International J. of Math. Sci. & Engg. Appls. (IJMSEA)
ISSN 0973-9424, Vol. 10 No. I (April, 2016), pp. 39-47

THE NUMBER OF HOMEOMORPHISM TYPES FOR ORIENTABLE SMALL COVERS OVER PRISMS AND POLYGONS

YANCHANG CHEN[1] AND MIMI CHEN[2]
[1,2] College of Mathematics and Information Science,
Henan Normal University, Xinxiang 453007,
P. R. China
E-mail: [1] cyc810707@163.com

Abstract

In this paper, we calculate the number of all orientable small covers over prisms and polygons up to homeomorphism.

1. Introduction

The notion of small covers is first introduced by Davis and Januszkiewicz [5], where a small cover is a smooth closed manifold M^n with a locally standard $(\mathbb{Z}_2)^n-$action such that its orbit space is a simple convex polytope. Nakayama and Nishimura found an orientability condition for a small cover [7]. In recent years, numerous studies have attempted to enumerate the number of equivalence classes of all small covers over a specific polytope. In [6], Garrison and Scott used a computer program to determine the

---

Key Words : *Small cover, Prism, Polygon, Orientable coloring.*
2010 AMS Subject Classification : 05C10.

number of homeomorphism classes of all small covers over a dodecahedron. Cai,Chen and Lü calculated the number of equivariant diffeomorphism classes of small covers over prisms [1]. Choi determined the number of equivariant homeomorphism classes of small covers over cubes [3]. However, there are few results about orientable small covers. Choi calculated the number of D-J equivalence classes of orientable small covers over cubes [4]. Cao and Lü showed that the cohomological rigidity holds for all small covers over prisms and they calculated the number of homeomorphism types for all small covers over prisms [2].

By $P^2(m)$ we denote a m-gon. Let $P^3(m)$ be a m-sided prism(i.e., the product of $P^2(m)$ and [0,1]). The main results are stated as follows.

Theorem 1.1 : The number of homeomorphism classes of all orientable small covers over $P^3(m)$ is

$$\begin{cases} 2, & if\ m = 4, \\ 3, & if\ m\ is\ even\ and\ m \neq 4, \\ 1, & if\ m\ is\ odd. \end{cases}$$

Theorem 1.2 : The number of homeomorphism classes of all orientable small covers over $P^2(m)$ is

$$\begin{cases} 1, & if\ m\ is\ even, \\ 0, & if\ m\ is\ odd. \end{cases}$$

The paper is organized as follows. In Section 2, we review the basic theory about small covers and some results about cohomological rigidity for small covers over prisms. In Section 3, we first reduce orientable colorings on a prism $P^3(m)$ to canonical forms, then using cohomological rigidity for small covers over prisms to distinguish them and finally prove Theorem 1.1 and Theorem 1.2.

2. Preliminaries

2.1. Orientable Colorings

An n-dimensional convex polytope P^n is said to be simple, if exactly n faces of codimension one meet at each of its vertices. A closed n-manifold M^n is said to be a small cover if it admits an effective $(\mathbb{Z}_2)^n-$action, which is locally isomorphic to the standard action of $(\mathbb{Z}_2)^n$ on R^n, and the orbit space of the action is a simple convex polytope P^n. Suppose that $\pi : M^n \to P^n$ is a small cover over a simple convex polytope P^n. Let $\mathcal{F}(P^n) = \{F_1, \cdots, F_l\}$ be the set of codimension-one faces (facets) of P^n. Then there are

