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A NOTE ON SPLIT EDGE DOMINATION NUMBER OF A GRAPH

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Abstract

A set F of a graph $G(V,E)$ is an edge dominating set if every edge in $E-F$ is adjacent to some edge in F. An edge domination number $\gamma^{'}(G)$ of G is the minimum cardinality of an edge dominating set. An edge dominating set F is called a split edge dominating set if the induced subgraph $\langle E-F\rangle$ is disconnected. The minimum cardinality of the split edge dominating set in G is its domination number and is denoted by $\gamma_s^{'}(G)$. We investigate several properties of split edge dominating sets and give some bounds on the split edge domination number.

1. Introduction

Let $G(V,E)$ be a graph with $p=\mid V\mid$ and $q=\mid E\mid$ denoting the number of vertices and

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edges respectively. All the graphs considered here are finite, non-trivial, undirected and connected without loops or multiple edges.

The degree of a vertex u is denoted by $d(u)$. The degree of an edge $e = uv$ of a graph G is the number defined by $deg(e) = degu + degv - 2$. An edge $e = uv$ is called an universal edge if $d(e) = q - 1$. The minimum(maximum) degree of an edge is denoted by $\delta^{'}(\Delta^{'})$. The induced subgraph of $X \subseteq E$ is denoted by $\langle X \rangle$. For a real number x, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x and $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. The edge independence number $\beta_1(G)$ is defined to be the number of edges in a maximum independent set of edges of G. A vertex of degree one is called a pendant vertex.

Let $G(V, E)$ be a connected graph. A subset S of V is called a dominating set of G if every vertex in $V - S$ is adjacent to at least one vertex in S. The concept of edge domination was introduced by Mitchell and Hedetniemi ([5], [6]). A subset F of E is called an edge dominating set of G if every edge not in F is adjacent to some edge in F. The minimum cardinality of an edge dominating set of G is called an edge domination number and is denoted by $\gamma^{'}(G)$.

A dominating set D of G is a split dominating set if the induced subgraph $\langle V - D \rangle$ is disconnected. The split domination number $\gamma_s(G)$ of G is the minimum cardinality of a split dominating set. This concept was introduced by Kulli, Janakiram in [8]. Any undefined term or notation in this paper can be found in Harary ([4], [9]). We need the following theorems.

Theorem 1.1 [10] : For any graph G with an end edge, $\gamma_s^{'} = \gamma^{'}(G)$. Furthermore, there exists a $\gamma_s^{'}$-set of G containing all edges adjacent to end-edges.

Theorem 1.2 [2] : For any connected graph G of even order p, $\gamma^{'}(G) = p/2$ if and only if G is isomorphic to K_p or $K_{p/2,p/2}$.

Theorem 1.3 [3] : For every n, $\gamma_{3,n}^{'} = n$.

In this paper we study the split edge domination number of a graph characterizing the problem for certain class of graphs.

2. Split Edge Domination Number of a Graph

Definition 2.1 : A set $F \subseteq E(G)$ is said to be split edge dominating set if F is an edge dominating set and induced subgraph $\langle E - F \rangle$ is disconnected. The minimum cardinality of split edge dominating set in G is the split edge domination number and is denoted by $\gamma_s^{'}(G)$ of G. This concept was introduced by K. M. Yogeesha, N. D.Soner, and Sirous Ghobadi [10] and later extended by V. R. Kulli and Radha Rajamani Iyer [7].

Now we study on split edge domination of subdivision of some standard graphs.

Theorem 2.2 [10] : For any path P_p with $p \geq 5$, the split edge domination number,

$$\gamma_s^{'}(P_p) = \begin{cases} \frac{p}{3} & \text{if } p \equiv 0(mod3), \\ \left\lfloor \frac{p}{3} \right\rfloor & \text{if } p \equiv 1(mod3), \\ \left\lfloor \frac{p}{3} \right\rfloor + 1 & \text{if } p \equiv 2(mod3). \end{cases}$$

Proof : Let $P_p = \{v_1, v_2, ..., v_p\}$ be any path and let $e_i = v_i v_{i+1}$ be an edge on P_p. Let

$$S_1 = \begin{cases} S & \text{if } p \equiv 0 \text{ or } 1(mod3) \ , \\ S \cup \{e_{p-1}\} & \text{if } p \equiv 2(mod3). \end{cases}$$

be an edge set on G where $S = \{e_j : j = 3k+2 \text{ for } 0 \leq k \leq \lfloor \frac{p}{3} \rfloor - 1\}$. Clearly S_1 is a split edge dominating set and $|S_1|$ will be the split edge domination number with minimum cardinality. Hence the proof. □

Theorem 2.3 [10] : For any cycle C_p with $p \geq 5$, the split edge domination number,

$$\gamma_s^{'}(C_p) = \begin{cases} \frac{p}{3} & \text{if } p \equiv 0(mod3), \\ \left\lfloor \frac{p}{3} \right\rfloor + 1 & \text{if } p \equiv 1 \text{ or } 2(mod3). \end{cases}$$

Proof : Let $C_p = \{v_1, v_2, ..., v_p\}$ be any cycle and let $e_i = v_i v_{i+1}$ be an edge on C_p. Let $S = \{e_j : j = 3k + 1 \text{ for } 0 \leq k \leq \lceil \frac{p}{3} \rceil - 1\}$ be any edge set on C_p. Clearly S is a split edge dominating set and $|S|$ will be its split edge domination number with minimum cardinality. Hence the proof. □

Theorem 2.4 : For any path P_p , $\gamma_s^{'}(P_p) + \gamma_s^{'}(S(P_p)) \leq p$. Equality holds for $p \equiv 0$ or $2(mod3)$.

