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SOME INTEGRALS INVOLVING I-FUNCTION OF TWO VARIABLES AND ITS APPLICATIONS

SACHIN SHARMA[1] AND A. K. RONGHE[2]
[1,2] Department of Mathematics,
S. S. L. Jain P. G. College, Vidisha (M.P.)-464001, India

Abstract

In the present paper, some double integrals for I-function of two variables and its applications have been evaluated.

1. Introduction

The double Mellin Barnes type contour integral occurring in this paper will be referred to as the I-function of two variables throughout the paper and will be defined and represented in the following manner. [6]

$$I\ [Z_1, Z_2] \equiv I^{0,n_1:m_2,n_2;m_3n_3}_{p_1,q_1:p_2,q_2;p_3,q_3}\left[\begin{array}{c|c} z_1 & (a_j;\alpha_j,A_j;\xi_j)_{1,p_1}:(c_j,C_j;U_j)_{1,p_2};(e_j,E_j;P_j)_{1,p_3} \\ \\ z_2 & (b_j;\beta_j,B_j;\eta_j)_{1,q_1}:(d_j,D_j;V_j)_{1,q_2};(f_j,F_j;Q_j)_{1,q_3}\end{array}\right]$$

$$= \frac{1}{(2\pi i)\ ^2}\int_{\mathcal{L}_s}\int_{\mathcal{L}_t}\phi(s,t)\theta_1(s)\theta_2(t)z_1^s z_2^t ds dt \qquad (1.1)$$

Key Words : *Hypergeometric function, I-function of two variables, Known integrals, etc.*

where $\phi(s,t), \theta_1(s), \theta_2(t)$ are given by

$$\phi(s,t) = \frac{\prod_{j=1}^{n_1} \Gamma^{\xi_j}(1-a_j+\alpha_j s+A_j t)}{\prod_{j=n_1+1}^{p_1} \Gamma^{\xi_j}(a_j-\alpha_j s-A_j t) \prod_{j=1}^{q_1} \Gamma^{\eta_j}(1-b_j+\beta_j s+B_j t)} \tag{1.2}$$

$$\theta_1(s) = \frac{\prod_{j=1}^{n_2} \Gamma^{U_j}(1-c_j+C_j s) \prod_{j=1}^{m_2} \Gamma^{V_j}(d_j-D_j s)}{\prod_{j=n_2+1}^{p_2} \Gamma^{U_j}(c_j-C_j s) \prod_{j=m_2+1}^{q_2} \Gamma^{V_j}(1-d_j+D_j s)}, \tag{1.3}$$

$$\theta_2(t) = \frac{\prod_{j=1}^{n_3} \Gamma^{P_j}(1-e_j+E_j t) \prod_{j=1}^{m_3} \Gamma^{Q_j}(f_j-F_j t)}{\prod_{j=n_3+1}^{p_3} \Gamma^{P_j}(e_j-E_j t) \prod_{j=m_3+1}^{q_3} \Gamma^{Q_j}(1-f_j+F_j t)}. \tag{1.4}$$

Also

$$\left.\begin{array}{c} z_1 \neq 0, \quad z_2 \neq 0 \\ i = \sqrt{-1} \\ \text{an empty product is interpreted as unity, etc.} \end{array}\right\} \tag{1.5}$$

2. Notation and Results Used

Throughout the present paper we shall use the following notations:

- $((P)) \equiv (a_j, \alpha_j, A_j; \xi_j)_{1,p_1}$, $((Q)) \equiv (c_j, C_j; U_j)1, p_2$, $((R)) \equiv (e_j, E_j; P_j)_{1,p_3}$
- $((S)) \equiv (b_j, \beta_j, B_j; \eta_j)_{1,q_1}$, $((T)) \equiv (d_j, D_j; V_j)_{1,q_2}$, $((U)) \equiv (f_j, F_j; Q_j)_{1,q_3}$
- $(a_j, \alpha_j, A_j; \xi_j)_{1,p}$ stands for $(a_1, \alpha_1, A_1; \xi_1), (a_2, \alpha_2, A_2; \xi_2), \cdots, (a_p, \alpha_p, A_p; \xi_p)$
- $(c_j, C_j; U_j)_{1,p_2}$ stands for $(c_1, C_1; U_1), (c_2, C_2, U_2), \cdots, (c_{p_2}, C_{p_2}; U_p)$.
- $(a_j; \alpha_j, A_j; 1)_{1,p}$ stands for $(a_1; \alpha_1, A_1; 1), (a_2; \alpha_2, A_2; 1), \cdots, (a_p; \alpha_p, A_p; 1)$.
- $(a_j; \alpha_j, A_j)\ _{1,p}$ stands for $(a_1; \alpha_1, A_1), (a_2; \alpha_2, A_2), \cdots, (a_p; \alpha_p, A_p)$.
- $(a_j; \alpha_j, 1)_{1,p}$ stands for $(a_1; \alpha_1), (a_2; \alpha_2, 1) \cdots, (a_p; \alpha_p, 1)$.
- $(a_j; \alpha_j)_{1,p}$ stands for $(a_1; \alpha_1), (a_2; \alpha_2), \cdots, (a_p; \alpha_p)$.