l connected submanifolds $M_1, \cdots, M_l$ determined by π and F_i (i.e.,$M_i = \pi^{-1}(F_i)$). Each submanifold M_i is fixed pointwise by the $\mathbb{Z}_2-$subgroup G_i of $(\mathbb{Z}_2)^n$, so that each facet F_i corresponds to the $\mathbb{Z}_2-$subgroup G_i. Obviously, such the $\mathbb{Z}_2-$subgroup G_i actually agrees with an element ν_i in $(\mathbb{Z}_2)^n$ as a vector space. For each face F of codimension u, since P^n is simple, there are u facets $F_{i_1}, \cdots, F_{i_u}$ such that $F = F_{i_1} \cap \cdots \cap F_{i_u}$. Then, the corresponding characteristic submanifolds $M_{i_1}, \cdots, M_{i_u}$ intersect transversally in the $(n-u)$-dimensional submanifold $\pi^{-1}(F)$, and the isotropy subgroup G_F of $\pi^{-1}(F)$ is a subtorus of rank u and is generated by $G_{i_1}, \cdots, G_{i_u}$ (or is determined by $\nu_{i_1}, \cdots, \nu_{i_u}$ in $(\mathbb{Z}_2)^n$). Thus, this actually gives a characteristic function (see [5])

$$\lambda : \mathcal{F}(P^n) \longrightarrow (\mathbb{Z}_2)^n$$

defined by $\lambda(F_i) = \nu_i$ such that for any face $F = F_{i_1} \cap \cdots \cap F_{i_u}$ of P^n, $\lambda(F_{i_1}), \cdots, \lambda(F_{i_u})$ are linearly independent in $(\mathbb{Z}_2)^n$. If we regard each nonzero vector of $(\mathbb{Z}_2)^n$ as being a color, then the characteristic function λ means that each facet is colored by a color. Thus, we also call λ a $(\mathbb{Z}_2)^n$-coloring on P^n here.

Davis and Januszkiewicz [5] gave a reconstruction process of M^n by using the $(\mathbb{Z}_2)^n$-coloring λ and the product bundle $(\mathbb{Z}_2)^n \times P^n$ over P^n, so that all small covers over P^n are classified in terms of all $(\mathbb{Z}_2)^n$-colorings on $\mathcal{F}(P^n)$. By $\Lambda(P^n)$ we denote the set of all $(\mathbb{Z}_2)^n$-colorings on P^n. Then we have

Theorem 2.1 (Davis-Januszkiewicz) : Let $\pi : M^n \to P^n$ be a small cover over a simple convex polytope P^n. Then all small covers over P^n are given by $\{M(\lambda) | \lambda \in \Lambda(P^n)\}$.

Nakayama and Nishimura [7] found an orientability condition for a small cover.

Theorem 2.2 : For a basis $\{e_1, \cdots, e_n\}$ of $(\mathbb{Z}_2)^n$, a homomorphism $\varepsilon : (\mathbb{Z}_2)^n \longrightarrow \mathbb{Z}_2 = \{0, 1\}$ is defined by $\varepsilon(e_i) = 1 (i = 1, \cdots, n)$. A small cover $M(\lambda)$ over a simple convex polytope P^n is orientable if and only if there exists a basis $\{e_1, \cdots, e_n\}$ of $(\mathbb{Z}_2)^n$ such that the image of $\varepsilon\lambda$ is $\{1\}$.

We call a $(\mathbb{Z}_2)^n$-coloring which satisfies the orientability condition in Theorem 2.2 an orientable coloring of P^n. In case n=2, it is easy to see that an orientable coloring of a polygon P^2 is just a 2-coloring of P^2 (i.e.,P is colored by two colors). In case n=3, a three-dimensional small cover $M(\lambda)$ over P^3 is orientable if and only if there exists a basis $\{\alpha, \beta, \gamma\}$ of $(\mathbb{Z}_2)^3$ such that the image of λ is contained in $\{\alpha, \beta, \gamma, \alpha + \beta + \gamma\}$.

Since each triple of $\{\alpha, \beta, \gamma, \alpha+\beta+\gamma\}$ is linearly independent, the orientable coloring of a 3-polytope P^3 is just a 4-coloring of P^3. By the four color theorem, we have the following lemma.

Lemma 2.3 : There exists an orientable small cover over every simple convex 3-polytope.

