-s}ds \tag{8}$$

For all z$\neq$ 0 where $\omega = \sqrt{(-1)}$

$$\Omega^{m,n}_{x_i,y_i,\tau_i,r}\left(s\right) = \frac{\prod\limits_{j=1}^{m}\Gamma\left(b_j+B_js\right)\prod\limits_{j=1}^{n}\Gamma\left(1-a_j-A_js\right)}{\sum\limits_{i=1}^{r}\tau_i\prod\limits_{j=m+1}^{y_i}\Gamma\left(1-b_{ji}-B_{ji}s\right)\prod\limits_{j=n+1}^{x_i}\Gamma\left(a_{ji}+A_{ji}s\right)} \tag{9}$$

The integration path $L = L_{i\gamma\infty}, \gamma\epsilon R$ extends from $\gamma - i\infty$ to $\gamma + i\infty$, and is such that the poles ,assumed to be simple of $\Gamma\left(1-\alpha_i-A_js\right), j = 1,\ldots,n$ do not coincide with the pole of $\Gamma\left(\beta_i+B_js\right), j = 1,\ldots,m$ the parameter x_i, y_i are non negative integers satisfying $0 \le n \le x_i, 0 \le m \le y_i, \tau_i > 0$ for $i = 1,\ldots,r$. The $A_j, B_j, A_{ji}, B_{ji} > 0$ and $a_j, b_j, a_{ji}, b_{ji}\epsilon C$, The empty product in (2) is interpreted as unity. The existence conditions for the defining integral (1) are giving below

$$\phi_l > 0, |arg\left(z\right)| < \frac{\pi}{2}\phi_l, l = 1,\ldots,r \tag{10}$$

$$\phi_l \ge 0, |arg\left(z\right)| < \frac{\pi}{2}\phi_l \quad and \quad R\left(\xi_l\right)+1 < 0 \tag{11}$$

where

$$\phi_l = \sum_{j=1}^{n}A_j + \sum_{j=1}^{m}B_j - \tau_l\left(\sum_{j=n+1}^{x_l}A_{jl} + \sum_{j=m+1}^{y_l}B_{jl}\right) \tag{12}$$

$$\xi_l = \sum_{j=1}^{m} b_j - \sum_{j=1}^{n} a_j + \tau_l \left(\sum_{j=m+1}^{y_l} b_{jl} - \sum_{j=n+1}^{x_l} a_{jl} \right) + \frac{1}{2}\left(x_l - y_l\right), (i-1, ..., r) \tag{13}$$

For details ccount of Aleph ($\aleph$)-function see [12] and [13].

The general polynomials of R variables given by Srivasthava [8,9] defined and represented as:

$$S_{n_1,...,n_R}^{m_1,...,m_R}\left[x_1,...,x_R\right] = \sum_{s_1=0}^{\frac{n_1}{m_1}} ... \sum_{s_R=0}^{\frac{n_R}{m_R}} \left\{ \Pi_{i=1}^{R} \frac{(-n_i)_{m_i s_i}}{\Gamma(s_i+1)} x^{s_i} \right\} A\left(n_1, s_1; ...; n_R s_R\right) \tag{14}$$

Where $n_i, m_i = 1, ..., R; m_i \neq 0, \forall i \epsilon 1, 2, ..., R$ the coefficients $A\left(n_1, s_1; ...; n_R s_R\right), (s_i \geq 0)$ are arbitrary constant, real or complex. The general class of polynomials [9] is capable of reducing to a number of familiar multivariable polynomials by suitable specializing the arbitrary coefficients $A\left(n_1, s_1; ...; n_R s_R\right), (s_i \geq 0)$

Fox H-function in series representation is given in [8], [9] is as follows:

$$H_{P,Q}^{M,N}\left[Z\right] = H_{P,Q}^{M,N}\left[z|_{(f_Q,F_Q)}^{(e_P,E_P)}\right] = \sum_{h=1}^{N}\sum_{\nu=0}^{\infty} \frac{(-1)^{\nu} X(\xi)}{\Gamma(\nu+1) E_h} \left(\frac{1}{z}\right)^{\xi} \tag{15}$$