- $(a_j;1)_{1,p}$ stands for $(a_1;1),(a_2;1),\cdots,(a_p;10.$

- $(a_p)=(a_j)_{1,p}$ stands for $(a_1),(a_2),\cdots,(a_p)$.

Following the results of Braaksma [3, p.278] and Rathie [6, 8], it can easily be shown that the function defined by

$$R=\sum_{j=1}^{p_1}\xi_j\alpha_j+\sum_{j=1}^{p_2}U_jC_j-\sum_{j=1}^{q_2}\eta_j\beta_j-\sum_{j=1}^{q_2}V_jD_j, \tag{2.1}$$

$$S=\sum_{j=1}^{p_1}\xi_jA_j+\sum_{j=1}^{p_3}P_jE_j-\sum_{j=1}^{q_1}\eta_jB_j-\sum_{j=1}^{q_3}Q_jF_j, \tag{2.2}$$

is an analytic function of z_1 and z_2 if $R<0$ and $S<0$. And the integral (1.1) is convergent if,

$$\Delta_1>0,\Delta_2>0,\ |arg(z_1)|<\frac{1}{2}\Delta_1\pi,\ |arg(z_2)|<\frac{1}{2}\Delta_2\pi$$

where,
$\Delta_1 =$

$$\left[\sum_{j=1}^{n_1}\xi_j\alpha_j-\sum_{j=n_1+1}^{p_1}\xi_j\alpha_j-\sum_{j=1}^{q_1}\eta_j\beta_j+\sum_{j=1}^{n_2}U_jC_j-\sum_{j=n_2+1}^{p_2}U_jC_j+\sum_{j=1}^{m_2}V_jD_j-\sum_{j=m_2+1}^{q_2}V_jD_j\right], \tag{2.3}$$

$\Delta_2 =$

$$\left[\sum_{j=1}^{n_1}\xi_jA_j-\sum_{j=n_1+1}^{p_1}\xi_jA_j-\sum_{j=1}^{q_1}\eta_jB_j+\sum_{j=1}^{n_3}P_jE_j-\sum_{j=n_3+1}^{p_3}P_jE_j+\sum_{j=1}^{m_3}Q_jF_j-\sum_{j=m_3+1}^{q_3}Q_jF_j\right]. \tag{2.4}$$

For more details, integral (1.1) is converges absolutely see the research paper of K. Shantha kumari, Vasudevan Nambisan and A.K. Rathie [6, p.290].

The following results Sharma and Rathie [8] will be required in the proof of integrals.

$$\int_0^1 x^{\rho-1}(1-x)^{\rho}[1+ax+b(1-x)]^{-2\rho-1}\cdot$$
$${}_2F_1\left[\alpha,\beta;\frac{1}{2}(\alpha+\beta+2):\frac{x(1+a)}{1+ax+b(1-x)}\right]dx$$
$$=2^{\alpha+\beta-2\rho}\left\{\Gamma\left(\rho-\frac{\alpha}{2}-\frac{\beta}{2}\right)\cdot\Gamma\left(\frac{\alpha+\beta+2}{2}\right)\cdot\Gamma\rho\right\}\cdot$$
$$\{(\alpha-\beta)\quad(1+a)^{\rho}(1+b)^{\rho}\Gamma(\alpha)\Gamma(\beta)\}^{-1}\times$$
$$\left[\frac{(2\rho-\alpha+\beta)\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{\beta}{2}\right)}{\Gamma\left(\rho-\frac{\alpha}{2}+1\right)\Gamma\left(\rho-\frac{\beta}{2}+\frac{1}{2}\right)}-\frac{(2\rho+\alpha-\beta)\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\beta+1}{2}\right)}{\Gamma\left(\rho-\frac{\alpha_2}{2}+\frac{1}{2}\right)\Gamma\left(\rho-\frac{\beta}{2}+1\right)}\right] \tag{2.5}$$

where, $Re(\rho)>0, Re(2\rho-\alpha-\beta)>0$, a and b are constant such that the expression $1+ax+b(1-x)$ is not zero and $0\leq x\leq 1$,

$$\int_0^1 x^{\rho-1}(1-x)^{\rho-1}[1+ax+b(1-x)]^{-2\rho+1}\cdot$$
$${}_2F_1\left[\alpha,\beta;\frac{1}{2}(\alpha+\beta+2):\frac{x(1+a)}{1+ax+b(1-x)}\right]dx$$
$$=2^{-2\rho+\alpha+\beta-1}\cdot\frac{\Gamma(\rho-1)\cdot\Gamma\left(\frac{\alpha+\beta}{2}\right)\cdot\left(\rho-\frac{\alpha}{2}-\frac{\beta}{2}-1\right)}{(1+a)^{\rho}(1+b)^{\rho}\Gamma(\alpha)\Gamma(\beta)}\times$$
$$\left[\frac{(2\rho+\alpha-\beta)\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\beta+1}{2}\right)}{\Gamma\left(\rho-\frac{\alpha}{2}-\frac{1}{2}\right)\Gamma\left(\rho-\frac{\beta}{2}\right)}\right]+\left[\frac{(2\rho+\alpha-\beta)\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\beta+1}{2}\right)}{\Gamma\left(\rho-\frac{\alpha_2}{2}-\frac{1}{2}\right)\Gamma\left(\rho-\frac{\beta}{2}\right)}\right] \tag{2.6}$$