Remark : Generally speaking, we can't make sure that there always exist small covers over a simple convex polytope P^n when $n \geq 4$. For example, see [5, Nonexample 1.22]. By $O(P^n)$ we denote the set of all orientable colorings on P^n. There is a natural action of $GL(n, \mathbb{Z}_2)$ on $O(P^n)$ defined by the correspondence $\lambda \longmapsto \sigma \circ \lambda$, and the action on $O(P^n)$ is free. Two small covers M_1 and M_2 over P^n are said to be Davis-Januszkiewicz equivalent (or simply, D-J equivalent) if there is an automorphism $\varphi : (\mathbb{Z}_2)^n \to (\mathbb{Z}_2)^n$ and a homeomorphism $f : M_1 \to M_2$ such that $f(t \cdot x) = \varphi(t) \cdot f(x)$ for every $t \in (\mathbb{Z}_2)^n$ and $x \in M_1$ and f covers the identity on P^n. We can see that two orientable small covers $M(\lambda_1)$ and $M(\lambda_2)$ over P^n are D-J equivalent if and only if there is $\sigma \in GL(n, \mathbb{Z}_2)$ such that $\lambda_1 = \sigma \circ \lambda_2$.

2.2. Stanley-Reisner Face Ring and Ordinary Cohomology

Let P^n be a simple convex polytope with $\mathcal{F}(P^n) = \{F_1, \cdots, F_l\}$. Following [5], the Stanley-Reisner face ring of P^n over $\mathbb{Z}_2$, denoted $\mathbb{Z}_2(P^n)$, is defined as follows:

$$\mathbb{Z}_2(P^n) = \mathbb{Z}_2[F_1, \cdots, F_l]/I$$

where the $F_i^{,}s$ are regarded as indeterminates of degree one, and I is a homogenous ideal generated by all square free monomials of the form $F_{i_1}, \cdots, F_{i_s}$ with $F_{i_1} \bigcap \cdots \bigcap F_{i_s} = \emptyset$. Let $\pi : M^n \to P^n$ be a small cover over a simple convex polytope P^n with $\mathcal{F}(P^n) = \{F_1, \cdots, F_l\}$, and $\lambda : \mathcal{F}(P^n) \longrightarrow (\mathbb{Z}_2)^n$ its $(\mathbb{Z}_2)^n$-coloring. Now we extend $\lambda : \mathcal{F}(P^n) \longrightarrow (\mathbb{Z}_2)^n$ to a linear map $\widetilde{\lambda} : (\mathbb{Z}_2)^l \longrightarrow (\mathbb{Z}_2)^n$ by replacing $\{F_1, \cdots, F_l\}$ by the basis $\{e_1, \cdots, e_l\}$ of $(\mathbb{Z}_2)^l$. Then $\widetilde{\lambda} : (\mathbb{Z}_2)^l \longrightarrow (\mathbb{Z}_2)^n$ is surjective, and $\widetilde{\lambda}$ can de regarded as an $n \times l$-matrix (λ_{ij}), which is written as follows:

