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AN INEQUALITY FOR EMANANT IN COMPLEX PLANE

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Abstract

Let $p(z)$ be a polynomial of degree n and let α be any real or complex number, then the polar derivative or "emanant" of $p(z)$ denoted by $D_\alpha p(z)$, is defined as

$$D_\alpha p(z) = np(z) + (\alpha - z)p'(z).$$

In this paper, we find a generalized result for emanant i.e. polar derivative of the polynomial $p(z) = \sum_{j=0}^{n} a_j z^j$ of degree n having no zeros in $|z| < k$, $k \geq 1$. Our results generalize and improve upon some earlier well known results proved in this direction.

1. Introduction and Statement Results

The following famous result is known as Bernstein [5] inequality.

Theorem A : If $p(z)$ is a polynomial of degree n, then

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|. \tag{1.1}$$

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The result is sharp and equality holds in (1.1) for $p(z) = \lambda z^n$, λ being a complex number. Inequality (1.1) can be sharpened if we restrict ourselves to the class of polynomials having no zeros in $|z| < 1$. In this connection, the following result was conjectured by Erdös [8] and later verified by Lax [11].

Theorem B : If $p(z)$ is a polynomial of degree n having no zero in $|z| < 1$, then

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{1.2}$$

The result is sharp and equality holds in (1.2) for $p(z) = \alpha + \beta z^n$, where $|\alpha| = |\beta|$. Simple proofs of (1.2) were given by de-Bruijn [6], and later by Aziz and Mohammad [2].

Malik [12] considered the class of polynomials $p(z) \neq 0$ in $|z| < k$, $k \geq 1$ and proved the following generalization of Theorem B.

Theorem C : If $p(z)$ is a polynomial of degree n having no zero in the disk $|z| < k$, $k \geq 1$, then

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1+k} \max_{|z|=1} |p(z)|. \tag{1.3}$$

The result is best possible and equality holds for $p(z) = (z+k)^n$.

Govil [9] improved upon Theorem C and proved the following result.

Theorem D : If $p(z)$ is a polynomial of degree n having no zero in $|z| < k$, $k \geq 1$, then

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1+k} \left[\max_{|z|=1} |p(z)| - \max_{|z|=k} |p(z)| \right]. \tag{1.4}$$

The result is sharp and equality holds for the polynomial $p(z) = (z+k)^n$.

Theorem C was generalized by Bidkham and Dewan [7], who proved the following result.

Theorem E : If $p(z)$ is a polynomial of degree n such that it has no zero in $|z| < k$, $k \geq 1$, then for $1 \leq R \leq k$,

$$\max_{|z|=R} |p'(z)| \leq n \frac{(R+k)^{n-1}}{(1+k)^n} \max_{|z|=1} |p(z)|. \tag{1.5}$$

The result is best possible and extremal polynomial is $p(z) = \left(\frac{z+k}{1+k}\right)^n$.

Let α be a complex number. If $p(z)$ is a polynomial of degree n, then polar derivative of $p(z)$ with respect to point α, denoted by $D_\alpha p(z)$, is defined by

$$D_\alpha p(z) = np(z) + (\alpha - z)p'(z). \tag{1.6}$$

Clearly $D_\alpha p(z)$ is a polynomial of degree $n-1$ and it generalizes the ordinary derivative (see Marden [13]) in the sense that

$$\lim_{\alpha\to\infty}\left[\frac{D_\alpha p(z)}{\alpha}\right]=p'(z). \tag{1.7}$$

Laguerre [10] called the polynomial $D_\alpha p(z)$ the "emanant" of $p(z)$, Pȯlya and Szegö [14] called "the derivative of $p(z)$ with respect to the point α" and Marden [13] simply called "the polar derivative of $p(z)$". It is obviously of interest to obtain estimates concerning the growth of $D_\alpha p(z)$.

For the class of polynomials not vanishing in the disc $|z|<k$, $k\geq 1$, Aziz [1] proved the following result, which extends Theorem C to the polar derivative of $p(z)$.

Theorem F : If $p(z)$ is a polynomial of degree n having no zeros in the disc $|z|<k$, $k\geq 1$, then for every real or complex number α with $|\alpha|\geq 1$,

$$\max_{|z|=1}|D_\alpha p(z)|\leq n\left(\frac{k+|\alpha|}{1+k}\right)\max_{|z|=1}|p(z)|. \tag{1.8}$$

The result is best possible and equality in (1.8) holds for the polynomial $p(z)=(z+k)^n$, with real $\alpha\geq 1$.

As a refinement of Theorem F, Aziz and Shah [3] proved the following result, which also extends Theorem D to the polar derivative of $p(z)$.

Theorem G : If $p(z)=\sum\limits_{j=0}^{n}a_jz^j$ is a polynomial of degree n having no zero in $|z|<k$, $k\geq 1$, then for every real or complex number α with $|\alpha|\geq 1$,

$$\max_{|z|=1}|D_\alpha p(z)|\leq\frac{n}{1+k}\left\{(|\alpha|+k)\max_{|z|=1}|p(z)|-(|\alpha|-1)\min_{|z|=k}|p(z)|\right\}. \tag{1.9}$$

The result is best possible and extremal polynomial is $p(z)=\left(\frac{z+k}{1+k}\right)^n$ for real $\alpha\geq k$.