where, $Re(\rho)>1, Re(2\rho-\alpha-\beta)>2$,

$$\int_0^{\frac{\pi}{2}} e^{i\pi(2\omega-1)\theta}\cdot(\sin\theta)^{\omega-1}\cdot(\cos\theta)^{\omega-1}.$$
$${}_2F_1\left[\alpha',\beta';\frac{(\alpha'+\beta')}{2};e^{i\theta}\cos\theta\right]d\theta$$
$$=e^{\frac{i\pi(\omega-1)}{2}}\frac{\Gamma(\omega-1)}{2^{2\omega-\alpha'-\beta'-1}}\quad\cdot\quad\Gamma\left(\frac{\alpha'}{2}+\frac{\beta'}{2}\right)\Gamma\left(\omega-\frac{\alpha'}{2}-\frac{\beta'}{2}-1\right)\cdot\{\Gamma(\alpha')\Gamma(\beta')\}^{-1}$$
$$\cdot\left[\frac{(2\omega-\alpha'+\beta'-2)\Gamma\left(\frac{\alpha'+1}{2}\right)\Gamma\left(\frac{\beta'}{2}\right)}{\Gamma\left(\omega-\frac{\alpha'}{2}\right)\Gamma\left(\omega-\frac{\beta'}{2}-\frac{1}{2}\right)}+\frac{(2\omega+\alpha'-\beta'-2)\Gamma\left(\frac{\alpha'}{2}\right)\Gamma\left(\frac{\beta'+1}{2}\right)}{\Gamma\left(\omega-\frac{\alpha'}{2}-\frac{1}{2}\right)\Gamma\left(\omega-\frac{\beta'}{2}\right)}\right] \tag{2.7}$$

where $Re(\omega) > 1, Re(2\omega - \alpha' - \beta') > 2$,

$$\int_0^{\frac{\pi}{2}} e^{i\pi(2\omega+1)\theta} \cdot (\sin\theta)^{\omega-1} \cdot (\cos\theta)^{\omega-1} \cdot {}_2F_1\left[\alpha', \beta'; \frac{(\alpha'+\beta'+2)}{2}; e^{i\theta}\cos\theta\right]$$
$$= e^{\frac{i\pi(\omega-1)}{2}} \frac{\Gamma(\omega)}{2^{2\omega-\alpha'-\beta'+2}} \cdot \Gamma\left(\omega - \frac{\alpha'}{2} - \frac{\beta'}{2}\right)\Gamma\left(\frac{\alpha'}{2} - \frac{\beta'}{2} + 1\right) \cdot \{\Gamma(\alpha'-\beta')\Gamma(\alpha')\Gamma(\beta')\}^{-1}$$
$$\cdot \left[\frac{(2\omega-\alpha'+\beta')\Gamma\left(\frac{\alpha'+1}{2}\right)\Gamma\left(\frac{\beta'}{2}\right)}{\Gamma\left(\omega-\frac{\alpha'}{2}+1\right)\Gamma\left(\omega-\frac{\beta'}{2}+\frac{1}{2}\right)} - \frac{(2\omega+\alpha'-\beta')\Gamma\left(\frac{\alpha'}{2}\right)\Gamma\left(\frac{\beta'+1}{2}\right)}{\Gamma\left(\omega-\frac{\alpha'}{2}+\frac{1}{2}\right)\Gamma\left(\omega-\frac{\beta'}{2}+1\right)}\right] \tag{2.8}$$

where, $Re(\omega) > 0, Re(2\rho - \alpha' - \beta') > 0$.

3. Double Integrals involving I-function of Two Variables

First Integral

$$\int_0^1 \int_0^{\frac{\pi}{2}} x^{\rho-1}(1-x)^{\rho}[1+ax+b(1-x)]^{-2\rho-1} \cdot {}_2F_1\left[\alpha, \beta; \frac{(\alpha+\beta+2)}{2}; \frac{x(1+a)}{1+ax+b(1-x)}\right] \cdot$$
$$e^{i\pi(2\omega+1)\theta} \cdot (\sin\theta)^{\omega} \cdot (\cos\theta)^{\omega} \cdot {}_2F_1\left[\alpha', \beta'; \frac{(\alpha'+\beta'+2)}{2}; e^{i\theta}\cos\theta\right]$$
$$I^{0,n_1:m_2,n_2;m_3,n_3}_{p_1,q_1:p_2,q_2;p_3,q_3}\left[\begin{array}{c|c} z_1 x^{\rho_1}(1-x)^{\rho_1}[1+ax+b(1-x)]^{-2\rho_1} e^{2\omega_1\theta_i}(\sin\theta)^{\omega_1}(\cos\theta)^{\omega_1} & ((P)),((Q)),((R)) \\ z_2 x^{\rho_2}(1-x)^{\rho_2}[1+ax+b(1-x)]^{-2\rho_2} e^{2\omega_2\theta_i}(\sin\theta)^{\omega_2}(\cos\theta)^{\omega_2} & ((S)),((T)),((U)) \end{array}\right] d\theta dx$$
$$= \frac{2^{\alpha+\beta-2\rho-2}\Gamma\left(\frac{\alpha+\beta+2}{2}\right)}{\Gamma(\alpha)\Gamma(\beta)(\alpha-\beta)(1+a)^{\rho}(1+b)^{\rho}} \frac{e^{\frac{i\pi(\omega+1)}{2}}\Gamma\left(\frac{\alpha'+\beta'}{2}+1\right)}{2^{2\omega-\alpha'-\beta'+2}\Gamma(\alpha'-\beta')\Gamma(\alpha')\Gamma(\beta')} \cdot$$
$$\left[\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{\beta}{2}\right)I_1 - \Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\beta+1}{2}\right)I_2\right] \cdot$$
$$\left[\Gamma\left(\frac{\alpha'+1}{2}\right)\Gamma\left(\frac{\beta'}{2}\right)I_3 - \Gamma\left(\frac{\alpha'}{2}\right)\Gamma\left(\frac{\beta'+1}{2}\right)I_4\right] \tag{3.1}$$