$$\{\lambda(F_1), \cdots, \lambda(F_l)\}.$$

One knows that $H_1(B\mathbb{Z}_2^l; \mathbb{Z}_2) = H_1(E\mathbb{Z}_2^n \times_{\mathbb{Z}_2^n} M^n; \mathbb{Z}_2) = \mathbb{Z}_2^l$ and $H_1(B\mathbb{Z}_2^n; \mathbb{Z}_2) = \mathbb{Z}_2^n$. So we have that $p_* : H_1(E\mathbb{Z}_2^n \times_{\mathbb{Z}_2^n} M^n; \mathbb{Z}_2) \longrightarrow H_1(B\mathbb{Z}_2^n; \mathbb{Z}_2)$ can be identified with $\widetilde{\lambda} : (\mathbb{Z}_2)^l \longrightarrow (\mathbb{Z}_2)^n$, where $p : E\mathbb{Z}_2^n \times_{\mathbb{Z}_2^n} M^n \longrightarrow B\mathbb{Z}_2^n$ is the fibration of the Borel construction associating to the universal principal $\mathbb{Z}_2^n$-bundle $E\mathbb{Z}_2^n \longrightarrow B\mathbb{Z}_2^n$. Furthermore, $p^* : H^1(B\mathbb{Z}_2^n;\mathbb{Z}_2) \longrightarrow H^1(E\mathbb{Z}_2^n \times_{\mathbb{Z}_2^n} M^n;\mathbb{Z}_2)$ is identified with the dual map $\widetilde{\lambda}^* : \mathbb{Z}_2^{n*} \longrightarrow \mathbb{Z}_2^{l*}$, where $\widetilde{\lambda}^* = \widetilde{\lambda}^T$ as matrices. Therefore, column vectors of $\widetilde{\lambda}^*$ can be understood as linear combinations of $F_1, \cdots, F_l$ in the face ring $\mathbb{Z}_2(P^n) = \mathbb{Z}_2[F_1, \cdots, F_l]/I$. Write

$$\lambda_i = \lambda_{i1}F_1 + \cdots + \lambda_{il}F_l.$$

Let J_λ be the homogeneous ideal $(\lambda_1, \cdots, \lambda_n)$ in $\mathbb{Z}_2[F_1, \cdots, F_l]$. Davis and Januszkiewicz calculated the ordinary cohomology of M^n, which is stated as follows.

Theorem 2.4 (Davis-Januszkiewicz) : Let $\pi : M^n \to P^n$ be a small cover over a simple convex polytope P^n. Then its ordinary cohomology

$$H^*(M^n;\mathbb{Z}_2) \cong \mathbb{Z}_2[F_1, \cdots, F_l]/I + J_\lambda.$$

2.3. Cohomological Rigidity for Small Covers Over Prisms

In [2], X. Cao and Z. L*ü* introduced sector method to obtain the following result:

Theorem 2.5 : Two orientable small covers $M(\lambda_1)$ and $M(\lambda_2)$ over $P^3(m)$ are homeomorphic if and only if their cohomologies $H^*(M(\lambda_1);\mathbb{Z}_2)$ and $H^*(M(\lambda_2);\mathbb{Z}_2)$ are isomorphic as rings.

By c and f we denote the top and bottom facets of $P^3(m)$ respectively, and by $s_1, \cdots, s_m$ we denote all sided facets of $P^3(m)$ in their general order. An orientable coloring on $P^3(m)$ will simply be described as a sequence by writing its sided facet colorings in order.

Definition 2.6 : An orientable coloring λ on $P^3(m)$ is said to be 2-independent if all $\lambda(s_i), i = 1, \cdots, m$, span a 2-dimensional subspace of $(\mathbb{Z}_2)^3$; otherwise it is said to be 3-independent. If $\lambda(c) = \lambda(f)$, then λ is said to be trivial; otherwise nontrivial.

Applying two sectors to the trivial orientable coloring λ, Cao and L*ü* give the following two operations on its coloring sequence $(\lambda(s_1), \cdots, \lambda(s_m))$ without changing the homeomorphism type and orientability of the small cover constructed from λ:

- O_1 Take two sided facets $s_i, s_j (i < j)$ with the same coloring and then reflect the coloring sequence of $s_i, s_{i+1}, \cdots, s_j$.
- O_2 Take $s_i, s_j (i < j)$ with $\lambda(s_i), \lambda(s_j), \lambda(c) (i.e. \lambda(f))$ independent, then reflect the coloring sequence of $s_i, s_{i+1}, \cdots, s_j$ and do a linear transform $(\lambda(s_i), \lambda(s_j), \lambda(c)) \longrightarrow (\lambda(s_j), \lambda(s_i), \lambda(c))$ to change the reflected coloring sequence.

