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PERMUTATION GENERATING METHOD AND STUDY OF ASSOCIATED RANDOM VARIABLE

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Abstract

This paper deals with a new method of forming a partial random permutation which is based on coin tossing experiment and discusses the properties of random variable associated with them.

1. Introduction

Permutations and Combinations have always played a significant role in many aspects of Discrete Mathematics and Statistics. The foundation of probability theory rests on these two important features. The study of random variables associated with randomly constructed permutations continues to attract the attention of many researchers worldwide.

In this article we propose a new method of forming a partial random permutation which is based on coin tossing experiment and study the properties of random variable associated with them. We describe the method in the next section.

Key Words : *Partial random permutation, Coin tossing experiment.*

2. Description of Method

The permutation of n integers $(1, 2, \cdots, n)$ is denoted by $\pi = (\pi_1\pi_2 \cdots \pi_n)$.

The positions corresponding to the n integers $(1, 2, \cdots, n)$ can be represented as as follows.

Position	1	2	$\cdots$	i	$i+1$	$i+2$	$\cdots$	n
Integer	$*$	$*$	$\cdots$	$*$	$*$	$*$	$\cdots$	$*$

The method of permutation generation depends upon the outcomes of an unbiased coin which is tossed $(n-1)$ times.

If the first toss results in a head (H), we write 1 at the n^{th} position whereas, if the first toss results in a tail (T), we write 1 at the first position. The next vacant positions are successively filled from the right or left end with the appropriate successive numbers depending upon the outcomes at the successive trials i.e, whenever the outcome is head (H), the number corresponding to that trial is written from the right end at the next available vacant position whereas the vacant position from the left end is filled with the appropriate successive number if the outcome is tail (T).

This process is followed for $(n-1)$ tosses. Finally, the last integer n is placed at the last empty place.

Illustration :

Suppose $n = 4$ i.e we are interested in obtaining a random permutation of the integers $(1, 2, 3, 4)$. Here an unbiased coin is tossed three times. The possible outcomes of the first three trials and the resulting permutation are presented below.

Sr. No.	Outcome Trial no.			Position filled				Resulting permutation
	1	2	3	1	2	3	4	
1	H	H	H	4	3	2	1	(4 3 2 1)
2	H	H	T	3	4	2	1	(3 4 2 1)
3	H	T	H	2	4	3	1	(2 4 3 1)
4	H	T	T	2	3	4	1	(1 2 3 4)
5	T	T	T	2	3	4	1	(1 2 3 4)
6	T	T	H	1	2	4	3	(1 2 4 3)
7	T	H	H	1	4	3	2	(1 4 3 2)
8	T	H	T	1	3	4	2	(1 3 4 2)

In this case, the eight distinct permutations are as follows (4 3 2 1) , (3 4 2 1) , (2 4 3 1) , (2 3 4 1) , (1 2 3 4) , (1 2 4 3) , (1 4 3 2) , and (1 3 4 2).

Remark : It may be noted that for $n = 4$, we obtain $2^3 = 8$ permutations. In general for n integers $(1, 2, \cdots, n)$, the number of distinct permutations possible are 2^{n-1}. Also, the permutations obtained are equiprobable.

In the next section we define the random variable associated with the permutations and obtain its probability distribution and other properties.

3. Probability Distribution of the Random Variable X

Let the random variable X denotes the position of the highest integer n in the permutation. Since the possible positions that the highest integer n can occupy are $(1, 2, \cdots, n)$, X can range from $1, 2, \cdots, n$.

Theorem 1 : The probability distribution of the random variable X is given by

$$P(X = x) = p(x) = \binom{n-1}{x-1} \left(\frac{1}{2}\right)^{n-1}, \quad x = 1, 2, \cdots, n. \tag{3.1}$$

Proof : The possible positions of the highest integer can be represented as follows:

Position	1	2	$\cdots$	$(x-1)$	x	$(x+1)$	$\cdots$	$(n-1)$	n
Integer	$*$	$*$	$\cdots$	$*$	n	$*$	$\cdots$	$*$	$*$

We use the following argument to find the probability of the highest integer occupying position x.

