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STUDY OF SPANNING TREE OF UNDIRECTED GRAPH WITH HELP OF TUTTE MATRIX TREE THEOREM

SUDHIR PRAKASH SRIVASTAVA
IET, Dr. R. M.L. Avadh University, Faizabad, India
E-mail: sudhir_ietfzd@yahoo.com

Abstract

In this paper we study about several properties about Laplacian matrix. Then describe Kirchhoff's Matrix-tree theorem for undirected graph and Tutte Matrix-tree theorem for digraph. These theorem play important role to count number of spanning tree in undirected graph and digraph. In this paper we conclude that if we convert every undirected graph in digraph with each vertex in and out edges then we can count spanning tree of undirected graph with help of Tutte Matrix tree theorem.

1. Introduction

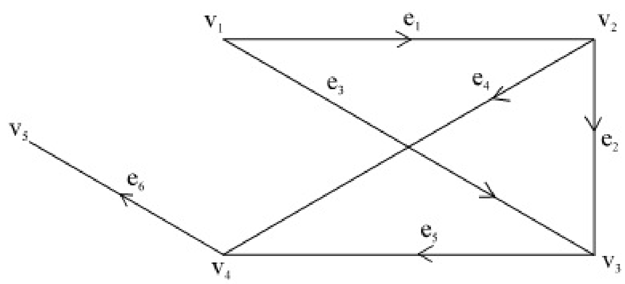
Let us suppose a simple graph (i.e. no loop and parallel edges) $G = (V, E)$ where V is the set of vertices and E is set of edges each of whose element is a pair of distinct vertices. We can assume that we will familiar with basic concept graph Theory. Let $V = \{1, 2, 3, \cdots, n\}$ and $E = \{e_1, e_2, e_3, \cdots, e_n\}$. The adjacency matrix $A(G)$ of G is $n \times n$ matrix with its row and columns indexed by V with the (i, j) entry equal to 1 if vertices i, j are adjacent and 0 otherwise.

Key Words : *Laplacian matrix, Adjacency matrix, Digraph, Undirected graph.*

Thus $A(G)$ is symmetry matrix with its i^{th} two or column sum equal to $di(G)$ which define as degree of vertex, Let $D(G)$ denoted the $n \times n$ diagonal matrix, whose i^{th} diagonal entry is $di(G), i = 1, 2, \cdots, n$. Then the Laplacin matrix of G denoted by $L(G)$ is define as

$$L(G) - D(G) - A(G).$$

For example Let a graph



Then vertex set $V = \{v_1, v_2, v_3, v_4, v_5\}$ and the edges set $E = \{e_1, e_2, e_3, e_4, e_5\}$ then

$$A(G) = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$D(G) = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$L(G) = D(G) - A(G) = \begin{bmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 & 0 \\ -1 & -1 & 3 & -1 & 0 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

There is another way to represent Laplacian matrix. Let now G is diagraph. Then we can take an incidence matrix of G is $Q(G)$ of $n \times m$. The row and column of $Q(G)$ is 0 if vertex i and edges e_j are not incident otherwise it is 1 or -1 for example in fig.

$$Q(G) = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Then we can determine $Q(G)^T$. If we define $Q(G).Q(G)^T$ then it is equal to $L(G)$. So we can say

$$L(G) = Q(G).Q(G)^T.$$

Then we can describe some basic properties of Laplacian matrix

(i) $L(G)$ is a symmetric matrix.

(ii) The non-diagonal element of Laplacian matrix is non-positive, mean non-diagonal element is either 0 or -1. That implies Laplacin matrix is Stieltjes Matrix.

Stieltjes Matrix : A matrix which all non-diagonal element is either 0 or negative and value of that matrix is positive then that matrix is known Stieltjes Matrix.

(iii) The rank of $L(G)$ is $(n-k)$, where k is number of connected component of G. In particular if G is connected then rank of $L(G)$ is $(n-1)$.

There are so many properties of Laplacian matrix known but it this paper we focused Kirchhoff's matrix theorem, which useful to determine spanning tree or tree nature.