where $\mathbf{I}_1, \mathbf{I}_2, \mathbf{I}_3$ and I_4 represent I-function of two variables, they are as follows:

$$\mathbf{I_1} \equiv \mathbf{I}^{0,n_1+3:m_2,n_2;m_3,n_3}_{p_1+3,q_1+3:p_2,q_2;p_3,q_3}\left[\begin{array}{c|c} z_1/(2)^{(2\rho_1+2)} \cdot (1+a)^{\rho_1}[1+b]^{\rho_1} & ((P)),P_1,((Q)),((R)) \\ z_2/(2)^{(2\rho_2+2)} \cdot (1+a)^{\rho_2}[1+b]^{\rho_2} & ((S)),P_2,((T)),((U)) \end{array}\right] \tag{3.2}$$

$$\mathbf{I_2} \equiv \mathbf{I}^{0,n_1+3:m_2,n_2;m_3,n_3}_{p_1+3,q_1+3:p_2,q_2;p_3,q_3}\left[\begin{array}{c|c} z_1/(2)^{(2\rho_1+2)}\cdot(1+a)^{\rho_1}[1+b)^{\rho_1} & ((P)),P_3,((Q)),((R)) \\ \\ z_2/(2)^{(2\rho_2+2)}\cdot(1+a)^{\rho_2}[1+b)^{\rho_2} & ((S)),P_4,((T)),((U)) \end{array}\right] \tag{3.3}$$

$$\mathbf{I_3} \equiv \mathbf{I}^{0,n_1+3:m_2,n_2;m_3,n_3}_{p_1+3,q_1+3:p_2,q_2;p_3,q_3}\left[\begin{array}{c|c} z_1 e^{\frac{i\pi\omega_1}{2}} & ((P)),P_5,((Q)),((R)) \\ \\ z_2 e^{\frac{i\pi\omega_2}{2}} & ((S)),P_6,((T)),((U)) \end{array}\right], \tag{3.4}$$

$$\mathbf{I_4} \equiv \mathbf{I}^{0,n_1+3:m_2,n_2;m_3,n_3}_{p_1+3,q_1+3:p_2,q_2;p_3,q_3}\left[\begin{array}{c|c} z_1 e^{\frac{i\pi\omega_1}{2}} & ((P)),P_7,((Q)),((R)) \\ \\ z_2 e^{\frac{i\pi\omega_2}{2}} & ((S)),P_8,((T)),((U)) \end{array}\right], \tag{3.5}$$

The integral (3.1) is valid if the following set of (sufficient) conditions is satisfied.

(i) $RE(\alpha) > 0, Re(\omega) > 0, |arg(z_1)| < \frac{1}{2}\Delta_1\pi, |arg(z_2)| < \frac{1}{2}\Delta_2\pi,$

(ii) $Re\left(2\rho - \alpha - \beta + 2\rho_1\left[\min_{1\le j\le m_3} Re\left(\frac{d_j}{\delta_j}\right)\right] + 2\rho_2\left[\min_{1\le j\le m_3} Re\left(\frac{d_j}{\delta_j}\right)\right]\right) > 0,$

(iii) $Re\left(2\omega - \alpha' - \beta' + 2\omega_1\left[\min_{1\le j\le m_3} Re\left(\frac{d_j}{\delta_j}\right)\right] + 2\omega_2\left[\min_{1\le j\le m_3} Re\left(\frac{d_j}{\delta_j}\right)\right]\right) > 0,$

where all $((P)), ((Q)), ((R)), ((S)), ((T))$ and $((U))$ sets of parameters are mentioned in section (2). Sets of parameter P_1 to P_8 are as follows:

