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CONTINUOUS FUNCTION OF FUZZY-VALUED VARIABLE AND SOME OF IT'S PROPERTIES

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Abstract

This paper relates to the definitions and properties of continuous function of fuzzy sets. Throughout the research, a fuzzy set is considered as a variable which is called fuzzy-valued variable. A layered approach is taken and emphasis is given on modularity. The concerned fuzzy-valued variable is made to sit at the top of the pyramid with semantically less vague terms comprise the base. With this approach, fuzzy continuous function is defined and different properties are proved. It is compared with classical real continuous function and its properties. All these have been done in the light of fuzziness so that the transition from crisp to fuzzy or fuzzy to crisp can be explained.

1. Introduction

Functions, particularly continuous functions, play a crucial role in building real and complex analysis in classical mathematics. Fuzzy mathematics, as an extension of classical mathematics should possess concepts of function, its continuity and other related

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Key Words : *Fuzzy-valued variable, Continuous function, Functional analysis.*

aspects. The building blocks of fuzzy mathematics are fuzzy sets which are nothing but real-valued context-and-concept-dependent functions defined on a crisp universal set. So, functions in fuzzy mathematics should ideally be defined as a kind of extension of functions of classical mathematics. We, here, cultivate continuous functions of fuzzy-valued variables. Fuzzy continuous functions are introduced by different researchers in different ways. We define it mainly as an extension of crisp continuous functions. Firstly, function of fuzzy- valued variables is defined in terms of existence of real valued functions valid for all elements of the universal set. Existence of such real valued functions are shown to be guaranteed. Next are explained the concept of limit, monotonicity and boundedness of fuzzy-valued variables. The well known bounded-ness property of crisp continuous function is proved to be true for fuzzy continuous function provided certain conditions are fulfilled. Some properties regarding monotonicity are also proved. The most important feature of the paper is that it relates continuity of fuzzy function with fuzziness involved in fuzzy set. Fuzziness is the key component that decides whether a set is a fuzzy set or a crisp one. Therefore, it gets nice if fuzzy counterpart of some well-known and well-discussed property of classical mathematics goes in sync with fuzziness. Section 2 of the paper recalls some known properties of continuous function of classical real analysis and a few characteristics of fuzzy set and fuzzy-valued variable. Section 3 deals with function of fuzzy-valued variable, its limit, continuity and relation with fuzziness. Section 4 draws a formal conclusion to the discussion.

Review of Previous Work : Continuous fuzzy functions are approximated by fuzzy polynomials (Liu, 2002). In (Azad, 1981), author studied fuzzy topological spaces with specific attention to the weaker forms of fuzzy continuity. In (Bhaumick et. al, 1993), a new class of functions called fuzzy weakly completely continuous functions is introduced and studied. The aim of the paper (Mukherjee, 1993) is to introduce the concept of fuzzy faintly continuous function. These functions have been characterized and investigated mainly in the light of the notions of quasi-concidence, q-neighborhood, fuzzy-interior and fuzzy-closure. It is seen that fuzzy continuity implies fuzzy faintly continuity but not conversely. The converse is also true if the co-domain of the function is a fuzzy regular space. Finally, a comparative study regarding the mutual interrelations among fuzzy R-map, fuzzy completely continuous, fuzzy almost continuous and fuzzy continuous functions along with fuzzy faintly continuous function is made. The paper by

(Mukherjee et. al, 1991), deals with the study of almost continuous and weakly continuous multi-functions in fuzzy setting. In (Guang-Quan, 1991), the authors introduces the concept of fuzzy distance of fuzzy numbers, fuzzy limit of fuzzy numbers, fuzzy function and fuzzy limit of a fuzzy function and fuzzy continuous function on the set of fuzzy numbers and give some of their elementary properties. He also discusses some important theorems of fuzzy numbers and the fuzzy continuous function on the M-closed interval. In (Hanafy, 2006), after giving the basic results related to the product of functions and the graph of functions in intuitionistic fuzzy topological spaces, Hanafy introduces and studies the concept of fuzzy completely continuous functions between intuitionistic fuzzy topological spaces. Each of (Georgiou et al, 1999), (Wang, 2006) and (Yalvac, 1987) deals with different aspects of fuzzy sets in connection with fuzzy topological spaces. In (Ekici, 2004), some properties of fuzzy continuous function and its various weaker and stronger forms are studied. In their paper (Thangaraj et. al, 2007), a comparative study regarding the interrelations among the fuzzy strongly continuous, fuzzy perfectly continuous, fuzzy a-continuous, fuzzy pre-continuous and fuzzy continuous functions with fuzzy contra-continuous functions is made by Thangaraj and Balasubramanium. Preservation to some fuzzy topological structures is examined under these functions. Function spaces play an important role in complex analysis in the theory of differential equations, in functional analysis and in almost every other branch of modern mathematics. Let, $FC(Y;Z)$ be the set of all fuzzy continuous functions from a fuzzy topological space Y into a fuzzy topological space Z. The aim of paper by (Ganster et. al, 2005) is to study the notion of group, fuzzy group, topological group and fuzzy topological group on the fuzzy function space $FC(Y,Z)$.