2. Kirchhoff's Matrix-Tree Theorem

It is very beautiful theorem that useful to count spanning tree in graph. It describe very good connection between graph theory and linear algebra. The result discovered by German Physicist Gustav Kirchhoff in 1847 during study of electrical circuit.

We well known about spanning tree that if a subgraph H of a Graph G contain every vertices of Graph G and that subgraph has no any cycle that such subgraph is known as spanning tree.

We can also define a Laplacian matrix another way

$$L_{ij} = \begin{cases} deg(v_j) & \text{if } i = j \\ -1 & \text{If } i \neq j \text{ and } (v_i, v_j) \\ 0 & \text{otherwise} \end{cases}$$

It is equivalently $L = D - A$.

Theorem : It $G(V,E)$ is an undirected graph and L is its Laplacian matrix, then number of spanning tree (N_T) contained in G is determine by following computation.

(i) Chosen a vertex (V_i) and eliminate the i^{th} row and ith column from L to get new matrix L_i.

(ii) Compute $N_T = det(L_i)$.

For Example

$$L_1 = \begin{bmatrix} 3 & -1 & -1 & 0 \\ -1 & 3 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \quad N_T = det(L_1) = 10$$

Application of Kirchhoff's Matrix that complete graph with n vertices. The Laplacian $L(K_n)$ is $n \times n$ matrix with $(n-1)$ an diagonal and -1 otherwise. It is very easy verify that any cofactor of $L(K_n)$ equal to n^{n-2}, which cofactor is number of spanning tree in K_n.

This is alternative, note that as Cayley's formula count's the number of distinct tabled trees of complete graph K_n. We need to compute any cofactor of the Laplacian matrix K_n. The Laplacian matrix in this case is -

$$L_1 = \begin{bmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & & & \\ -1 & -1 & \cdots & n-1 \end{bmatrix}$$

Any cofactor of the above matrix is n^{n-2}, which is Cayley formula.

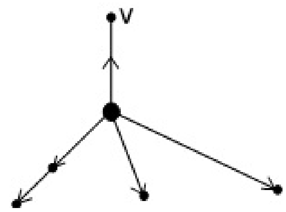
3. Tutte Matrix-Tree Theorem

After Kirchhoff result in 1948. W. T. Tutte discovered a result for directed graph or diagraph. To study that result we can define some important definition.

Definition : A vertex $v \in V$ in a diagraph $G(V,E)$ is a root if very other vertex is accessible from v.

Definition : A graph $G(V,E)$ is a directed tree or arborescence if G contain a root and the graph G that one obtains by ignoring the directedness of the edges is a tree.

Definition : A subgraph $T(V, E^1)$ of a diagraph $G(V, E)$ is a spanning arborescence if T is arborescence that contain all the vertices of G.



The graph is an arborescence whose v is root vertex.

Theorem : If $G(V, E)$ is a diagraph with vertex set $v = \{v_1, v_2, \cdots, v_n\}$ and L is an $n \times n$ matrix whose entries are given by

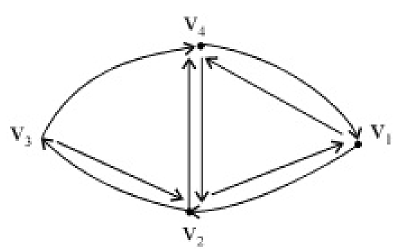
$$L_{ij} = \begin{cases} deg_(in)(v_j) & \text{if } \ i = j \\ -1 & \text{If } i \neq j \text{ and } (v_i, v_j) \in E \\ 0 & \text{otherwise} \end{cases}$$

Then number of spanning arborescence with root v_j is

$$N_j = det(L_j)$$

where L_j is matrix produced by deleting the j^{th} row and column from L.