Proof : Let S be the $\gamma_s^{'}$ set of P_p. Therefore by Theorem 2.2,

$$\gamma_s^{'}(P_p) = |S| \tag{1}$$

If $S^{'}$ is the $\gamma_s^{'}$ set of subdivision of P_p, then by Theorem 2.2 we get,

$\gamma_s^{'}(S(P_p)) = |S^{'}|$ that is

$$\gamma_s^{'}(S(P_p)) = \begin{cases} \frac{2p}{3} & \text{if } p \equiv 0(mod3), \\ \left\lfloor \frac{2p}{3} \right\rfloor + 2 & \text{if } p \equiv 1(mod3), \\ \left\lfloor \frac{2p}{3} \right\rfloor + 1 & \text{if } p \equiv 2(mod3). \end{cases} \tag{2}$$

Consider a particular case where $p \equiv 1(mod3)$. Adding (1) and (2), we get

$$\begin{aligned} \gamma_s^{'}(P_p) + \gamma_s^{'}(S(P_p)) &= |S| + |S^{'}|, \\ &= \left\lfloor \frac{p}{3} \right\rfloor + \left\lfloor \frac{2p}{3} \right\rfloor + 4, \\ &\leq p + 3. \end{aligned}$$

The other two cases are obvious. Hence the proof. □

Theorem 2.5 : For any cycle C_p , $\gamma_s^{'}(C_p) + \gamma_s^{'}(S(C_p)) \leq p+1$. Equality holds for $p \equiv 1$ or $2(mod3)$.

Proof : Let S be the $\gamma_s^{'}$ set of C_p. Therefore by Theorem 2.3,

$$\gamma_s^{'}(C_p) = |S| \tag{3}$$

If $S^{'}$ is the $\gamma_s^{'}$ set of subdivision of C_p, then by Theorem 2.3 we have

$$\gamma_s^{'}(S(C_p)) = \begin{cases} \frac{2p}{3} & \text{if } p \equiv 0(mod3), \\ \left\lfloor \frac{2p}{3} \right\rfloor + 1 & \text{if } p \equiv 1 \text{ or } 2(mod3). \end{cases} \tag{4}$$

Consider a particular case where $p \equiv 1(mod3)$. Adding (3) and (4), we get

$$\begin{aligned} \gamma_s^{'}(C_p) + \gamma_s^{'}(S(C_p)) &= |S| + |S^{'}|, \\ &= \left\lfloor \frac{p}{3} \right\rfloor + 1 + \left\lfloor \frac{2p}{3} \right\rfloor + 1, \\ &\leq p + 1. \end{aligned}$$

The other two cases are obvious. Hence the proof. □

3. Bounds on the Split Edge Domination Number

Theorem 3.1 : For any connected graph G with $q \geq 3$, $\gamma_s^{'}(G) \geq 1$. Equality holds if and only if there exists only one cutset e in G with degree $q-1$.

Proof : Suppose $\gamma_s^{'}(G) = 1$ and $S = \{e\}$ is a cut-set of G. Clearly $E - S$ is disconnected and e dominates all the other edges of G. Hence e is a cut-set with degree $q-1$. Suppose there exists another cut-set e_1 of degree $q-1$. Then e_1 is adjacent to all the remaining edges of G. In this case $\langle E-S\rangle$ is connected, a contradiction to $S=\{e\}$ is a $\gamma_s^{'}$-set of G.

Converse part is obvious. Hence the proof. □

Theorem 3.2 : Let G be any graph with $\delta(G)>1$ and e be an edge in a graph G with degree k such that $\langle N(e)\rangle$ is disconnected. Then split edge domination number, $\gamma_s^{'}(G)\leq q-k$.

Proof : If e is an edge of degree k and $\langle N(e)\rangle$ is disconnected, then $E-N(e)$ is a split edge dominating set. Therefore $|E-N(e)|>\gamma_s^{'}(G)$. Hence the proof. □

Theorem 3.3 : If F is a $\gamma_s^{'}$-set of a graph G, then $E-F$ is a dominating set of G and hence $\gamma^{'}(G)+\gamma_s^{'}(G)\leq q$.

Proof : Suppose $E-F$ is not a dominating set of a graph G, then there exists an edge e in F which is not adjacent to any of the edges in $E-F$. Thus by Theorem **??** $F-\{e\}$ is a split edge dominating set of G, a contradiction to the minimality of F. Further $E-F$ is a dominating set of G and so $|E-F|\geq\gamma^{'}(G)$. Hence the proof. □

Theorem 3.4 : For any path P_p with $p\geq 6, \gamma_s^{'}(\overline{P}_p)=2(p-4)$.