2. Real Continuous Function, Fuzzy-valued Variable and Aggregation Operation

In this section we memorize some ideas of classical as well as fuzzy mathematics those will help the readers to understand the following section.

Definition 2.1 : A real continuous function f in classical mathematics is a rule from a crisp set A to a crisp set B such that for any small real number $\epsilon > 0$, there exists a real number $\delta > 0$ so that

$$\forall \ x_1, x_2 \in A, \ |x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \epsilon \tag{1}$$

where $f(x) \in B$, the f-image of $x \in A$ exists for all $x \in A$ and we write, $y = f(x)$.
We mention below some important properties of real continuous function without proof, those are relevant to our topic of discussion. The proofs are available in any book of mathematical analysis.

Theorem 2.2 : For a real continuous function $y = f(x)$ defined over a closed interval $[a, b]$

(i) f is bounded on $[a, b]$ and attains its bounds, and

(ii) if $f(a) \neq f(b)$ then f attains every value between $f(a)$ and $f(b)$ at least once in the open interval (a, b).

Now, we recapitulate some definitions in the fuzzy context. Though the definition of our main tool viz., fuzzy-valued variable has been given in the previous chapter, we are giving it once again here to help the readers.

Definition 2.3 : A fuzzy-valued variable X is defined as a function $X : F \to F$ where F is the set of normal, convex fuzzy sets defined over a finite universal set $U = \{x_1, x_2, \cdots, x_p\}$.

Definition 2.4 : Aggregation function on n fuzzy sets is defined as a function $h : [0,1]^n \to [0,1], (n \geq 2)$. When applied to fuzzy sets $X_1, X_2, X_3, \cdots, X_n$ defined over universal set U, it produces an aggregate fuzzy set X say, such that for all $x_k \in U, k = 1, 2, \cdots, p$ [?]

$$\mu_X(x_k) = h(\mu_{X_1}(x_k), \mu_{X_2}(x_k), \cdots, \mu_{X_n}(x_k)). \tag{2}$$

In order to qualify as an intuitively meaningful aggregation function, h must satisfy at least three following axioms.

Axiom 1 : (Boundary condition) $h(0, 0, \cdots, 0) = 0$ and $h(1, 1, \cdots, 1) = 1$.

Axiom 2 : (Monotonicity property) For any pair $(a_1, a_2, \cdots, a_n)$ and $(b_1, b_2, \cdots, b_n)$ of n-tuples where $a_i, b_i \in [0,1], i = 1, 2, \cdots, n$,

$$a_i \leq b_i, \quad \text{for all} \quad i = 1, 2, \cdots, n$$

$$\Rightarrow h(a_1, a_2, \cdots, a_n) \leq h(b_1, b_2, \cdots, b_n).$$

Axiom 3 : (Continuity) h is continuous.

Aggregation function on fuzzy sets is a function by which several fuzzy sets are combined in a desirable way to produce a single fuzzy set that can meaningfully express an overall idea about the system.

3 Continuous Function of Fuzzy-valued Variable

Let F be the set of all normal, convex fuzzy sets defined over a finite universal set $U = \{x_1, x_2, \cdots, x_p\}$.

Let X and Y be two fuzzy-valued variables defined over F which can take the sets $X_1, X_2, \cdots, X_n$ and $Y_1, Y_2, \cdots, Y_n$ respectively as their values where X_i's, Y_i's are fuzzy sets defined over U.

Definition 3.1 : We denote the aggregate of fuzzy-valued variable X by X_A and define it as a fuzzy set defined over U such that for all x_k in U,

$$\mu_{X_A}(x_k) = h(\mu_{X_1}(x_k), \mu_{X_2}(x_k), \cdots, \mu_{X_n}(x_k)). \tag{3}$$

for $k = 1, 2, \cdots, p$, where h is an aggregation function from $[0,1]^n \to [0,1]$.