By applying these two operations on the coloring sequence $(\lambda(s_1), \cdots, \lambda(s_m))$, we can reduce a colored polytope $(P^3(m), \lambda)$ to $(P^3(m), \lambda')$. In this case, $(P^3(m), \lambda)$ and $(P^3(m), \lambda')$ are said to be sector-equivalent.

3. The Number of Orientable Small Covers

First let us consider the case in which the simple convex polytope is a prism $P^3(m)$. The argument is divided into two cases: (I) λ is trivial; (II) λ is nontrivial.

3.1. Trivial Orientable Colorings

Let $\{e_1, e_2, e_3\}$ be a basis of $(Z_2)^3$. Given a colored polytope $(P^3(m), \lambda)$, throughout suppose that λ is trivial with $\lambda(c) = \lambda(f) = e_1$. We have

Lemma 3.1 : If the trivial orientable coloring λ is 2-independent, then m is even and λ is sector-equivalent to the canonical form λ_{C_1} with the coloring sequence $C_1 = (e_2, e_3, \cdots, e_2, e_3)$.

Proof : Since the trivial orientable coloring λ is 2-independent, we only can choose two colors from $\{e_2, e_3, e_1 + e_2 + e_3\}$ up to D-J equivalence. So λ is unique up to D-J equivalence and m is even.

Next we consider the case in which λ is 3-independent.

Lemma 3.2 : If the trivial orientable coloring λ is 3-independent, then when m is even, λ is sector-equivalent to λ_{C_2} with the coloring sequence $C_2 = (e_1 + e_2 + e_3, e_2, e_3, e_2, \cdots, e_3, e_2)$ and when m is odd, λ is sector-equivalent to λ_{C_3} with the coloring sequence $C_3 = (e_1 + e_2 + e_3, e_2, e_3, \cdots, e_2, e_3)$.

Proof : Since λ is 3-independent, $e_2, e_3, e_1 + e_2 + e_3$ must wholly appear in its coloring sequence up to D-J equivalence. One may assume that the time number r of $e_1 + e_2 + e_3$ appearing in the coloring sequence of λ is less than $\frac{m}{2}$ up to D-J equivalence. By the definition of λ, we easily see that any two $e_1 + e_2 + e_3^{,}s$ in the coloring sequence cannot become neighbors. Let $e_1 + e_2 + e_3, x_1, \cdots, x_r, e_1 + e_2 + e_3, y$ with $x_i, y \neq e_1 + e_2 + e_3$ be a subsequence of the coloring sequence. If $r > 1$, we proceed as follows:

(1) when $x_1 = y$, by doing the operation O_1 on $x_1, \cdots, x_r, e_1 + e_2 + e_3, y$, we may only change the subsequence $e_1 + e_2 + e_3, x_1, \cdots, x_r, e_1 + e_2 + e_3, y$ into $e_1 + e_2 + e_3, y, e_1 + e_2 + e_3, x_r, \cdots, x_1$ in the coloring sequence, and the value of r is unchanged.

(2) when $x_1 \neq y$, with no loss suppose that $x_1 = e_2, y = e_3$ and x_1, y, e_1 are linearly

independent. Then by doing the operation O_2 on $x_1 = e_2, x_2, \cdots, x_r, e_1 + e_2 + e_3, y = e_3$, we may only change the subsequence $e_1 + e_2 + e_3, e_2, x_2, \cdots, x_r, e_1 + e_2 + e_3, e_3$ into $e_1 + e_2 + e_3, e_2, e_1 + e_2 + e_3, x'_r, \cdots, x'_2, e_3$ with $x'_i \neq e_1 + e_2 + e_3$, and the value of r is unchanged.