Where $\xi = \frac{(e_h - 1 - h)}{E_h}$ and h=1,2,...,N And

$$X(\xi) = \frac{\prod_{j=1}^{M} \Gamma\left(f_j + F_j \xi\right) \prod_{j=1}^{N} \Gamma\left(1 - e_j - E_j \xi\right)}{\prod_{j=M+1}^{Q} \Gamma\left(1 - f_j - F_j \xi\right) \prod_{j=N+1}^{P} \Gamma\left(e_j + E_j \xi\right)} \tag{16}$$

Shukla and Prajapati [1], defined and investigated the function $E_{\beta,\rho}^{\gamma,q}(z)$ as

$$E_{\beta,\rho}^{\gamma,q}(z) = \frac{1}{\Gamma(\gamma)} H_{1,2}^{1,1}\left[-z|_{(0,1),(1-\rho,\beta)}^{(1-\gamma,\rho)}\right], \beta, \gamma, \rho \epsilon C, R(\beta) > 0, q \epsilon (0,1) \cup N \tag{17}$$

$= \frac{1}{2\pi i \Gamma(\gamma)} \int_{-\infty}^{\infty} \frac{\Gamma(s)\Gamma(\gamma - qs)}{\Gamma(\rho - \beta s)} (-z)^{-s} ds$

The relation of Wright's function ${}_p\psi_q$ and H-function is given by R.K. Saxena and Mathai [5], [13].

$${}_p\psi_q \left[{}^{(a_1,A_1),...,(a_p,A_p)}_{(b_1,B_1),...,(b_q,B_q)}|z\right] = H_{p,q+1}^{1,p}\left[-z|_{(0,1),(1-b_1,B_1),...,(1-b_q,B_q)}^{(1-a_1,A_1),...,(1-a_p,A_p)}\right] \tag{18}$$

Theorem : By the set of sufficient conditions (10), (11), (12) and (17), let $\eta, u, u_1, u_R \epsilon C, R(\delta) > 0, Re\left(1 + \frac{\eta}{(1-\alpha)}\right) > 0, R(\eta, u, u_1, u_R) > 0,\ \mu > 0, \beta,$ $\gamma \epsilon C,\ R(\beta), R(\eta), R(\gamma), R(\rho)\ >\ 0, \lambda \epsilon (0,1) \cup N$ and m_i is an arbitrary integral and coefficients $(n_1, s_1; ...; n_R s_R)$ are arbitrary constants,real or complex.

$$P_{0+}^{(\eta,\alpha)}\left[\left\{\left(x^{u-1} S_{n_1,\ldots,n_R}^{m_1,\ldots,m_R}\left(x^{u_1,\ldots,u_R}\right)\right) \aleph_{x_i,y_i,\tau_i,r}^{m,n}\left(dx^{\delta}\Big|_{(b_j,B_j)_{1,m},\ldots,[\tau_i(b_j,B_j)]_{m+1,y_i}}^{(a_j,A_j)_{1,n},\ldots,[\tau_i(a_j,A_j)]_{n+1,x_i}}\right)\right\} \times\right.$$
$$\left.\left\{H_{P,Q}^{M,N}\left[cx^{\mu}\Big|_{(f_Q,F_Q)}^{(e_P,E_P)}\right] E_{\beta,\rho}^{\gamma,\lambda}\left(bx^{\beta}\right)\right\}\right] = \frac{x^{\eta+u+u_1 s_1+\ldots+u_R s_R}}{[a(1-\alpha)]^{\sigma+u+u_1 s_1+\ldots+u_R s_R}} \times$$
$$\Gamma\left(\frac{\eta}{1-\alpha}\right) \frac{1}{\Gamma(\gamma)} \sum_{s_1=0}^{\frac{n_1}{m_1}} \ldots \sum_{s_R=0}^{\frac{n_R}{m_R}} \left\{\Pi_{i=1}^{R} \frac{(-n_i)_{m_i s_i}}{\Gamma(s_i+1)} x^{s_i}\right\} A(n_1, s_1; \ldots; n_R s_R)$$
$$H_{P,Q}^{M,N}\left[\frac{cx^{\mu}}{[a(1-\alpha)]^{\mu}}\Big|_{(f_Q,F_Q)}^{(e_P,E_P)}\right] \aleph_{x_i,y_i,\tau_i,r}^{m,n}\left[\frac{dx^{\delta}}{[a(1-\alpha)]^{\delta}}\Big|_{(b_j,B_j)_{1,m},\ldots,[\tau_i(b_j,B_j)]_{m+1,y_i}}^{(a_j,A_j)_{1,n},\ldots,[\tau_i(a_j,A_j)]_{n+1,x_i}}\right] \times$$
$$_2\psi_2\left[{}^{(\gamma,\lambda),(u+u_1 s_1+\ldots+u_R s_R-\mu\xi,\beta)}_{(\rho,\beta),\left(1+\frac{\eta}{1-\alpha}+u+u_1 s_1+\ldots+u_R s_R-\mu\xi,\beta\right)}\Big|\frac{bx^{\beta}}{[a(1-\alpha)]^{\beta}}\right] \tag{19}$$