Definition 3.2 : $f : F \to F$ is called a function of fuzzy-valued variable X and we write $Y = f(X)$ if there exists a continuous function $f_r : [0,1] \to [0,1]$ for each x_k belonging to U such that

$$\mu_{Y_A^j}(x_k) = f_r(\mu_{X_A^j}(x_k)). \tag{4}$$

where X_A^j and Y_A^j are aggregates of fuzzy-valued variables X^j and Y^j respectively by the same aggregation function h. For different j, X^j represents the same linguistic variable X measured in different ways and Y^j is the corresponding representation of Y. We call Y, the f-image of X. For each x_k belonging to U, we can find points $(\mu_{X_A^j}(x_k), \mu_{Y_A^j}(x_k))$ for finite number of j and by polynomial interpolation f_r can be constructed. Obviously, for larger number of j, better would be the accuracy of the interpolating polynomial $f_r : X_A^j, Y_A^j$ are called the pivotal aggregates.

As for example, fuzzy-valued variable "personality"' is a function of "intelligence", "health" is a function of "nutrition" and "beauty" is a function of "health" and so on. Moreover, min operator, max operator and arithmetic mean are some of the aggregation functions for which function of fuzzy-valued variable exists.

Let us consider the min operator as the aggregation operator concerned. For all x_k in

U,

$$\mu_{X_A}(x_k) = \min(\mu_{X_1}(x_k), \mu_{X_2}(x_k), \cdots, \mu_{X_n}(x_k)) \tag{5}$$

and

$$\mu_{Y_A}(x_k) = \min(\mu_{Y_1}(x_k), \mu_{Y_2}(x_k), \cdots, \mu_{Y_n}(x_k)) \tag{6}$$

for $k = 1, 2, \cdots, p$.

For each x_k we construct $f_r : [0,1] \to [0,1]$ by interpolating points $(\mu_{X_A}(x_k), \mu_{Y_A}(x_k))$ where

$$\min(\mu_{Y_1}(x_k), \mu_{Y_2}(x_k), \cdots, \mu_{Y_n}(x_k)) = f_r(\min(\mu_{X_1}(x_k), \mu_{X_2}(x_k), \cdots, \mu_{X_n}(x_k)))$$

i.e.,

$$\mu_{Y_A}(x_k) = f_r(\mu_{X_A}(x_k)) \tag{7}$$

and the interpolating points are obtained by using X available from different sources. Moreover, if X and Y are intuitively directly proportional, we consider additional points $(0,0)$ and $(1,1)$. Otherwise, we consider points $(0,1)$ and $(1,0)$ as two more interpolating points while constructing f_r. The totality of f_r for all x_k is said to constitute the function of fuzzy-valued variable X. Therefore, a function $f : F \to F$ for fuzzy-valued variable X exists with the minimum operator as the underlying aggregation function.

Similar results can also be obtained for other aggregation operators. Let $f : F \to F$ be a function of fuzzy-valued variable X such that $Y = f(X)$. Therefore for certain aggregation function $h : [0,1]^n \to [0,1]$ we get different aggregates X_A and Y_A from different representations of fuzzy-valued variables X and Y respectively. Also, for each x_k there exists continuous function $f_r : [0,1] \to [0,1]$ such that

$$\mu_{Y_A}(x_k) = f_r(\mu_{X_A}(x_k)).$$

Limit of function $Y = f(X)$ is said to exist and equal to $L \in F$ as $X \to P$, where $P \in F$ is a representation of X if limit of f_r exists element-wise i.e.,

$$\lim_{\mu_{X_A}(x_k) \to \mu_{P_A}(x_k)} f_r(\mu_{X_A}(x_k)) = \mu_{L_A}(x_k) \tag{8}$$

holds for all x_k in U. Formally, if for any pre-assigned $\epsilon > 0$, there exists $\delta > 0$ such that for all x_k in U,

$$|\mu_{Y_A}(x_k) - \mu_{L_A}(x_k)| < \epsilon$$

whenever

$$|\mu_{X_A}(x_k) - \mu_{P_A}(x_k)| < \delta$$

holds, we say

$$\lim_{\mu_{X_A}(x_k) \to \mu_{P_A}(x_k)} f_r(\mu_{X_A}(x_k)) = \mu_{L_A}(x_k)$$

holds for all x_k in U, i.e.,

$$\lim_{X \to P} f(X) = L$$

exists. Since f_r is continuous, limit of f_r exists always and hence limit of f also exists. Then L would be another representation of Y.