Proof : Consider an edge $e=uv$ with minimum degree in $\overline{P}_p$. Clearly the set $F=N(e)$ is an edge dominating set of $\overline{P}_p$. Also $\langle E(\overline{P}_p)-F\rangle$ is a disconnected graph with two components K_2 and $\overline{G}_{p-2}$. Thus F itself is a split edge dominating set of $\overline{P}_p$. Therefore

$$\begin{aligned}\gamma_s^{'}(\overline{P}_p)&=|F|,\\&=|N(e)|,\\&=d(u)+d(v)-2,\\&=2p-8.\end{aligned}$$

Hence the proof. □

Theorem 3.5 : For any Cycle C_p with $p\geq 6, \gamma_s^{'}(\overline{C}_p)=2(p-4)$.

Proof of this Theorem is similar to the above one.

Theorem 3.6 : If F is an independent edge dominating set of a graph G with $|F|>1$, then $E-F$ is a split edge dominating set of G. In particular $\gamma_s^{'}(G)+\beta_1(G)\leq q$.

Proof : Since F is an independent edge dominating set of $G, E-F$ is a edge dominating set of G. Further $\langle F\rangle=\langle E-(E-F)\rangle$ is disconnected. Hence $E-F$ is a split edge dominating set of G. This gives that $|E-F|\geq\gamma_s^{'}(G)$. In particular, if $|F|=\beta_1(G)$, then $q - \beta_1(G) \geq \gamma_s^{'}(G)$ and hence the proof. □

Proposition 3.7 : Let F be a $\gamma_s^{'}-$set of a graph G. If $E - F$ is a split dominating set of G, then $\gamma_s^{'}(G) \leq \frac{q}{2}$.

Proof : Since $E - F$ is a split dominating set of $G, |E - F| \geq \gamma_s^{'}(G)$ and hence proposition follows. □

4. Split Edge Domination Number on Corona and Join of Graphs

Here we discuss the results on split edge domination number of corona and join of two graphs. The corona $G = H \circ K_1$ is a graph constructed from a copy of H, where for each vertex $v \in V(H)$, a new vertex $v^{'}$ and a pendant edge $vv^{'}$ are added. The following Theorem gives a sharp bound for the cototal edge domination number of $(G_{p_1} \circ G_{p_2})$.

Theorem 4.1 : Let G be a connected graph with p_1 vertices and K_{1,p_2} be any star. The Split edge domination number of the corona of G and K_{1,p_2}, $\gamma_s^{'}(G \circ K_{1,p_2}) = p_1 + \delta(G)$.

Proof : Let $u_i \in V_1$ for $1 \leq i \leq p_1$ be a vertex set of G and v be a support vertex of K_{1,p_2}. Let $S = \{u_i v : \forall u_i \in G\}$ be an edge set in the corona $(G \circ K_{1,p_2})$. Clearly S is an edge dominating set. Let $E_1 = \{e_1, e_2, \cdots$ be the set of edges incident on a vertex v_1 where v_1 is one among the vertex with minimum degree in G. Then $F = S \cup E_1$ is a split edge dominating set where the induced subgraph $\langle E - F \rangle$ is disconnected. Hence F is a split edge dominating set with minimum cardinality. Therefore $\gamma_s^{'}(G \circ K_{1,p_2}) = p_1 + \delta(G)$. Hence the proof. □

Theorem 4.2 : Let $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ be two connected graphs. The split edge domination number of the corona of G_1 and G_2 is, $\gamma_s^{'}(G_1 \circ G_2) \leq p_1 + p_1\lfloor \frac{p_2}{2} \rfloor + \delta(G_1)$.

Proof : Let $u_i \in V_1$ for $1 \leq i \leq p_1$ and let $v_j \in V_2$ for $1 \leq j \leq p_2$ be the vertex set of G_1 and G_2 respectively. Let $S = \{u_i v_k : \forall u_i \in G_1$ and any one vertex $v_k \in V_2\}$ in the corona $(G_1 \circ G_2)$. Let E_1 be an edge set incident on a vertex v_1 where v_1 is one among the vertex with minimum degree in G_1 and E_2 be an edge dominating set for the graph G_2. Then set S along with p_1 copies of E_2 in the corona will form an edge dominating set F of $(G_1 \circ G_2)$ and $F = S \cup E_2 \cup E_1$ forms a split edge domination set of $(G_1 \circ G_2)$ with minimum cardinality. Therefore

$$\begin{aligned} \gamma_s^{'}(G_1 \circ G_2) &\leq |S| + p_1(|E_2|) + |E_1|, \\ &\leq p_1 + p_1\left\lfloor \frac{p_2}{2} \right\rfloor + \delta(G_1). \end{aligned}$$

Hence the proof. □

Theorem 4.3 : Let G be a graph with $\delta(G) = 1$. Then split edge domination number, $\gamma_s^{'}(G) = p - 1$ if and only if $G = H \circ K_1$ where H is a complete graph.

Proof : Consider an edge $e = uv$ where $u \in K_p$ and $v \in K_1$ of G. Clearly $F = N(e)$ is an edge dominating set of $G = H \circ K_1$. Also $G - F$ is a disconnected graph with two components. Thus F itself is a split edge dominating set of G with minimum cardinality. Therefore

$$\begin{aligned} \gamma_s^{'}(G) &= |F|, \\ &= |N(e)| = d(u), \\ &= p - 1. \end{aligned}$$

Hence the proof. □

The forthcoming Theorems gives a result on the join of some standard graphs.

Now we define the join of two graphs. For disjoint graphs G_1 and G_2, the join $G = G_1 + G_2$ is the graph with $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1) \wedge v \in V(G_2)\}$. The forth coming Theorems gives the result on the join of some standard graphs.