We prove the following result which in turn gives generalization of Theorem G. More precisely, we prove

Theorem 1 : If $p(z)=\sum\limits_{j=0}^{n}a_jz^j$ is a polynomial of degree n having no zeros in $|z|<k$, $k\geq 1$, then for every real or complex number α with $|\alpha|\geq k$ and for $1\leq r\leq k$, we have

$$\max_{|z|=r}|D_\alpha p(z)|\leq n\left[\frac{(|\alpha|+k)(r+k)^{n-1}}{(1+k)^n}\max_{|z|=1}|p(z)|-\left\{\frac{(|\alpha|+k)(r+k)^{n-1}}{(1+k)^n}-1\right\}\min_{|z|=k}|p(z)|\right]. \tag{1.10}$$

If we put $r=1$ in (1.10), we get following result due to Aziz and Shah [3].

Corollary 2 : If $p(z)=\sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n having no zero in $|z|<k$, $l\geq 1$, then for every real or complex number α with $|\alpha|\geq 1$,

$$\max_{|z|=1}|D_\alpha p(z)|\leq \frac{n}{1+k}\left\{(|\alpha|+k)\max_{|z|=1}|p(z)|-(|\alpha|-1)\min_{|z|=k}|p(z)|\right\}. \tag{1.11}$$

The result is best possible and extremal polynomial is $p(z)=\left(\frac{z+k}{1+k}\right)^n$ for real $\alpha\geq k$. Dividing both sides of inequality (1.10) by $|\alpha|$ and letting $|\alpha|\to\infty$, we get the following result which is an improvement of Theorem E.

Corollary 3 : If $p(z)=\sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n having no zeros in $|z|<k$, $k\geq 1$, then for $1\leq r\leq k$,

$$\max_{|z|=r}|p'(z)|\leq \frac{n(r+k)^{n-1}}{(1+k)^n}\left[\max_{|z|=1}|p(z)|-\min_{|z|=k}|p(z)|\right]. \tag{1.12}$$

The result is sharp and equality holds for the polynomial $p(z)=(z+k)^n$.

2. Lemmas

We need the following lemma due to Aziz and Zargar [4] to prove the Theorem 1.

Lemma 2.1 : If $p(z)$ is a polynomial of degree n having no zeros in $|z|<k$, $k\geq 1$, then for $1\leq R\leq k^2$, $m=\min_{|z|=k}|p(z)|$, we have

$$\max_{|z|=R}|p(z)|\leq \left(\frac{R+k}{1+k}\right)^n\max_{|z|=1}|p(z)|-\left\{\left(\frac{R+k}{1+k}\right)^n-1\right\}m. \tag{2.1}$$

3. Proof of the Theorem

Proof of Theorem 1 : Let $1\leq r\leq k$. Since $p(z)$ has no zeros in $|z|<k$, $k\geq 1$, the polynomial $p(rz)$ has no zeros in $|z|<\frac{k}{r}$, $\frac{k}{r}\geq 1$, therefore applying Theorem G to the polynomial $p(rz)$ with $\frac{|\alpha|}{r}\geq 1$, we get

$$\max_{|z|=1}\left|D_{\frac{\alpha}{r}}p(rz)\right|\leq \frac{n}{1+\frac{k}{r}}\left\{\left(\frac{|\alpha|}{r}+\frac{k}{r}\right)\max_{|z|=1}|p(rz)|-\left(\frac{|\alpha}{r}-1\right)\min_{|z|=\frac{k}{r}}|p(rz)|\right\},$$

implies

$$\max_{|z|=1}\left|np(rz)+\left(\frac{\alpha}{r}-z\right)rp'(rz)\right|\leq \frac{n}{(k+r)}\left\{(|\alpha|+k)\max_{|z|=1}|p(rz)|-(|\alpha|-r)\min_{|z|=\frac{k}{r}}|p(rz)|\right\}. \tag{3.1}$$

Inequality (3.1) is equivalent to

$$\max_{|z|=1}|D_\alpha p(z)| \leq \frac{n}{(k+r)}\left\{(|\alpha|+k)\max_{|z|=r}|p(z)| - (|\alpha|-r)\min_{|z|=k}|p(z)|\right\}. \tag{3.2}$$

For $1 \leq r \leq k$, inequality (3.2) when combined with Lemma 2.1, gives

$$\begin{aligned}
\max_{|z|=r}|D_\alpha p(z)| &\leq \frac{n}{(k+r)}\Bigg[(|\alpha|+k)\left\{\left(\frac{r+k}{1+k}\right)^n \max_{|z|=1}|p(z)| \right.\\
&\quad \left. - \left(\left(\frac{r+k}{1+k}\right)^n - 1\right)m\right\} - (|\alpha|-r)m\Bigg]\\
&= \frac{n}{(k+r)}\Bigg[\frac{(|\alpha|+k)(r+k)^n}{(1+k)^n}\max_{|z|=1}|p(z)|\\
&\quad - \left\{(\alpha|+k)\left(\left(\frac{r+k}{1+k}\right)^n - 1\right) + (|\alpha|-r)\right\}m\Bigg]\\
&= \frac{n}{(k+r)}\Bigg[\frac{(|\alpha|+k)(r+k)^n}{(1+k)^n}\max_{|z|=1}|p(z)|\\
&\quad - \left\{(\alpha|+k)\left(\frac{r+k}{1+k}\right)^n - (|\alpha|+k) + (|\alpha|-r)\right\}m\Bigg]\\
&= n\left[\frac{(|\alpha|+k)(r+k)^{n-1}}{(1+k)^n}\max_{|z|=1}|p(z)| - \left\{\frac{(|\alpha|+k)(r+k)^{n-1}}{(1+k)^n} - 1\right\}m\right].
\end{aligned}$$

From which is the required result follows.

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