Thus, we may reduce the orientable coloring λ to another orientable coloring with the following coloring sequence

$$(e_1 + e_2 + e_3, y_1, e_1 + e_2 + e_3, y_2, \cdots, e_1 + e_2 + e_3, y_{s-1}, e_1 + e_2 + e_3, y_s, z_1, \cdots, z_{m-2s}) \text{ with } m - 2s > 0. \quad (3.1)$$

With no loss, one may assume that $y_{s-1} = e_2$. If $y_s = e_3$, by doing the operation O_2 on $e_1+e_2+e_3, y_{s-1}, e_1+e_2+e_3, y_s$, one may change $e_1+e_2+e_3, y_{s-1}, e_1+e_2+e_3, y_s$ into $e_1 + e_2 + e_3, y_s, y_{s-1}, , y_s$. Thus the coloring sequence (3.1) can be reduced to $(e_1 + e_2 + e_3, y_1, e_1+e_2+e_3, y_2, \cdots, e_1+e_2+e_3, y_{s-2}, e_1+e_2+e_3, y_s, y_{s-1}, , y_s, z_1, \cdots, z_{m-2s})$. If $y_s \neq e_3$, then $y_s = e_2$. By doing the operation O_2 on $e_1+e_2+e_3, y_{s-1}, e_1+e_2+e_3, y_s, z_1 (z_1 = e_3)$, one may change $e_1+e_2+e_3, y_{s-1}, e_1+e_2+e_3, y_s, z_1$ into $e_1+e_2+e_3, y_s, z_1, y_{s-1}, z_1$. So we have managed to reduce the number r of $e_1 + e_2 + e_3's$ by 1. We continue this process until we reach $s = 1$.

Thus, when m is even, up to D-J equivalence λ is unique, as desired. When m is odd, the situation is similar.

3.2. Nontrivial Orientable Colorings

Here let $\{e_1, e_2, e_3\}$ be also a basis of $(\mathbb{Z}_2)^3$. Given a colored polytope $(P^3(m), \lambda)$, throughout suppose that λ is nontrivial, i.e. $\lambda(c) \neq \lambda(f)$. Without loss of generality, suppose $\lambda(c) = e_1$ and $\lambda(f) = e_2$. We have

Lemma 3.3 : Let λ be a nontrivial orientable coloring on $P^3(m)$. Then m is even and λ is sector-equivalent to λ_{C_4} with the coloring sequence $C_4 = (e_3, e_1+e_2+e_3, \cdots, e_3, e_1 + e_2 + e_3)$.

Proof : Since λ is a nontrivial orientable coloring, it is easy to see that λ is unique up to D-J equivalence and m is even.

So when m is odd, there is only a homeomorphism class of orientable small covers over $P^3(m)$. And when m is even, there are 3 homeomorphism classes of orientable small covers over $P^3(m)$ at most. Below we use cohomological rigidity for small covers over prisms to distinguish three canonical forms $\lambda_{C_1}, \lambda_{C_2}$ and λ_{C_4}.

Given a colored polytope $(P^3(m), \lambda)$, we knows that the mod 2 cohomology ring of

$M(\lambda)$ is

$$H^*(M(\lambda);\mathbb{Z}_2)=\mathbb{Z}_2[c,f,s_1,\cdots,s_m]/I+J_\lambda$$

where I is the ideal generated by cf and s_is_j with $s_i\bigcap s_j=\emptyset$, and J_λ is the ideal generated by three linear relations (determined by the $3\times(m+2)$ matrix $(\lambda(c),\lambda(f),\lambda(s_1),\cdots,\lambda(s_m)))$.

Lemma 3.4 : When $m\neq 4$, arbitrary two small covers of $M(\lambda_{C_1}),M(\lambda_{C_2})$ and $M(\lambda_{C_4})$ aren't homeomorphic.

When $m=4$, $M(\lambda_{C_1})$ and $M(\lambda_{C_2})$ are't homeomorphic and $M(\lambda_{C_1})$ and $M(\lambda_{C_4})$ are't homeomorphic, but $M(\lambda_{C_2})$ and $M(\lambda_{C_4})$ are homeomorphic.