Theorem 4.4 : Let C_{p_1} and C_{p_2} be any two cycles of order p_1 and p_2 respectively. Then the split edge domination number, $\gamma_s^{'}[C_{p_1} + C_{p_2}] = p_1 + p_2 + 2$.

Proof : Let C_{p_1} and C_{p_2} be the two cycles labelled in order as $u_1 e_1 u_2 e_2 \cdots u_{p_1}$ $e_{p_1} u_1$ and $v_1 e_1^{'} v_2 e_2^{'} \cdots v_{p_2} e_{p_2}^{'} v_1$ respectively. Let $e = u_i v_j$ be an edge in $E(G)$. Let $E_1 = \{uv : u \text{ or } v \neq u_i \text{ and } u \text{ or } v \neq v_j\}$ be the set of edges in $E(G)$. Then $E - \{E_1 \cup \{e\}\}$ forms a split edge dominating set in the join of the two cycles C_{p_1} and C_{p_2} respectively. Thus

$$\begin{aligned} \gamma_s^{'}[C_{p_1} + C_{p_2}] &\leq |E| - \Big[|E_1| + 1\Big], \\ &\leq p_1 + 1 + p_2 + 1, \\ &\leq p_1 + p_2 + 2. \end{aligned}$$

Equality is obvious. Hence the proof. □

Theorem 4.5 : Let P_{p_1} and P_{p_2} be any two paths of order p_1 and p_2 respectively. Then the split edge domination number, $\gamma_s^{'}[P_{p_1} + P_{p_2}] = p_1 + p_2$.

Proof : Let P_{p_1} and P_{p_2} be the two paths labelled in order as $u_1 e_1 u_2 e_2$ $\cdots e_{p-1} u_{p_1}$ and $v_1 e_1^{'} v_2 e_2^{'} \cdots e_{p_2-1}^{'} v_{p_2}$ respectively. Let $e = u_i v_j$ be an edge in $E(G)$ where u_i and v_j are the pendant vertices of P_{p_1} and P_{p_2} respectively. Let $E_1 = \{uv : u$

or $v \neq u_i$ and u or $v \neq v_j\}$ be the set of edges in $E(G)$. Then $E - \{E_1 \cup \{e\}\}$ forms a split edge dominating set in the join of the two paths P_{p_1} and P_{p_2}. Thus

$$\gamma_s^{'}[P_{p_1} + P_{p_2}] \leq |E| - \Big[|E_1| + 1\Big],$$
$$\leq p_1 + p_2.$$

Equality is obvious. Hence the proof. □

Theorem 4.6 : Let P_{p_1} be a path and C_{p_2} be a cycle of order p_1 and p_2 respectively. Then the split edge domination number, $\gamma_s^{'}[P_{p_1} + C_{p_2}] = p_1 + p_2 + 1$.

Proof : Let P_{p_1} and C_{p_2} be a path and a cycle labelled in order as $u_1 e_1 u_2 e_2 \cdots e_{p-1} u_{p_1}$ and $v_1 e_1^{'} v_2 e_2^{'} \cdots e_{p_2-1}^{'} v_{p_2-1} e_{p_2}^{'} v_{p_2}$ respectively. Let $e = u_i v_j$ be an edge in $E(G)$ where u_i is a pendant vertex of the path P_{p_1} and v_j be any vertex in the cycle C_{p_2}. Let $E_1 = \{uv : u$ or $v \neq u_i$ and u or $v \neq v_j\}$ be the set of edges in $E(G)$. Then $E - \{E_1 \cup \{e\}\}$ forms a split edge dominating set in the join of a path P_{p_1} and a cycle C_{p_2}. Thus

$$\gamma_s^{'}[P_{p_1} + C_{p_2}] \leq |E| - \Big[|E_1| + 1\Big],$$
$$\leq p_1 + p_2 + 1.$$

Equality is obvious. Hence the proof. □

5. Adding an End Edge

In this section we observe some properties of graphs obtained by adding K_2 to a cycle C_p. If $e = uv$ is and edge of a graph G with $deg(u) = 1$ and $deg(v) > 1$, then e is called an end edge and u an end vertex.

Theorem 5.1 : Let $G^{'}$ be the graph obtained by adding k end edges $u_1 v_j$ for $j = 1, 2, \cdots k$ to a cycle C_p where $u_1 \in C_p$ and $\{v_1, v_2, \cdots v_k\} \notin C_p$. Then the split edge domination number, $\gamma_s^{'}(G^{'}) = \lceil \frac{p}{3} \rceil$.

Proof : Let $C_p = \{u_1, u_2, \cdots, u_p\}$ be a cycle with p vertices and $G^{'}$ be the graph obtained by adding k end edges $\{u_1 v_1, u_1 v_2, \cdots u_1 v_k\}$ such that $u_1 \in C_p$ and $\{v_1, v_2, \cdots v_k\} \notin C_p$. Let $e_i = u_i u_{i+1}$ be an edge on cycle.

