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COMMON FIXED POINT FOR GENERALIZED F-WEAK CONTRACTIVE MAPPINGS

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Abstract

The aim of this work is to define generalized F-weak contractive mappings and prove that there is a unique common fixed point for generalized F-weak contractive mappings, the presented work is a generalization of Wardowski and Dung [12] . An example is given to show that our result is a proper extension of Wardowski and Dung [12, theorem 2.4].

1. Introduction and Preliminaries

Fixed point is one of the most fruitful and effective tools in mathematics which has enormous applications within as well as outside mathematics. In 1922 Banach established the famous fixed point theorem which is called the Banach contraction principle. This principle is a forceful tool in nonlinear analysis. Recently many results of the fixed point have been proved [1-10]. In [11], Wardowski has introduced the concept of an F-contraction as follows:

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Definition 1.1 : Let $\mathcal{F}$ be the family of all functions $F : (0, +\infty) \to \mathbb{R}$ such that

(F1) F is strictly increasing, that is, for all $\alpha, \beta \in (0, +\infty)$ if $\alpha < \beta$ then $F(\alpha) < F(\beta)$;

(F2) For each sequence $\{\alpha_n\}$ of positive numbers, the following holds;

$$\lim_{n\to\infty} \alpha_n = 0 \text{ if and only } \lim_{n\to\infty} F(\alpha_n) = -\infty;$$

(F3) There exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} (\alpha^k F(\alpha_n)) = 0$.

Let (X, d) be a metric space. A map $T : X \to X$ is said to be an F-contraction on (X, d) if there exists $F \in \mathcal{F}$ and $\tau > 0$ such that for all $x, y \in X$,

$$d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \le F(d(x, y)).$$

The following functions $F : (0, +\infty) \to \mathbb{R}$ are the elements of $\mathcal{F}$:

(1) $F\alpha = ln\ \alpha$

(2) $F\alpha = ln\ \alpha + \alpha$

(3) $F\alpha = -\frac{1}{\sqrt{\alpha}}$

(r) $F\alpha = ln(\alpha^2 + \alpha)$

In 2014 Wardowski and Dung [12] introduced the notion of an F-contraction into an F-weak contraction as follows :

Definition 1.2 : Let (X, d) be a metric space. A map $T : X \to X$ is said to be an F-weak contraction on (X, d) if there exists $F \in \mathcal{F}$ and $\tau > 0$ such that for all $x, y \in X$ satisfying $d(Tx, Ty) > 0$, the following holds :

$$\tau + F(d(Tx, Ty)) \le F\left(\max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\}\right).$$

By using the notion of F-weak contraction, the author has proved a fixed point theorem which generalizes the notion of an F-contraction into F-weak contraction as follows:

Theorem 1.3 : Let (X, d) be a metric space. A map $T : X \to X$ be an F-weak contraction if T or F is continuous, then we have

(1) T has a unique fixed point $x^* \in X$.

(2) For all $x \in X$, the sequence $\{T^n x\}$ is convergent to x^*.

In this paper we define generalized F-weak contractive mappings and prove some common fixed point results for F-weak contractive mappings which are the generalizations of results given in Wardowski and Dung [12]. An example is given to show that our result is a proper extension of theorem 2.4 [12].

2. Main Results

Definition 2.1 : Let (X, d) be a metric space and S, T be two self-maps on (X, d) are said to F-weak contractive mappings if there exists $F \in \mathcal{F}$ and $\tau > 0$ such that for all $x, y \in X$ satisfying $(Sx, Ty) > 0$, the following holds :

$$\tau + F(d(Sx, Ty)) \leq F\left(\max\left\{d(x, y), d(x, Sx), d(y, Ty), \frac{d(x, Ty), d(y, Sx)}{2}\right\}\right). \quad (2.1)$$

Theorem 2.2 : Let (X, d) be a metric space and S, T be two self-maps on (X, d) which satisfy condition (2.1) for all $x, y \in X$. If S and T or F is continuous then S and T have a unique common fixed point in X.