$$\begin{aligned}
P_1 &\equiv \left\{\left(1-\rho+\frac{\alpha+\beta)}{2};\rho_1,\rho_2;1\right),(1-\rho;\rho_1,\rho_2;1),(\alpha-\beta-2\rho;2\rho_1,2\rho_2;1)\right\},\\
P_2 &\equiv \left\{(1-2\rho+\alpha+\beta;2\rho_1,2\rho_2;1),\left(\frac{\alpha}{2}-\rho;\rho_1,\rho_2;1\right),\left(\frac{1}{2}+\frac{\beta}{2}-\rho;\rho_1,\rho_2;1\right)\right\},\\
P_3 &\equiv \left\{(1-\rho;\rho_1,\rho_2;1),(\beta-\alpha-2\rho;2\rho_1,2\rho_2;1),\left(1-\beta+\frac{(\alpha+\beta)}{2};\rho_1,\rho_2;1\right)\right\},\\
P_4 &\equiv \left\{(1-\alpha+\beta-2\rho;2\rho_1,2\rho_2;1),\left(\frac{1}{2}+\frac{\alpha}{2}-\rho;\rho_1,\rho_2;1\right),\left(\frac{\beta}{2}-\rho;\rho_1,\rho_2;1\right)\right\},\\
P_5 &\equiv \left\{(1-\omega;\omega_1,\omega_2;1),\left(1-\omega;\frac{(\alpha'+\beta')}{2};\omega_1,\omega_2;1\right),(\alpha'-\beta'-2\omega;2\omega_1,2\omega_2;1)\right\}\\
P_6 &\equiv \left\{\left(\frac{\alpha'}{2}-\omega;\omega_1,\omega_2;1\right),\left(\frac{1+\beta'}{2}-\omega;\omega_1,\omega_2;1\right),(1+\alpha'-\beta'+2\omega;2\omega_1,2\omega_2;1)\right\},\\
P_7 &\equiv \left\{(1-\omega;\omega_1,\omega_2;1),\left(1-\omega+\frac{(\alpha'+\beta')}{2};\omega_1,\omega_2;1\right),(\beta'-\alpha'+2\omega;2\omega_1,2\omega_2;1)\right\},\\
P_8 &\equiv \left\{(1+\beta'-\alpha'-2\omega;2\omega_1,2\omega_2;1),\left(\frac{1+\alpha'}{2}-\omega;\omega_1,\omega_2;1\right),\left(\frac{\beta'}{2}-\omega;\omega_1,\omega_2;1\right)\right\}.
\end{aligned}$$

Second Integral

$$\int_0^1 \int_0^{\frac{\pi}{2}} x^{\rho-1}(1-x)^{\rho-1}[1+ax+b(1-x)]^{-2\rho+1} \cdot$$
$${}_2F_1\left[\alpha, \beta; \frac{(\alpha+\beta+2)}{2}; \frac{x(1+a)}{1+ax+b(1-x)}\right] \times$$
$$e^{i\pi(2\omega-1)\theta} \cdot (\sin\theta)^{\omega-1} \cdot (\cos\theta)^{\omega-1} \cdot \ {}_2F_1\left[\alpha', \beta'; \frac{(\alpha'+\beta')}{2}; e^{i\theta}\cos\theta\right] \times$$
$$I^{0,n_1:m_2,n_2;m_3,n_3}_{p_1,q_1:p_2,q_2;p_3,q_3}$$
$$\left[\begin{array}{c|c} z_1 x^{\rho_1}(1-x)^{\rho_1}[1+ax+b(1-x)]^{-2\rho_1} e^{i\pi(2\omega_1-1)\theta_1}(\sin\theta)^{\omega_1}(\cos\theta)^{\omega_1} & ((P)),((Q)),((R)) \\ z_2 x^{\rho_2}(1-x)^{\rho_2}[1+ax+b(1-x)]^{-2\rho_2} e^{i\pi(2\omega_2-1)\theta_2}(\sin\theta)^{\omega_2}(\cos\theta)^{\omega_2} & ((S)),((T)),((U)) \end{array}\right] d\theta dx$$
$$= \frac{2^{\alpha+\beta-2\rho-1}\Gamma\left(\frac{\alpha+\beta+2}{2}\right)}{\Gamma(\alpha)\Gamma(\beta)(1-a)^{\rho}(1-b)^{\rho}} \cdot \left[\Gamma\left(\frac{\alpha+\beta}{2}\right)\Gamma\left(\frac{\beta}{2}\right) I_5 + \Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha+\beta}{2}\right) I_6\right] \times$$
$$\frac{e^{\frac{i\pi(\omega-1)}{2}}\Gamma\left(\frac{\alpha'+\beta'}{2}+1\right)}{2^{2\omega-\beta'+1}\Gamma(\alpha'-\beta')\Gamma(\alpha')\Gamma(\beta')} \cdot \left[\Gamma\left(\frac{\alpha'+1}{2}\right)\Gamma\left(\frac{\beta'}{2}\right) I_7 - \Gamma\left(\frac{\alpha'}{2}\right)\Gamma\left(\frac{\beta'+1}{2}\right) I_8\right] \quad (3.6)$$

where $\mathbf{I}_5, \mathbf{I}_6, \mathbf{I}_7$ and I_8 are as follows:

$$\mathbf{I_5} \equiv \mathbf{I}^{0,n_1+3:m_2,n_2;m_3,n_3}_{p_1+3,q_1+3:p_2,q_2;p_3,q_3} \left[\begin{array}{c|c} z_1/\cdot(1+a)^{\rho_1}[1+b)^{\rho_1}(2)^{2\rho_1} & ((P)),P_9,((Q)),((R)) \\ z_2/(1+a)^{\rho_2}[1+b)^{\rho_2}(2)^{2\rho_2} & ((S)),P_{10},((T)),((U)) \end{array}\right] \quad (3.7)$$