For example



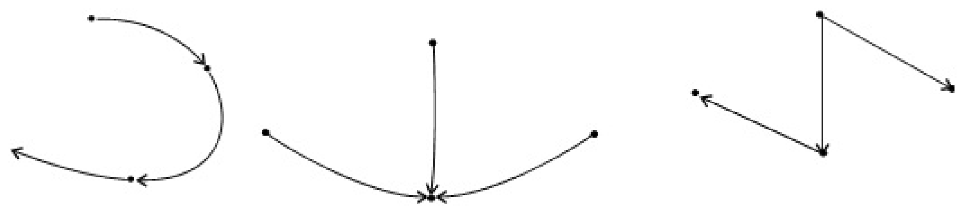
$$L = D_{in} - A \ = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 1 & -1 \\ -1 & -1 & 0 & 2 \end{bmatrix}$$

So,

$$N_1 = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & 0 & 2 \end{bmatrix} \quad \text{the det} \quad (N) = 2.$$

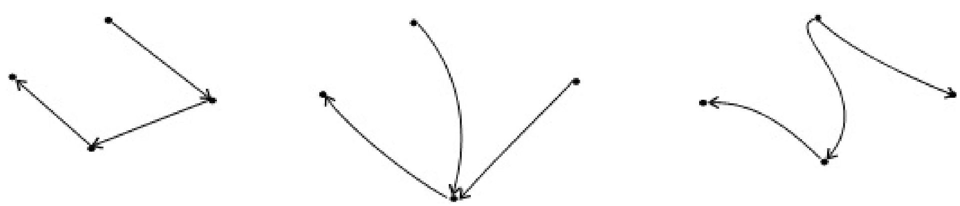
Similarly we determine N_2, N_3 and N_4 calculated det $(N_2) = 4$, $det(N_3) = 7$, $det(N_4) = 3$. The diagraph of vertex v_4 spanning tree is :



Similarly we determine of vertex v_1, v_2 and v_3 spanning tree also we can draw.

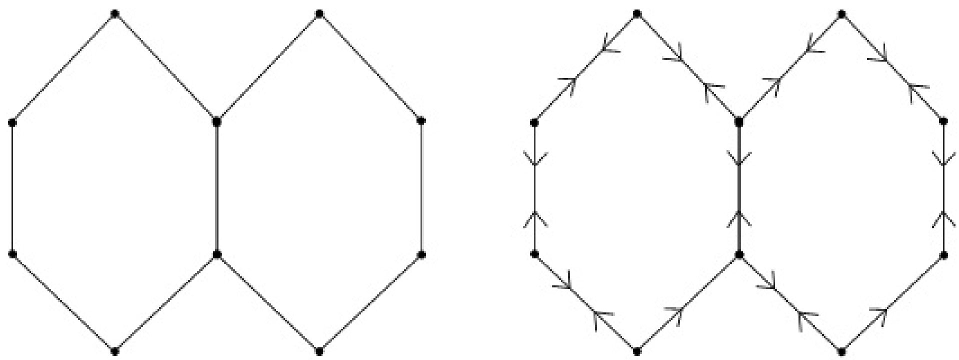
$$N_1 = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & 0 & 2 \end{bmatrix} \quad \text{the det} \quad (N) = 2.$$

Similarly we determine N_2, N_3, N_4 {i.e. $det(N_2) = 4$, $det(N_3) = 7$, $det(N_4) = 7$}. All Spanning free for V_4 vertex is given below.



4. Conclusion

If we convert an undirected graph $G(V, E)$ to a directed graph G^1 then very easy to count the spanning tree of G with help of G^1. For example let a graph



G G'

So we can describe a relation between Tutte Matrix Tree theorem and Krichhoff's theorem. Here Tutte matrix tree theorem is describe about directional graph or digraph, but Kirchhoff's Matrix theorem describe about undirected graph. Means if we want to counting spanning tree in an undirected graph with help of directional theorem. Then we should first make directed graph $G^1(V, E^1)$ (i.e. same vertex G and twice edges). Means if G has edges $e_1 = (v_1, v_2)$ then G^1 has two edges (v_1, v_2) and (v_2, v_1).
We can concluded for diagraph G^1 which includes the edges (v_1, v_2) and (v_2, v_1) wherever the original, undirected graph contain (v_1, v_2) we have

$$deg_{in}(v) = deg_{out}(v) = deg_G(v) \quad \forall \quad v \in V.$$

This implies that the Laplacian matrix L appearing in Tutte Theorem is equal to the graph Laplacian matrix appearing in Kirchhoff theorem.
So if we use Tutte method to compute the number of spanning arborescence in G^1. The result is same as we will used Kirchhoff theorem to count spanning in G.

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