Proof : Let x_0 be an arbitrary point in X and define $x_{2k+1} = Sx_{2k}, x_{2k+2} = Tx_{2k+1}, k = 0, 1, 2, \cdots$. Then

$$\begin{aligned}
&\tau + F(d(x_{2k+1}, x_{2k+2})) = \tau + F(d(Sx_{2k}, Tx_{2k+1})) \\
&\leq F(\max\{d(x_{2k}, x_{2k+1}), d(x_{2k}, Sx_{2k}), d(x_{2k+1}, Tx_{2k+1}), \\
&\left.\left.\frac{d(x_{2k}, Tx_{2k+1}) + d(x_{2k+1}, Sx_{2k})}{2}\right\}\right) \\
&\leq F(\max\{d(x_{2k}, x_{2k+1}), d(x_{2k}, x_{2k+1}), d(x_{2k+1}, x_{2k+2}), \\
&\left.\left.\frac{d(x_{2k}, x_{2k+2}) + d(x_{2k+1}, x_{2k+1})}{2}\right\}\right) \\
&\leq F\left(\max\left\{d(x_{2k}, x_{2k+1}), d(x_{2k+1}, x_{2k+2}), \frac{d(x_{2k}, x_{2k+1}) + d(x_{2k+1}, x_{2k+2})}{2}\right\}\right) \\
&\leq F(\max\{d(x_{2k}, x_{2k+1}), d(x_{2k+1}, x_{2k+2})\}). \qquad (2.2)
\end{aligned}$$

If there exists $k \in \mathbb{N}$ such that $\max\{d(x_{2k}, x_{2k+1}), d(x_{2k+1}, x_{2k+2})\} = d(x_{2k+1}, x_{2k+2})$ then (2.2) becomes

$$F(d(x_{2k+1}, x_{2k+2})) \leq F(d(x_{2k+1}, x_{2k+2})) - \tau < F(d(x_{2k+1}, x_{2k+2})).$$

It is a contradiction. Therefore,

$$\max\{d(x_{2k}, x_{2k+1}), d(x_{2k+1}, x_{2k+2})\} = d(x_{2k}, x_{2k+1}).$$

Thus from (2.2), we have

$$F(d(x_{2k+1}, x_{2k+2})) \leq F(d(x_{2k}, x_{2k+1})) - \tau.$$

Similarly,

$$\begin{aligned}
&\tau + F(d(x_{2k+3}, x_{2k+2})) = \tau + F(d(Sx_{2k+2}, Tx_{2k+1})) \\
&\leq F\left(\max\left\{d(x_{2k+2}, x_{2k+1}), d(x_{2k+2}, Sx_{2k+2}), d(x_{2k+1}, Tx_{2k+1}),\right.\right. \\
&\left.\left.\frac{d(x_{2k+2}, Tx_{2k+1}) + d(x_{2k+1}, Sx_{2k+2})}{2}\right\}\right) \\
&\leq F\left(\max\left\{d(x_{2k+2}, x_{2k+1}), d(x_{2k+2}, x_{2k+3}), d(x_{2k+1}, x_{2k+2}),\right.\right. \\
&\left.\left.\frac{d(x_{2k+2}, x_{2k+2}) + d(x_{2k+1}, x_{2k+3})}{2}\right\}\right) \\
&\leq F\left(\max\left\{d(x_{2k+2}, x_{2k+1}), d(x_{2k+2}, x_{2k+3}), \frac{d(x_{2k+1}, x_{2k+2}) + d(x_{2k+2}, x_{2k+3})}{2}\right\}\right) \\
&\leq F(\max\{d(x_{2k+2}, x_{2k+1}), d(x_{2k+2}, x_{2k+3})\}. \qquad (2.3)
\end{aligned}$$

If there exists $k \in \mathbb{N}$ such that

$$\max\{d(x_{2k+2}, x_{2k+1}), d(x_{2k+2}, x_{2k+3})\} = d(x_{2k+2}, x_{2k+3}),$$

then (2.3) becomes

$$F(d(x_{2k+3}, x_{2k+2})) \leq F(d(_{2k+2}, x_{2k+3})) - \tau < F(d(x_{2k+2}, x_{2k+3})).$$

It is a contradiction. Therefore,

$$\max\{d(x_{2k+2}, x_{2k+1}), d(x_{2k+2}, x_{2k+3})\} = d(x_{2k+1}, x_{2k+2}).$$