Proof : First we consider the case $m\neq 4$. $f^2=c^2=0$ in $H^*(M(\lambda_{C_1});\mathbb{Z}_2)$, but $f^2,c^2\neq 0$ in $H^*(M(\lambda_{C_2});\mathbb{Z}_2)$, so $H^*(M(\lambda_{C_1});\mathbb{Z}_2)$ and $H^*(M(\lambda_{C_2});\mathbb{Z}_2)$ aren't isomorphic as rings. Therefore, by Theorem 2.5, $M(\lambda_{C_1})$ and $M(\lambda_{C_2})$ are't homeomorphic.

In $H^*(M(\lambda_{C_1});\mathbb{Z}_2)$ $\sum\limits_{i\ is\ odd}s_i=\sum\limits_{i\ is\ even}s_i=0$, but $\sum\limits_{i\ is\ odd}s_i\neq 0$ and $\sum\limits_{i\ is\ even}s_i\neq 0$ in $H^*(M(\lambda_{C_4});\mathbb{Z}_2)$, thus $H^*(M(\lambda_{C_1});\mathbb{Z}_2)$ and $H^*(M(\lambda_{C_4});\mathbb{Z}_2)$ aren't isomorphic as rings. So $M(\lambda_{C_1})$ and $M(\lambda_{C_4})$ are't homeomorphic.

In $H^*(M(\lambda_{C_2});\mathbb{Z}_2)$ $f^2,c^2\neq 0$, but $f^2=c^2=0$ in $H^*(M(\lambda_{C_4});\mathbb{Z}_2)$, so $H^*(M(\lambda_{C_2});\mathbb{Z}_2)$ and $H^*(M(\lambda_{C_4});\mathbb{Z}_2)$ aren't isomorphic as rings. Therefore, $M(\lambda_{C_2})$ and $M(\lambda_{C_4})$ are't homeomorphic.

In the similar way, we also can give the proof of the case m=4.

Combining Lemma 3.1-3.4, we give the proof of Theorem 1.1.

Finally, we consider the case in which the simple convex polytope is a m-gon $P^2(m)$. Let e_1,e_2 be a basis of $(\mathbb{Z}_2)^2$. Here an orientable coloring on $P^2(m)$ will be also described as a sequence by writing its edge colorings in order.

The proof of Theorem 1.2 : One knows that an orientable coloring of $P^2(m)$ is just a 2-coloring of $P^2(m)$. Thus when m is odd, there aren't orientable colorings on $P^2(m)$ and there aren't orientable small covers over $P^2(m)$.

When m is even, it is easy to see that up to D-J equivalence, the orientable coloring λ is unique and sector-equivalent to λ_{C_5} with the coloring sequence $C_5=(e_1,e_2,\cdots,e_1,e_2)$. Thus, there is only an orientable small cover over $P^2(m)$ up to homeomorphism.

References

[1] Cai M., Chen X., Lü Z., Small covers over prisms, Topology Appl., 154 (2007), 2228-2234.

[2] Cao X., Lü Z., Cohomological rigidity and the number of homeomorphism types for small covers over prisms, Topology Appl., 158 (2011), 813-834.

[3] Choi S., The number of small covers over cubes, Algebr. Geom. Topol., 8 (2008), 2391-2399.

[4] Choi S., The number of orientable small covers over cubes, Proc. Japan Acad., Ser. A., 86 (2010), 97-100.

[5] Davis M., Januszkiewicz T., Convex polytopes, Coxeter orbifolds and torus actions, Duke Math J., 62 (1991), 417-451.

[6] Garrison A., Scott R., Small covers of the dodecahedron and the 120-cell, Proc. Amer. Math. Soc., 131 (2002), 963-971.

[7] Nakayama H., Nishimura Y., The orientability of small covers and coloring simple polytopes, Osaka J. math., 42 (2005), 243-256.