Let $S = \{e_j : j = 3l + 1$ for $0 \leq l \leq \lfloor \frac{p}{3} \rfloor - 1\}$ and

$$S_1 = \begin{cases} S & \text{if } p \equiv 0 \text{ or } 2(mod3), \\ S \cup \{e_{p-1}\} & \text{if } p \equiv 1(mod3). \end{cases}$$

be an edge set on S. Then S_1 is an edge dominating set and the induced subgraph $\langle E - S_1 \rangle$ is disconnected. Therefore S_1 is a split edge dominating set and $|S_1|$ will be the split edge domination number for the graph $G^{'}$.

The converse part of the Theorem is obvious. Hence the proof. □

Theorem 5.2 : Let $G^{'}$ be the graph obtained by adding k end edges u_iv_j to a cycle C_p with $d(u_i) \geq 3$ where $u_i \in C_p$ for $i = 1, 2, \cdots p$ and $v_j \notin C_p$ for $j = 1, 2, \cdots k$. Then the split edge domination number, $\gamma_s^{'}(G^{'}) = \lceil \frac{p}{2} \rceil$.

Proof : Let $C_p = \{u_1, u_2, \cdots, u_p\}$ be a cycle and $G^{'}$ be the graph obtained by adding k end edges u_iv_j where $u_i \in C_p$ for $i = 1, 2, \cdots p$ and $v_j \notin C_p$ for $j = 1, 2, \cdots k$. Let $e_i = u_iu_{i+1}$ be an edge of $G^{'}$.

Let $S = \{e_j : j = 2l + 1 \text{ for } 0 \leq l \leq \lfloor \frac{p}{2} \rfloor - 1\}$ and

$$S_1 = \begin{cases} S & \text{if } p \equiv 0 (mod 2), \\ S \cup \{u_pv_1\} & \text{if } p \equiv 1 (mod 2). \end{cases}$$

be an edge set of $G^{'}$. Then S_1 is an edge dominating set and the induced subgraph $\langle E - S_1 \rangle$ is disconnected. Therefore S_1 is a split edge dominating set and $|S_1|$ will be the split edge domination number for the graph $G^{'}$.

The converse part of the Theorem is obvious. Hence the proof. □

Corollary 5.3 : Let $G^{'}$ be the graph obtained by adding k end edges u_iv_j to a cycle C_p of order $p \geq 3$ in any manner. Then from the above Theorems $\lceil \frac{p}{3} \rceil \leq \gamma_s^{'}(G^{'}) \leq \lceil \frac{p}{2} \rceil$.

6. Cartesian Product of Split Edge Domination Number of a Graph

In this section we define "Independent split edge domination number" a new parameer of a graph. An edge dominating set F is called an independent edge dominating set if no two edges of F are adjacent [1]. The Independent edge domination number $\gamma_i^{'}(G)$ of G is the minimum cardinality taken over all independent edge dominating sets of G. The split edge dominating set is said to be an independent split edge dominating set if the induced subgraph $\langle F \rangle$ is an independent edge set.

The Cartesian product of G and H, denoted $G \times H$, has vertex set $V(G) \times V(H)$. Two vertices $(u, v), (u^{'}, v^{'})$ in $V(G) \times V(H)$ are adjacent if either $u = u^{'}$ and $vv^{'} \in E(H)$, or $v = v^{'}$ and $uu^{'} \in E(G)$. The graph $(P_n \times P_m)$ has m copy of the graph P_n in m columns. Let $\gamma_{is}^{'}(P_n \times P_m)$ denotes the size of minimum independent split edge dominating set of two paths $(P_n \times P_m)$ where $n \leq m$. In the sequel (list) we give the values of $\gamma'_{is}(P_n \times P_m)$ and cartesian product for few other graphs.

Theorem 6.1 : Let P_3 be a path of length 2 and P_m be any path with $m \geq 3$. Then Independent split edge dominating number of the cartesian product of two paths, $\gamma'_{is}[P_3 \times P_m] = m + 1$.

Proof : Consider an independent edge set
$S_s = \Big\{\{(i,1),(i,2)\}/i = 1,2,3,\cdots m\Big\}$. Also
$S_i = \Big\{\{(1,4+3k),(1,5+3k)\},\{(2,3+3k),(2,4+3k)\},\{(3,4+3k),(3,5+3k)\} : k = 0,1,2\cdots(\lfloor \frac{m}{3} \rfloor - 2)\Big\}$, see Figure(1).

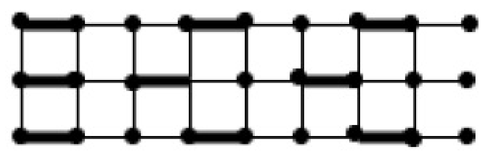

Figure 1: Cartesian product of $P_3 \times P_m$

Let us discuss the following cases:

Case(i) Let $m \equiv 0(mod3)$.
Let $F = S_s \cup S_i \cup \{(2, m-1),(2,m)\}$ is an independent edge dominating set and the induced subgraph $\langle E - F \rangle$ is disconnected. By Theorem 1.3 F is an independent split edge dominating set of $P_3 \times P_m$ with minimum cardinality. Therefore $|F| = |S_s| + |S_i| + 1$.

Case(ii) Let $m \equiv 1(mod3)$.
Let $F_1 = S_s \cup S_i \cup \{\{(1,m-1),(2,m-1)\},\{(2,m),(3,m)\}\}$ is an independent edge dominating set and the induced subgraph $\langle E - F_1 \rangle$ is disconnected. By Theorem 1.3 F_1 is an independent split edge dominating set of $P_3 \times P_m$ with minimum cardinality. Therefore $|F_1| = |S_s| + |S_i| + 2$.