Thus from (2.3), we have

$$F(d(x_{2k+3}, x_{2k+2})) \leq F(d(x_{2k+1}, x_{2k+2})) - \tau.$$

Thus for all $n \in \mathbb{N}$

$$F(d(x_{n+1}, x_n)) \leq F(d(x_n, x_{n-1})) - \tau,$$

and also it implies that

$$F(d(x_{n+1}, x_n)) \leq F(d(x_1, x_0)) - n\tau. \tag{2.4}$$

Taking the limit as $n \to \infty$ in (2.4), we get

$$\lim_{n\to\infty} F(d(x_{n+1}, x_n)) = -\infty$$

that together with (F2) gives

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{2.5}$$

From (F3), there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty} (d(x_{n+1}, x_n))^k F(d(x_{n+1}, x_n)) = 0 \cdots \tag{2.6}$$

It follows from (2.4) , that

$$(d(x_{n+1}, x_n))^k (F(d(x_{n+1}, x_n)) - F(d(x_1, x_0))) \leq -(d(x_{n+1}, x_n))^k n\tau \leq 0 \tag{2.7}$$

for all $n \in \mathbb{N}$. By using (2.5), (2.6) and taking the limit as $n \to \infty$ in (2.7), we get

$$\lim_{n\to\infty} (n(d(x_{n+1}, x_n))^k) = 0.$$

Then there exists $n_1 \in \mathbb{N}$ such that $n(d(x_{n+1}, x_n))^k \leq 1$ for all $n \geq n_1$, that is

$$d(x_{n+1}, x_n) \leq \frac{1}{n^{1/k}} \quad \text{for all} \quad n \geq n_1. \tag{2.8}$$

For all $m \geq n \geq n_1$ by using (2.8) and the triangle inequality, we get

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m-1}) + \cdot + d(x_{n+1}, x_n) \\ &< \sum_{i=n}^{\infty} d(x_{i+1}, x_i) \\ &\leq \sum_{i=n}^{\infty} \tfrac{1}{i^{1/k}}. \end{aligned} \tag{2.9}$$

Since the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^{1/k}}$ is convergent, taking the limit as $n \to \infty$ in (2.9), we get $\lim\limits_{m,n\to\infty} d(x_m, x_n) = 0$. This proves that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $u \in X$ such that $\lim\limits_{n\to\infty} x_n = u$. We shall prove that u is a common fixed point of S and T, by two following cases.

Case 1 : S and T are continuous. We have

$$d(u, Su) = \lim_{k\to\infty} d(x_{2k}, Sx_{2k}) = \lim_{k\to\infty} d(x_{2k}, x_{2k+1}) = 0.$$

Similarly $d(u, Tu) = 0$. This proves that u is a common fixed point of S and T.

Case 2 : F is continuous. In this case , we consider two following sub-cases.

Sub-case 2.1 : For each $n \in \mathbb{N}$ there exists $i_n \in \mathbb{N}$ such that $x_{2i_n+1} = Su$, $x_{2i_n+2} = Tu$ and $i_n > i_{n-1}$ where $i_0 = 1$.

Then we have

$$\begin{aligned} u &= \lim_{n\to\infty} x_{2i_n+1} = \lim_{n\to\infty} x_{2i_n+2} \\ &= \lim_{n\to\infty} Su = \lim_{n\to\infty} Tu \\ &= Su = Tu. \end{aligned}$$

Sub-case 2.2 : There exists $n_0 \in \mathbb{N}$ such that $x_n \neq Su$ for all $n \geq n_0$. That is $d(Tx_{2k+1}, Su) > 0$ for some k.

It follows from (2.1), that

$$\begin{aligned} &\tau + F(d(x_{2k+2}, Su)) = \tau + F(d(Tx_{2k+1}, Su)) \\ &\leq F\left(\max\left\{d(u, x_{2k+1}), d(u, Su), d(x_{2k+1}, Tx_{2k+1}), \frac{d(u, Tx_{2k+1}) + d(x_{2k+1}Su)}{2}\right\}\right) \\ &\leq F\left(\max\left\{d(u, x_{2k+1}), d(u, , Su), d(x_{2k+1}, x_{2k+2}), \frac{d(u, x_{2k+2}) + d(x_{2k+1}, Su)}{2}\right\}\right). \end{aligned}$$

Since F is continuous, taking $k \to \infty$ and if $d(u, Su) > 0$, then

$$\begin{aligned} \tau + F(d(x_{2k+2}, Su)) &\leq F\left(\max\left\{d(u, Su), \frac{d(u, Su)}{2}\right\}\right) \\ &\leq F(d(u, Su)) \end{aligned}$$

which is a contradiction. Therefore $d(u, Su) = 0$.