Proof : Using the definitions (1), (8), (14), (15) and (17) then by interchange the order of integrations and summations (which is permissible under the conditions stated above), evaluate inner integral by making use of beta and gamma function formula, we arrive at the desired results.

Special Cases :

1. If we have putting $\tau_i = 1, i = 1, 2, ..., r$ in (19) then we reduce the following results in term of I- function [14].

$$P_{0+}^{(\eta,\alpha)}\left[\left\{\left(x^{u-1} S_{n_1,\ldots,n_R}^{m_1,\ldots,m_R}\left(x^{u_1,\ldots,u_R}\right)\right) I_{x_i,y_i,1,r}^{m,n}\left(dx^{\delta}\Big|_{(b_j,B_j)_{1,m},\ldots,[(b_j,B_j)]_{m+1,y_i}}^{(a_j,A_j)_{1,n},\ldots,[(a_j,A_j)]_{n+1,x_i}}\right)\right\} \times\right.$$
$$\left.\left\{H_{P,Q}^{M,N}\left[cx^{\mu}\Big|_{(f_Q,F_Q)}^{(e_P,E_P)}\right] E_{\beta,\rho}^{\gamma,\lambda}\left(bx^{\beta}\right)\right\}\right] = \frac{x^{\eta+u+u_1 s_1+\ldots+u_R s_R}}{[a(1-\alpha)]^{\sigma+u+u_1 s_1+\ldots+u_R s_R}} \times$$
$$\Gamma\left(\frac{\eta}{1-\alpha}\right) \frac{1}{\Gamma(\gamma)} \sum_{s_1=0}^{\frac{n_1}{m_1}} \ldots \sum_{s_R=0}^{\frac{n_R}{m_R}} \left\{\Pi_{i=1}^{R} \frac{(-n_i)_{m_i s_i}}{\Gamma(s_i+1)} x^{s_i}\right\} A(n_1, s_1; \ldots; n_R s_R)$$
$$H_{P,Q}^{M,N}\left[\frac{cx^{\mu}}{[a(1-\alpha)]^{\mu}}\Big|_{(f_Q,F_Q)}^{(e_P,E_P)}\right] I_{x_i,y_i,1,r}^{m,n}\left[\frac{dx^{\delta}}{[a(1-\alpha)]^{\delta}}\Big|_{(b_j,B_j)_{1,m},\ldots,[(b_j,B_j)]_{m+1,y_i}}^{(a_j,A_j)_{1,n},\ldots,[(a_j,A_j)]_{n+1,x_i}}\right] \times$$
$$_2\psi_2\left[{}^{(\gamma,\lambda),(u+u_1 s_1+\ldots+u_R s_R-\mu\xi,\beta)}_{(\rho,\beta),\left(1+\frac{\eta}{1-\alpha}+u+u_1 s_1+\ldots+u_R s_R-\mu\xi,\beta\right)}\Big|\frac{bx^{\beta}}{[a(1-\alpha)]^{\beta}}\right] \tag{20}$$