Case(iii) Let $m \equiv 2(mod3)$.
Let $F_2 = S_s \cup S_i \cup \Big\{\{(1,m-1),(1,m)\},\{(2,m-2),(2,m-1)\},\{(3,m-1),(3,m)\}\Big\}$ is an independent edge dominating set and the induced subgraph $\langle E - F_2 \rangle$ is

disconnected. By Theorem 1.3 F_2 is an independent split edge dominating set of $P_3 \times P_m$ with minimum cardinality. Therefore $|F_2| = |S_s| + |S_i|$.

Thus $\gamma'_{is}[P_3 \times P_m] = m + 1$. Hence the proof. □

Theorem 6.2 : Let P_n be a path of length $n-1$ and P_m be any path with $m \geq 3$. Then independent split edge domination number of the cartesian product of two paths,

$$\gamma'_{is}[P_n \times P_m] \leq \begin{cases} \frac{n(m-2)}{3} + n & \text{if } m \equiv 2(mod3), \\ \left\lceil \frac{n(m-2)}{3} \right\rceil + n + 1 & \text{if } m \equiv 0 \text{ or } 1(mod3). \end{cases}$$

Equality holds for $m \equiv 2(mod3)$.

Proof : Consider an independent edge set

$S_s = \{\{(i,1),(i,2)\}/i = 1,2,3,\cdots n\}$.

Also, if n is odd, then

$$S_1 = \begin{cases} \bigcup_{i=1,3,\cdots n} & \{(i,3k+4),(i,3k+5)\} : k = 0,1,\cdots \lfloor \frac{m}{3} \rfloor - 2, \\ \bigcup_{i=2,4,\cdots n-1} & \{(i,3k+3),(i,3k+4)\} : k = 0,1,\cdots \lfloor \frac{m}{3} \rfloor - 2. \end{cases}$$

If n is even, then

$$S_1 = \begin{cases} \bigcup_{i=1,3,\cdots n-1} & \{(i,3k+4),(i,3k+5)\} : k = 0,1,\cdots \lfloor \frac{m}{3} \rfloor - 2, \\ \bigcup_{i=2,4,\cdots n} & \{(i,3k+3),(i,3k+4)\} : k = 0,1,\cdots \lfloor \frac{m}{3} \rfloor - 2. \end{cases}$$

See Figure(2)

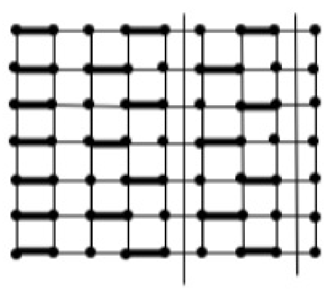

Figure 2

Let us discuss the following cases:

Case(i) If $m \equiv 2(mod3,)$

Let, if n is odd, then

$$S_2 = \begin{cases} \bigcup_{i=1,3,\cdots n} & \{(i,m-1),(i,m)\}, \\ \bigcup_{i=2,4,\cdots n-1} & \{(i,m-2),(i,m-1)\}. \end{cases}$$

If n is even, then

$$S_2 = \begin{cases} \bigcup_{i=1,3,\cdots n-1} & \{(i,m-1),(i,m)\}, \\ \bigcup_{i=2,4,\cdots n} & \{(i,m-2),(i,m-1)\}. \end{cases}$$

Then $F = S_1 \cup S_2 \cup S_s$ forms an independent edge dominating set and the induced subgraph $\langle E - F \rangle$ of $P_n \times P_m$ is disconnected. Thus F is an independent split edge dominating set of $P_n \times P_m$.

$$\begin{aligned} \gamma'_{is}[P_n \times P_m] &\leq |F|, \\ &\leq |S_1| + |S_2| + |S_s|. \end{aligned}$$

Case(ii) If $m \equiv 0(mod3)$, Let

$$S_3 = \begin{cases} \Big\{\{(1,m),(2,m)\},\{(3,m),(4,m)\},\cdots\{(n-2,m),(n-1,m)\}\Big\} & \text{if } n \text{ is odd,} \\ \Big\{\{(1,m),(2,m)\},\{(3,m),(4,m)\},\cdots\{(n-1,m),(n,m)\}\Big\} & \text{if } n \text{ is even.} \end{cases}$$

Then $F_1 = S_1 \cup S_3 \cup S_s$ is an independent edge dominating set and the induced subgraph $\langle E - F_1 \rangle$ of $P_n \times P_m$ is disconnected. Thus $|F_1| \leq |S_1| + |S_3| + |S_s|$. Therefore F_1 is an independent split edge dominating set of $P_n \times P_m$.

Case(iii) Let $m \equiv 1(mod3)$.

For $n \geq 4$, we can partition the set of m columns of $P_n \times P_m$ in such a way that two columns at the B_i blocks for beginning, $B_i, \Big(i = 1,2,\cdots \lfloor \frac{m}{3} \rfloor - 1\Big)$ at the middle and two columns at the end. The set $S_s \cup S_1$ will dominate the first two columns and B_i blocks. In addition we can determine a set isomorphic toS_R which dominates m and $m-1$ columns by a set isomorphic to S_R. Let $n = 4q + l : 1 \leq q \leq \lfloor \frac{n}{4} \rfloor, 0 \leq l \leq 3$. Consider the following two cases to find S_R as shown in Figure (3).