$$\mathbf{I_6} \equiv \mathbf{I}^{0,n_1+3:m_2,n_2;m_3,n_3}_{p_1+3,q_1+3:p_2,q_2;p_3,q_3} \left[\begin{array}{c|c} z_1/(2)^{(2\rho_1)}\cdot(1+a)^{\rho_1}[1+b)^{\rho_1} & ((P)),P_{11},((Q)),((R)) \\ z_2/(2)^{(2\rho_2)}\cdot(1+a)^{\rho_2}[1+b)^{\rho_2} & ((S)),P_{12},((T)),((U)) \end{array}\right] \quad (3.8)$$

$$\mathbf{I_7} \equiv \mathbf{I}^{0,n_1+3:m_2,n_2;m_3,n_3}_{p_1+3,q_1+3:p_2,q_2;p_3,q_3} \left[\begin{array}{c|c} z_1 e^{\frac{i\pi\omega_1}{2}}/2\omega_1 & ((P)),P_{13},((Q)),((R)) \\ z_2 e^{\frac{i\pi\omega_2}{2}}/2\omega_2 & ((S)),P_{14},((T)),((U)) \end{array}\right], \quad (3.9)$$

$$\mathbf{I_8} \equiv \mathbf{I}^{0,n_1+3:m_2,n_2;m_3,n_3}_{p_1+3,q_1+3:p_2,q_2;p_3,q_3} \left[\begin{array}{c|c} z_1 e^{\frac{i\omega_1\pi}{2}} & ((P)),P_{15},((Q)),((R)) \\ z_2 e^{\frac{i\omega_2\pi}{2}} & ((S)),P_{16},((T)),((U)) \end{array}\right], \quad (3.10)$$

The integral (3.6) is valid if the following set of conditions (sufficient) are satisfied.

(i) $Re(\rho) > 0, Re(\omega) > 1, |arg(z_1)| < \frac{1}{2}\Delta_1\pi, |arg(z_2)| < \frac{1}{2}\Delta_2\pi,$

(ii) $Re(\alpha) > 0, Re(\omega) > 0,$

(iii) $Re\left(2\rho-\alpha-\beta+2\rho_1\left[\min_{1\le j\le m_3} Re\left(\frac{d_j}{\delta_j}\right)\right]+2\rho_2\left[\min_{1\le j\le m_3} Re\left(\frac{d_j}{\delta_j}\right)\right]\right)>0,$

etc. where all $((P)),((Q)),((R)),((S)),((T))$ and $((U))$ sets of parameters are mentioned in section (2). Sets of parameter P_9 to P_{16} are as follows:

$$P_9 \equiv \left\{ \begin{array}{l} (2-\rho;\rho_1,\rho_2;1),\left(2-\rho+\frac{(\alpha+\beta)}{2};\rho_1,\rho_2;1\right), \\ (2+\alpha-\beta;2\rho_1,2\rho_2;1) \end{array} \right\},$$

$$P_{10} \equiv \left\{ \begin{array}{l} \left(1+\frac{\alpha}{2}-\rho;\rho_1,\rho_2;1\right),\left(\frac{3}{2}-\rho+\frac{\beta}{2};\rho_1,\rho_2;1\right) \\ (3-2\rho-\alpha-\beta;\rho_1,\rho_2;1) \end{array} \right\}$$

$$P_{11} \equiv \left\{(2-\rho;\rho_1,\rho_2;1),\left(2-\rho+\frac{(\alpha+\beta)}{2};\rho_1,\rho_2;1\right)\right\}$$

$$P_{12} \equiv \left\{\left(1-\rho+\frac{\beta}{2};\rho_1,\rho_2;1\right),\left(\frac{3}{2}-\rho+\frac{\alpha}{2};\rho_1,\rho_2;1\right),(3-2\rho-\alpha-\beta;\rho_1,\rho_2;1),\right\},$$

$$P_{13} \equiv \left\{(2-\omega;\omega_1,\omega_2;1),\left(2-\omega+\frac{\alpha'}{2}+\frac{\beta'}{2};\omega_1,\omega_2;1\right),(2+\alpha'-\beta'-2\omega;2\omega_1,2\omega_2;1),\right\},$$

$$P_{14} \equiv \left\{\left(1+\frac{\alpha'}{2}-\omega;\omega_1,\omega_2;1\right),\left(\frac{3}{2}-\omega+\frac{\beta'}{2};\omega_1,\omega_2;1\right),(3-2\omega+\alpha'-\beta';2\omega_1,2\omega_2;1)\right\},$$

$$P_{15} \equiv \left\{ \begin{array}{l} (2-\omega;\omega_1,\omega_2;1),\left(2-\omega+\frac{\alpha'}{2}+\frac{\beta'}{2};\omega_1,\omega_2;1\right), \\ (2-2\omega-\alpha'-\beta';2\omega_1,2\omega_2;1) \end{array} \right\},$$

$$P_{16} \equiv \left\{\left(1-\omega+\frac{\beta'}{2};\omega_1,\omega_2;1\right),\left(\frac{3}{2}-\omega+\frac{\alpha'}{2};\omega_1,\omega_2;1\right),(3-2\omega-\alpha'-\beta';2\omega_1,2\omega_2;1)\right\}.$$