That is u is a fixed point of S. Similarly we can show that u is a fixed point of T.

Uniqueness. Let v $(\neq u)$ be another common fixed point of S and T i.e. $d(u, v) > 0$. It follows from (2.1) that

$$\begin{aligned} F(d(u,v)) &< \tau + F(d(Su,Tv)) \\ &\leq F\left(\max\left\{d(u,v), d(u,Su), d(v,Tv), \frac{d(u,Tv)+d(v,Su)}{2}\right\}\right) \\ &\leq F\left(\max\left\{d(u,v), d(u,u), d(v,v), \frac{d(u,v)+d(v,u)}{2}\right\}\right) \\ & F(d(u,v)) < F(d(u,v)) \end{aligned}$$

which is a contradiction. Then $(u, v) = 0$. i.e. $u = v$.

This proves that the common fixed point of S and T is unique.

We conclude this paper with an illustrative example which demonstrates theorem (2.1).

Example 2.3 : Let $S, T : [0, 1] \to [0, 1]$ be given by :

$$Sx = \begin{cases} 0, & \text{if} \quad x \in [0,1) \\ \frac{1}{4} & \text{if} \quad x = 1 \end{cases} \qquad \text{and} \qquad Tx = \begin{cases} 0, & \text{if} \quad x \in [0,1) \\ \frac{1}{5}, & \text{if} \quad x = 1 \end{cases}$$

Since S and T are not continuous, so are not F-contraction, but for $x \in [0, 1)$ and $y = 1$, we have

$$d(Sx,Ty) = d(Sx,T1) = \left|0 - \frac{1}{5}\right| = \frac{1}{5} > 0 \quad \text{and}$$

$$\max\left\{d(x,1), d(x,Sx), d(1,T1), \frac{d(x,T1)+d(1,Sx)}{2}\right\} \geq d(1,T1) = \left|1 - \frac{1}{5}\right| = \frac{4}{5}.$$

Also for $x = 1$ and $y \in [0, 1)$, we have

$$d(Sx,Ty) = d(S1,Ty) = \left|\frac{1}{4} - 0\right| = \frac{1}{4} > 0$$

and

$$\max\left\{d(1,y), d(1,S1), d(y,Ty), \frac{d(1,Ty)+d(y,S1)}{2}\right\} \geq d(1,S1) = \left|1 - \frac{1}{4}\right| = \frac{3}{4}.$$

Therefore, by choosing $F\alpha = ln\ \alpha, \alpha \in (0, +\infty)$ and $ln\ 4$, we see that S and T satisfy the condition (2.1). Thus S and T have a unique common fixed point.

Corollary 2.4 : Let (X, d) be a metric space and S, T be two self-maps on (X, d) such that for all $x, y \in X$ satisfying $(Sx, Ty) > 0$, the following holds :

$$\tau + F(d(Sx,Ty)) \leq F(ad(x,y) + bd(x,Sx) + cd(y,Ty) + e[d(x,Ty) + d(y,Sx)]) \cdots \quad (2.10)$$

wher $F \in \mathcal{F}, \tau > 0, a, b, c, e \geq 0$ and $a + b + c + 2e < 1$. If S and T or F is continuous then S and T have a unique common fixed point in X.

Proof : For all $x, y \in X$, we have

$$\begin{aligned} & ad(X,y) + bd(x,Sx) + cd(y,Ty) + e[d(x,Ty) + d(y,Sx)] \\ & \leq (a+b+c+2e)\left(\max\left\{d(x,y), d(x,Sx), d(y,Ty), \frac{d(x,Ty)+d(y,Sx)}{2}\right\}\right) \\ & \leq \left(\max\left\{d(x,y), d(x,Sx), d(y,Ty), \frac{d(x,Ty)+d(y,Sx)}{2}\right\}\right). \end{aligned}$$

Then by (F1), we see that (2.1) is a consequence of (2.10). Thus corollary is proved.

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