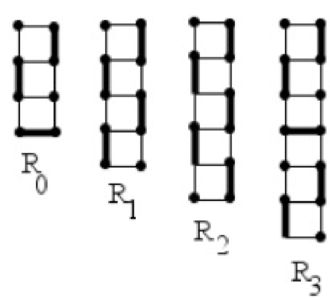

Figure 3

i) If $q = 1$ then $S_R = \{R_l : 0 \leq l \leq 3\}$.

ii) If $q > 1$ then $S_R = \{(\lfloor \frac{n}{4} \rfloor - 1)R_0 + R_l : 0 \leq l \leq 3\}$.

Therefore $S_4 = S_1 \cup S_s \cup S_R$ is an independent split edge dominating set and the induced subgraph $\langle E - S_4 \rangle$ of $P_n \times P_m$ is disconnected. Thus S_4 is an independent split edge dominating set with minimum cardinality. Therefore $|S_4| \leq |S_1| + |S_R| + |S_s|$.

Hence the proof. □

Theorem 6.3 : Let C_3 be a cycle of order 3 and C_m be any cycle with $m \geq 3$. Then split edge domination number of the cartesian product of two cycles,

$$\gamma_s^{'}[C_3 \times C_m] \leq \begin{cases} m+3 & \text{if } m \text{ is even,} \\ m+4 & \text{if } m \text{ is odd.} \end{cases}$$

Proof : Consider an edge set
$S_s = \Big\{\{(1,1),(1,2)\}, \{(1,1),(3,1)\}, \{(1,1),(1,m)\}, \{(2,1),(2,2)\},$
$\{(2,1),(3,1)\}, \{(2,1),(2,m)\}\Big\}$. Also let us discuss the following cases:

Case(i) If m be an even number.
Let $S_1 = \Big\{\{(1,2k+4),(2,2k+4)\} : k = 0,1,\cdots \lfloor \frac{m}{2} \rfloor - 2\Big\}$
$\bigcup \Big\{\{(2,2k+3),(3,2k+3)\} : k = 0,1,\cdots \lfloor \frac{m}{2} \rfloor - 2\Big\}$.
Then $F = S_1 \cup S_s$ is an edge dominating set and the induced subgraph $\langle E - F \rangle$ of $C_3 \times C_m$ is disconnected. Thus F is a split edge dominating set of $C_3 \times C_m$.

$$\gamma_s^{'}[C_3 \times C_m] \leq |F|,$$
$$\gamma_s^{'}[C_3 \times C_m] \leq |S_1| + |S_s|.$$

Case(ii) If m be an odd number.
Let $S_2 = \Big\{\{(1,2k+4),(2,2k+4)\} : k = 0,1,\cdots \lfloor \frac{m}{2} \rfloor - 1\Big\} \bigcup \Big\{\{(2,2k+3),(3,2k+3)\} : k = 0,1,\cdots \lfloor \frac{m}{2} \rfloor\Big\}$.
Then $F_1 = S_2 \cup S_s$ is an edge dominating set and the induced subgraph $\langle E - F_1 \rangle$ of $C_3 \times C_m$ is disconnected. Thus F_1 is a split edge dominating set of $C_3 \times C_m$.

$$\gamma_s^{'}[C_3 \times C_m] \leq |F_1|,$$
$$\gamma_s^{'}[C_3 \times C_m] \leq |S_2| + |S_s|.$$

Hence the proof. □

Theorem 6.4 : Let C_4 be a cycle of order 4 and C_m be any cycle with $m \geq 4$. Then split edge domination number of the cartesian product of two cycles,
$\gamma_s^{'}[C_4 \times C_m] \leq m + \lceil \frac{m}{2} \rceil + 3$.

Proof : Consider an edge set

$$S_s = \Big\{\{(1,1),(1,2)\},\{(1,1),(4,1)\},\{(1,1),(1,m)\},\{(2,1),(2,2)\},\{(2,1),(3,1)\},\{(2,1),(2,m)\}\Big\}.$$

Let $S_1 = \Big\{\{(1,2k+2),(2,2k+2)\} : k = 1,2,\cdots \lceil \frac{m}{2} \rceil - 2\Big\}$
$\bigcup \Big\{(2,2k+1),(3,2k+1)\} : k = 1,2,\cdots \lceil \frac{m}{2} \rceil - 1\Big\}$.
Also let us discuss the following cases:

Case(i) If m be an odd number.
Let $S_2 = \Big\{\{(4,2k),(4,2k+1)\} : k = 1,2,\cdots \lfloor \frac{m}{2} \rfloor\Big\}$. Then $F = S_1 \cup S_2 \cup S_s$ is an edge dominating set and the induced subgraph $\langle E-F \rangle$ of $C_4 \times C_m$ is disconnected. Thus F is a split edge dominating set of $C_4 \times C_m$.

$$\gamma_s^{'}[C_4 \times C_m] \leq |F|,$$
$$\gamma_s^{'}[C_4 \times C_m] \leq |S_1| + |S_2| + |S_s|.$$