2. If we choosing $\tau_i = 1, i = 1, 2, ..., r$ and $r = 1$ in (19) then we reduce the following results in term of H- function [15].

$$P_{0+}^{(\eta,\alpha)}\left[\left\{\left(x^{u-1}S_{n_1,\ldots,n_R}^{m_1,\ldots,m_R}\left(x^{u_1,\ldots,u_R}\right)\right)H_{p,q,1,1}^{m,n}\left(dx^{\delta}\Big|_{(b_j,B_j)_{1,q}}^{(a_j,A_j)_{1,p}}\right)\right\}\times\right.$$
$$\left.\left\{H_{P,Q}^{M,N}\left[cx^{\mu}\Big|_{(f_Q,F_Q)}^{(e_P,E_P)}\right]E_{\beta,\rho}^{\gamma,\lambda}\left(bx^{\beta}\right)\right\}\right]=\frac{x^{\eta+u+u_1s_1+\ldots+u_Rs_R}}{[a(1-\alpha)]^{\sigma+u+u_1s_1+\ldots+u_Rs_R}}\times$$
$$\Gamma\left(\frac{\eta}{1-\alpha}\right)\frac{1}{\Gamma(\gamma)}\sum_{s_1=0}^{\frac{n_1}{m_1}}\cdots\sum_{s_R=0}^{\frac{n_R}{m_R}}\left\{\Pi_{i=1}^{R}\frac{(-n_i)_{m_is_i}}{\Gamma(s_i+1)}x^{s_i}\right\}A\left(n_1,s_1;\ldots;n_Rs_R\right)$$
$$H_{P,Q}^{M,N}\left[\frac{cx^{\mu}}{[a(1-\alpha)]^{\mu}}\Big|_{(f_Q,F_Q)}^{(e_P,E_P)}\right]H_{p,q}^{m,n}\left[\frac{dx^{\delta}}{[a(1-\alpha)]^{\delta}}\Big|_{(b_j,B_j)_{1,q}}^{(a_j,A_j)_{1,p}}\right]\times$$
$${}_2\psi_2\left[{}_{(\rho,\beta),\left(1+\frac{\eta}{1-\alpha}+u+u_1s_1+\ldots+u_Rs_R-\mu\xi,\beta\right)}^{(\gamma,\lambda),(u+u_1s_1+\ldots+u_Rs_R-\mu\xi,\beta)}\Big|\frac{bx^{\beta}}{[a(1-\alpha)]^{\beta}}\right] \tag{21}$$

3. If we choosing $m_1=m_2=\ldots=m_R=n_1=n_2=\ldots=n_R=1, m=n=1, x_i=y_i=1, \tau_i=1=r$, where i=1,2,...,r And $\lambda=1$ then equation (19) Reduces to [15] in equation (21)

$$P_{0+}^{(\eta,\alpha)}\left[\left\{\left(x^{u-1}\right)H_{P,Q}^{M,N}\left[cx^{\mu}\Big|_{(f_Q,F_Q)}^{(e_P,E_P)}\right]E_{\beta,\rho}^{\gamma,1}\left(bx^{\beta}\right)\right\}\right]=$$
$$\frac{x^{\eta+u}}{[a(1-\alpha)]^{\sigma+u}}\Gamma\left(\frac{\eta}{1-\alpha}\right)\frac{1}{\Gamma(\gamma)}H_{P,Q}^{M,N}\left[\frac{cx^{\mu}}{[a(1-\alpha)]^{\mu}}\Big|_{(f_Q,F_Q)}^{(e_P,E_P)}\right]\times$$
$${}_2\psi_2\left[{}_{(\rho,\beta),\left(1+\frac{\eta}{1-\alpha}+u-\mu\xi,\beta\right)}^{(\gamma,1),(u-\mu\xi,\beta)}\Big|\frac{bx^{\beta}}{[a(1-\alpha)]^{\beta}}\right] \tag{22}$$

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