The method of constructing the permutation involves writing the successive integers from the left end whenever the outcome is tail (T) and filling the consecutive vacant positions from the right end with successive integers whenever the outcome is head (H). The highest integer n will occupy position x only if there are $(x - 1)$ tails in $(n - 1)$ tosses. This also means that there should be $(n - x)$ heads in these $(n - 1)$ tosses.

Since the permutation generation method involves a coin tossing experiment , it is easy to see that the random variable X follows Binomial distribution with parameters $(n-1)$ and $\frac{1}{2}$, with the origin shifted to 1.

$$\text{i.e.} \quad X \sim B\left(n-1, \frac{1}{2}\right) \quad \text{(with origin shifted to 1)}$$

Hence

$$\begin{aligned} P(X = x) &= p(x) \\ &= \begin{pmatrix} n-1 \\ \\ x-1 \end{pmatrix} \left(\tfrac{1}{2}\right)^{n-1}, \quad x = 1, 2, \cdots, n. \end{aligned} \tag{3.2}$$

This proves Theorem 1.

Further, if we define $X' = X - 1$ then

$$X' \sim B\left(n-1, \frac{1}{2}\right), \qquad x' = 0, 1, 2, \cdots, n-1$$

with

$$\begin{aligned} P(X' = x') &= p(x') \\ &= \begin{pmatrix} n-1 \\ \\ x' \end{pmatrix} \left(\tfrac{1}{2}\right)^{n-1}, \quad x' = 0, 1, 2, \cdots, n-1. \end{aligned} \tag{3.3}$$

4. Moments of X

Using the fact that $X' \sim B\left(n-1, \frac{1}{2}\right), x' = 0, 1, 2, \cdots, n-1$ and $X' = X - 1$, the following results easily follow.

1. $E[X] = \frac{n+1}{2}$ (4.1)

2. $V[X] = \frac{n-1}{4}$. (4.2)

   The higher order moments can also be similarly obtained. We give below expressions for the third and fourth raw as well as central moments of the random variable X.

3. The third raw moment of X is given by

$$E[X^3] = \frac{n^3 + 6n^2 + 3n - 2}{8} \tag{4.3}$$

4. The fourth raw moment of X is given by

$$E[X^4] = \frac{n^4 + 10n^3 + 15n^2 - 10n}{16} \tag{4.4}$$

5. The third central moment of X is given by

$$\mu_3[X] = 0. \tag{4.5}$$

6. The fourth central moment of X is given by

$$\mu_4[X] = \frac{3n^2 - 8n + 5}{16}. \tag{4.6}$$

5. Numerical Results

To study the nature of the distribution of X, numerical results were obtained using the theoretical expressions for various values of n. As $\mu_3[X]$ equals zero, the coefficient of skewness (β_1) equals zero for all values. We present the values of mean (μ_1'), variance (μ_2), fourth central moment (μ_4) and the coefficient of kurtosis (β_2) for various values of n in the following table.

Table 1 : Moments of X and Coefficient of kurtosis β_2

n	μ_1'	μ_2	μ_4	β_2
2	1.5000	0.2500	0.0625	1.0000
3	2.0000	0.5000	0.5000	2.0000
4	2.5000	0.7500	1.3125	2.3333
5	3.0000	1.0000	2.5000	2.5000
6	3.5000	1.2500	4.0625	2.6000
7	4.0000	1.5000	6.0000	2.6667
8	4.5000	1.7500	8.3125	2.7143
9	5.0000	2.0000	11.000	2.7500
10	5.5000	2.2500	14.0625	2.7778

It is evident from above table that $E[X]$ as well as $V[X]$ are increasing functions of n. Also, coefficient of kurtosis (β_2) increases with an increase in n. The distribution is platykurtic for small values of n whereas it becomes mesokurtic as n increases.

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