Case(ii) If m be an even number.
Let $S_3 = \Bigg\{\Big\{\{(4,2k),(4,2k+1)\} : k = 1,2,\cdots \lfloor \frac{m}{2} \rfloor\Big\} \bigcup \Big\{(3,m),(4,m)\Big\}\Bigg\}$.
Then $F_1 = S_1 \cup S_3 \cup S_s$ is an edge dominating set and the induced subgraph $\langle E - F_1 \rangle$ of $C_4 \times C_m$ is disconnected. Thus F_1 is a split edge dominating set of $C_4 \times C_m$.

$$\gamma_s^{'}[C_4 \times C_m] \leq |F_1|,$$
$$\gamma_s^{'}[C_4 \times C_m] \leq |S_1| + |S_3| + |S_s|.$$

Hence the proof. □

Theorem 6.6 : Let P_3 be a path with 3 vertices and C_m be any cycle with $m \geq 3$. Then split edge domination number of the cartesian product,
$\gamma_s^{'}[P_3 \times C_m] = m + 2$.

Proof : Consider an edge set
$S_s = \Big\{\{(1,1),(2,1)\},\{(1,2),(2,2)\},\{(1,1),(1,m)\},\{(1,2),(1,3)\},\{(3,2),(3,3)\}\Big\}.$

$$S_1 = \begin{cases} \bigcup_{i=1,3} & \{(i,3k+5),(i,3k+6)\} : k = 0,1,\cdots \lfloor \frac{m}{3} \rfloor - 1, \\ \bigcup_{i=2} & \{(i,3k+4),(i,3k+5)\} : k = 0,1,\cdots \lfloor \frac{m}{3} \rfloor - 1. \end{cases}$$

Also let us discuss the following cases:

Case(i) If $m \equiv 0(mod 3)$.
Then $F = S_1 \cup S_s$ is an edge dominating set and the induced subgraph $\langle E - F \rangle$ of $P_3 \times C_m$ is disconnected. Thus F is a split edge dominating set of $P_3 \times C_m$ with minimum cardinality. Therefore $\gamma_s^{'}[P_3 \times C_m] = |S_1| + |S_s|$.

Case(ii) If $m \equiv 1(mod 3)$.
Let $F_1 = S_1 \cup S_s \cup \{(2,m),(3,m)\}$ is an edge dominating set and the induced subgraph $\langle E - F_1 \rangle$ is disconnected. Thus F_1 is a split edge dominating set of $P_3 \times C_m$ with minimum cardinality.
Therefore $\gamma_s^{'}[P_3 \times C_m] = |S_1| + |S_s| + 1$.

Case(iii) If $m \equiv 2(mod 3)$.
Let $F_2 = S_1 \cup S_s \cup \Big\{\{(2,m-2),(2,m-1)\},\{(3,m-1),(3,m)\}\Big\}$ is an edge dominating set and the induced subgraph $\langle E - F_2 \rangle$ is disconnected. Thus F_2 is a split edge dominating set of $P_3 \times C_m$ with minimum cardinality. Therefore $\gamma_s^{'}[P_3 \times C_m] = |S_1| + |S_s| + 2$.

Hence the proof. □

Theorem 6.7 : Let P_n be a path of length 2 and K_m be any complete graph with $m \geq 3$. Then split edge domination number of the cartesian product,

$$\gamma_s^{'}[P_3 \times K_m] \leq \begin{cases} 3m - 3 & \text{if } m \text{ is an even number,} \\ 3m - 4 & \text{if } m \text{ is an odd number.} \end{cases}$$

Proof : Consider an edge set
$S_s = \Bigg\{\Big\{\{(1,1),(2,1)\},\{(1,2),(2,2)\}\Big\} \bigcup \Big\{\{(1,1),(1,i)\},\{(1,2),(1,i)\} : i = 3,4,\cdots m.\Big\}\Bigg\}.$
Let us discuss the following cases:

Case(i) If m is an even number.
Let $S_1 = \Big\{\{(2, 2k+3), (2, 2k+4)\} : k = 0, 1, \cdots \frac{m}{2} - 2\Big\} \bigcup \Big\{\{(3, 2k+1), (3, 2k+2)\} : k = 0, 1, \cdots \lfloor \frac{m}{2} \rfloor - 1\Big\}$. Then $S_1 \cup S_s$ is an edge dominating set and the induced subgraph $\langle E - S_1 \cup S_s\rangle$ of $P_3 \times K_m$ is disconnected. Thus $S_1 \cup S_s$ is a split edge dominating set of $P_3 \times K_m$ with minimum cardinality. Therefore $\gamma_s'[P_3 \times K_m] = |S_1| + |S_s|$.

Case(ii) If m is an odd number.
Let $S_2 = \Bigg\{\Big\{\{(2, 2k+4), (2, 2k+5)\} : k = 0, 1, \cdots \frac{m}{2} - 2\Big\}$
$\bigcup \Big\{\{(3, 2k+1), (3, 2k+2)\} : k = 0, 1, \cdots \lfloor \frac{m}{2} \rfloor - 1\Big\}\Bigg\}$. Then $S_2 \cup S_s$ is an edge dominating set and the induced subgraph $\langle E - (S_2 \cup S_s)\rangle$ of $P_3 \times K_m$ is disconnected. Thus $S_2 \cup S_s$ is a split edge dominating set of $P_3 \times K_m$ with minimum cardinality. Therefore $\gamma_s'[P_3 \times K_m] = |S_2| + |S_s|$.

Hence the proof. □

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