Third Integral

$$\int_0^1\int_0^{\frac{\pi}{2}} x^{\rho-1}(1-x)^{\rho-1}[1+ax+b(1-x)]^{1-2\rho}\cdot$$
$${}_2F_1\left[\alpha,\beta;\frac{(\alpha+\beta+2)}{2};\frac{x(1+a)}{1+ax+b(1-x)}\right]\times$$
$$e^{i(2\omega-1)\theta}\cdot(\sin\theta)^{\omega-1}\cdot(\cos\theta)^{\omega-1}\cdot\ {}_2F_1\left[\alpha',\beta';\frac{(\alpha'+\beta')}{2};e^{i\theta}\cos\theta\right]\times$$
$$I^{0,n_1:m_2,n_2;m_3,n_3}_{p_1,q_1:p_2,q_2;p_3,q_3}$$

$$\left[\begin{array}{c|c} z_1 x^{\rho_1}(1-x)^{\rho_1}[1+ax+b(1-x)]^{-2\rho_1} e^{2\omega_1 i\theta}(\sin\theta)^{\omega_1}(\cos\theta)^{\omega_1} & ((P)),((Q)),((R)) \\ z_2 x^{\rho_2}(1-x)^{\rho_2}[1+ax+b(1-x)]^{-2\rho_2} e^{i\omega_2\theta}(\sin\theta)^{\omega_2}(\cos\theta)^{\omega_2} & ((S)),((T)),((U)) \end{array}\right] d\theta dx$$

$$= \frac{2^{\alpha+\beta-2\rho-2}}{(1+a)^{\rho}(1+b)^{\rho}(\alpha-\beta)\Gamma(\alpha)\Gamma(\beta)} \cdot \left[\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{\beta}{2}\right) I_1 - \Gamma\left(\frac{\beta+1}{2}\right)\Gamma\left(\frac{\alpha}{2}\right) I_2\right] \times$$

$$\frac{e^{\frac{i\pi(\omega-1)}{2}}\Gamma\left(\frac{\alpha'+\beta'}{2}1\right)}{2^{2\omega+\alpha'+\beta'}\Gamma(\alpha')\Gamma(\beta')} \cdot \left[\Gamma\left(\frac{\alpha'+1}{2}\right)\Gamma\left(\frac{\beta'}{2}\right) I_7 - \Gamma\left(\frac{\beta'+1}{2}\right)\Gamma\left(\frac{\alpha'}{2}\right) I_8\right]$$

where $\mathbf{I}_1, \mathbf{I}_2, \mathbf{I}_7$ and $\mathbf{I}_8$ are mentioned in (3.2), (3.3), (3.9) and (3.10) respectively and set of conditions are as follows:

$Re(\rho) > 0, Re(\omega) > 1, |arg(z_1)| < \frac{1}{2}\Delta_1\pi, |arg(z_2)| < \frac{1}{2}\Delta_2\pi,$

$Re\left(2\rho - \alpha - \beta + 2\rho_1\left[\min_{1\le j\le m_3} Re\left(\frac{d_j}{\delta_j}\right)\right] + 2\rho_2\left[\min_{1\le j\le m_3} Re\left(\frac{d_j}{\delta_j}\right)\right]\right) > 0,$

$Re\left(2\omega - \alpha' - \beta' + 2\omega_1\left[\min_{1\le j\le m_3} Re\left(\frac{d_j}{\delta_j}\right)\right] + 2\omega_2\left[\min_{1\le j\le m_3} Re\left(\frac{d_j}{\delta_j}\right)\right]\right) > 0.$

Fourth Integral

$$\int_0^1 \int_0^{\frac{\pi}{2}} x^{\rho-1}(1-x)^{\rho-1}[1+ax+b(1-x)]^{-2\rho+1} \cdot$$

$${}_2F_1\left[\alpha, \beta; \frac{(\alpha+\beta+2)}{2}; \frac{x(1+a)}{1+ax+b(1-x)}\right] \cdot$$

$$e^{i(2\omega+1)\theta} \cdot (\sin\theta)^{\omega-1} \cdot (\cos\theta)^{\omega-1} \cdot {}_2F_1\left[\alpha', \beta'; \frac{(\alpha'+\beta'+2)}{2}; e^{i\theta}\cos\theta\right] \cdot$$

$$I^{0,n_1:m_2,n_2;m_3,n_3}_{p_1,q_1:p_2,q_2;p_3,q_3}$$

$$\left[\begin{array}{c|c} z_1 x^{\rho_1}(1-x)^{\rho_1}[1+ax+b(1-x)]^{-2\rho_1} e^{2\omega_1 i\theta}(\sin\theta)^{\omega_1}(\cos\theta)^{\omega_1} & ((P)),((Q)),((R)) \\ z_2 x^{\rho_2}(1-x)^{\rho_2}[1+ax+b(1-x)]^{-2\rho_2} e^{i\omega_2\theta}(\sin\theta)^{\omega_2}(\cos\theta)^{\omega_2} & ((S)),((T)),((U)) \end{array}\right] d\theta dx$$

