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PROPERTY OF FERMAT NUMBER $E_n \; \forall \; n \geq 1$

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Abstract

The Fermat numbers corresponding to $n = 1, 2, 3, 4$ are:

$$\begin{aligned} F_1 &= 5 &&= 6 \times 1 - 1 \\ F_2 &= 17 &&= 6 \times 3 - 1 \\ F_3 &= 257 &&= 6 \times 43 - 1 \\ F_4 &= 65537 &&= 6 \times 10923 - 1 \end{aligned}$$

In the present paper an attempt has been made to establish the property that every Fermat Number $F_n, n \geq 1$ is of the form $6m - 1$ where m is a positive integer.

Introduction

In the theory of numbers, a Fermat Number is a number of the form $2^{2^n} + 1$ where n is a non-negative integer.

Fermat numbers were named after *Pierre de Fermat* (August 17, 1601 - January 12, 1665), a French magistrate and government official. He worked in several areas of Mathematics including Number Theory. He was also the first to study the Fermat numbers.

Key Words : *Fermat Number, Prime Number.*

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By this definition, the first five Fermat number are:

$$\begin{aligned}
F_0 &= 2^{2^0}+1=3\\
F_1 &= 2^{2^1}+1=5\\
F_2 &= 2^{2^2}+1=17\\
F_3 &= 2^{2^3}+1=257\\
F_4 &= 2^{2^4}+1=65537\\
&\quad (n=0,1,2,3,4)
\end{aligned}$$

Fermat conjectured that all the numbers of the form $2^{2^n}+1$ were primes. Later on, Euler proved him wrong. If Fermat's conjecture were true, we'd immediately be able to conclude that there are infinitely many prime numbers. However, Fermat's conjecture is not true since

$$2^{2^5}+1=2^{32}+1=4294967297=641\times 6700417,$$

which definitely is not a prime.

Although Fermat numbers are not all primes, they are relatively prime to one another.

Proof : We have to prove that every Fermat number is of the form $6m-1$.

We have,

$$\begin{aligned}
F_n &= 2^{2^n}+1\\
F_n+1 &= 2^{2^n}+2\\
&= 2\times(2^{2^n}+1)\\
&= 2\times(2+1)(2^{2^n-2}-2^{2^n-3}+2^{2^n-4}-\cdots+2^{2^n-(2^n-2)}-2^{2^n-(2^n-1)}+1)
\end{aligned}$$

Using

$$\begin{aligned}
a^{2m-1}+b^{2m-1} &= (a+b)(a^{2m-2}-a^{2m-3}b+a^{2m-4}b^2+\cdots-ab^{2m-3}+b^{2m-2})\\
&= 2\times 3\times(2^{2^n-3}+2^{2^n-5}+2^{2^n-7}+\cdots+2^{2^n-(2^n-1)}+1)
\end{aligned}$$

$$\begin{aligned}
F_n+1 &= 6\times\{(2^{2^n-3}+2^{2^n-5}+2^{2^n-7}+\cdots+2^3+2^1)+1\}\\
F_n &= 6\times\{(2^{2^n-3}+2^{2^n-5}+2^{2^n-7}+\cdots 2^3+2)+1\}-1.
\end{aligned}$$

Hence $F_n = 6m-1$ where $m = 2^{2^n-3}+2^{2^n-5}+2^{2^n-7}+\cdots+2^3+2+1 = (2^{2^n-1}-2)/3+1$.

Discussion

$$\begin{aligned}
F_1 &= 2^{2^1}+1 = 5 &&= 6\times 1-1\\
F_2 &= 2^{2^2}+1 = 17 &&= 6\times 3-1\\
F_3 &= 2^{2^3}+1 = 257 &&= 6\times 43-1\\
F_4 &= 2^{2^4}+1 = 65537 &&= 6\times 10923-1\\
F_5 &= 2^{2^5}+1 = 4294967297 &&= 6\times 715827883-1
\end{aligned}$$

and so on.

Result

Every Fermat number $F_n, n \geq 1$ is of the form $6m-1$.

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