$$= \frac{2^{\alpha+\beta-2\rho-2}\Gamma\left(\frac{\alpha+\beta}{2}\right)\Gamma\left(\frac{\alpha+\beta+2}{2}\right)}{(1+a)^{\rho}(1+b)^{\rho}(\alpha-\beta)\Gamma(\alpha)\Gamma(\beta)} \left[\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{\beta}{2}\right) I_5 + \Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\beta+1}{2}\right) I_6\right] \times$$

$$\frac{e^{\frac{i\pi(\omega+1)}{2}}\Gamma\left(\frac{\alpha'+\beta'+2}{2}\right)}{2^{2\omega-\alpha'-\beta'+2}\Gamma(\alpha'+\beta')\Gamma(\alpha')\Gamma(\beta')} \cdot \left[\Gamma\left(\frac{\alpha'+1}{2}\right)\Gamma\left(\frac{\beta'}{2}\right) I_3 - \Gamma\left(\frac{\beta'+1}{2}\right)\Gamma\left(\frac{\alpha'}{2}\right) I_4\right]$$

where $\mathbf{I}_3, \mathbf{I}_4, \mathbf{I}_5$ and $\mathbf{I}_6$ are mentioned in (3.4), (3.5), (3.7) and (3.8) respectively and the set of conditions are as follows:

$Re(\rho) > 0, Re(\omega) > 1, |arg(z_1)| < \frac{1}{2}\Delta_1\pi, |arg(z_2)| < \frac{1}{2}\Delta_2\pi,$

and Δ_1 and Δ_2 given in (2.3) and (2.4) respectively and other sets of condition mention with (3.6) are satisfied.

Proof : To establish (3.1) expressing the I-function of two variables on the left hand side using (1.1) are double Mellin-Barnes contour integrals and interchanging the order of integration which is justifiable due to absolute convergence of integrals.
We have:

$$= \frac{-1}{4\pi^2}\int_{\mathcal{L}_s}\int_{\mathcal{L}_t}\left[\phi(s,t),\theta_1(s)\cdot\theta_2(t)\cdot z_1^s\cdot z_2^t\cdot\right.$$
$$\int_0^1 x^{\rho+\rho_1 s+\rho_2 t-1}(1-x)^{\rho+\rho_1 s+\rho_2 t}[1+ax+b(1-x)]^{-2\rho-2\rho_1 s-2\rho_2 t-1}\cdot$$
$${}_2F_1\left[\alpha,\beta;\frac{1}{2}(\alpha+\beta+2):\frac{x(1+a)}{1+ax+b(1-x)}\right]dx$$
$$\int_0^{\frac{\pi}{2}} e^{i\pi(2\omega+2\omega_1 s+2\omega_2 t+1)\theta}\cdot(\sin\theta)^{\omega+\omega_1 s+\omega_2 t-1}\cdot(\cos\theta)^{\omega+\omega_1 s+\omega_2 t-1},$$
$$\left.{}_2F_1\left[\alpha,\beta;\frac{(\alpha+\beta+2)}{2};e^{i\theta}\cos\theta\right]d\theta\right]dsdt.$$

Evaluate the inner integrals with the help of (2.5) and (2.8) and then applying (1.1) we get the R.H.S. in terms of product of I-function of two variables and hypergeometric functions.
Note : The proof of second, third and fourth integrals are similar to the first with the only difference that here we make use of known integrals, (2.1), (2.4) for second result, (2.1) and (2.3) for third result and (2.2) and (2.3) for fourth result respectively instead of (2.5) and (2.8).

4. Applications

In this section, we mention some interesting and useful applications of I-function of two variables:

- If all the exponents $\xi_j(j=1,\cdots,p_1),\eta_j(j=1,\cdots,q_1),U_j(j=1,2,\cdots,p_2),V_j(j=1,c\ldots,q_2)$, $P_j(j=1,\cdots,P_3),Q_j$ $(j=1,\cdots,q_3)$ are equal to unity, then (3.1), (3.6), (3.10) and (3.11) reduces to the H-function of two variables.

- If $p_1=q_1=n_1=0$ in (3.1), (3.6), (3.10) and (3.11) then it degenerate into product of two I-function of one variable introduced by Saxena [9].

- If $\alpha_j=A_j=\xi_j=1,C_j=U_j=1,E_j=P_j=1,D_j=V_j=1,F_j=Q_j=1$, then (3.1), (3.6), (3.10) and (3.11) reduces to the G-function of two variables.

- If we take $m_3 = 1, n_3 = p_3, f_1 = 0, (f_j)_{1,q_3} = 1, (A_j)_{1,p_1} = (B_j)_{1,q_1} = (E_j)_{1,p_3} = (F_j)_{1,q_3} = 1$ equate the exponents $P_j\ (j = 1, \cdots, p_3), Q_j\ (j = 1, \cdots, q_3)$ to unity, replace q_3 by $q_3 + 1$ and let $z_2 \to 0$ in (3.1), (3.6), (3.10) and (3.11) we get the known relation Ronghe [1].

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