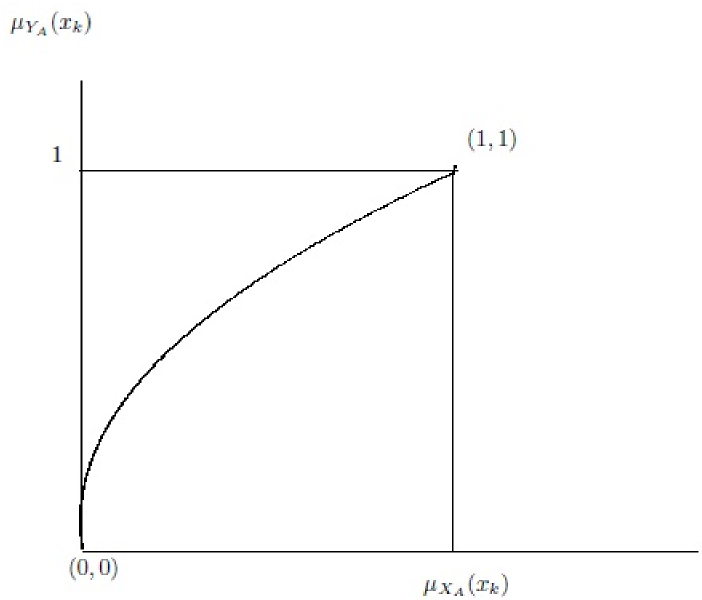

Fig 3.1: f_r for x_k

Definition 3.3 : $f : F \to F$ is continuous for P in F where P is a representation of X if for each x_k, f_r-image of $\mu_{P_A}(x_k)$ lies on the constructed f_r irrespective of the method of interpolation used.

Theorem 3.4 : If X is consistent in the sense that two different representations of X do not vary much when membership values of elements are considered, then Y is also consistent.

Proof : For function $f : F \to F$ there exists continuous functions f_r for each x_k. Let, X_1 and X_2 be two different representations of X. For a particular x_k, since f_r is continuous, we get,

$$|_{X_A^1}(x_k) - \mu_{X_A^2}(x_k)| < \delta$$

$$\Rightarrow |f_r(\mu_{X_A^1}(x_k)) - f_r(\mu_{X_A^2}(x_k))| < \epsilon$$

$$\Rightarrow |\mu_{Y_A^1}(x_k) - \mu_{Y_A^2}(x_k)| < \epsilon. \tag{9}$$

i.e., X is consistent implies that Y is consistent. Here, Y^1 and Y^2 are the f-images of X^1 and X^2 respectively.

Definition 3.5 : A function $f : F \to F$ is said to be monotone if for each $x_k \in U$, f_r is monotone in the sense that either

(i) (monotone increasing)

$$\mu_{X_A^1}(x_k) \leq \mu_{X_A^2}(x_k) \Leftrightarrow f_r(\mu_{X_A^1}(x_k)) \leq f_r(\mu_{X_A^2}(x_k)) \tag{10}$$

or,

(ii) (monotone decreasing)

$$\mu_{X_A^1}(x_k) \leq \mu_{X_A^2}(x_k) \Leftrightarrow f_r(\mu_{X_A^1}(x_k)) \geq f_r(\mu_{X_A^2}(x_k)) \tag{11}$$

where X_A^1, X_A^2 are the aggregates of two representations X^1, X^2 of X respectively under certain aggregation function.

Definition 3.6 : A crisp set S of different representations of fuzzy-valued variables X is said to be bounded if there exists representations X^M and X^m of X such that for each $x_k \in U$

$$\mu_{X_A^m}(x_k) \leq \mu_{X_A}(x_k) \leq \mu_{X_A^M}(x_k)$$

for X in S.

Theorem 3.7 : Let f be a continuous function of fuzzy-valued variable on $S \subset F$, the set of all fuzzy sets defined over $U = \{x_1, x_2, \cdots, x_p\}$ in the sense as described earlier. Let S be bounded. Then, the set $T = \{Y \in F/Y = f(X), X \in S\}$ is bounded. Moreover, if f is a monotone function then bounds of Y are f-images of bounds of X.

Proof : Given, S is bounded in the sense that there exist representations of X, X^m and X^M in S, such that, for each $x_k \in U$

$$\mu_{X_A^m}(x_k) \leq \mu_{X_A}(x_k) \leq \mu_{X_A^M}(x_k)$$

for all X in S where X_A is the aggregate of X for some aggregation function.
Now, f_r is always continuous on $[0, 1]$.
Let for $x_k \in U$

$$\mu_{Y_A^m}(x_k) = f_r(\mu_{X_A^m}(x_k))$$

and

$$\mu_{Y_A^M}(x_k) = f_r(\mu_{X_A^M}(x_k)).$$

By bound-property of real continuous function $f_r : [0, 1] \to [0, 1]$, there exists $\alpha_k, \beta_k \in [0, 1]$ such that $\alpha_k \leq f_r(\mu_{X_A}(x_k)) \leq \beta_k$ for all X in S, $k = 1, 2, \cdots, p$. Now we take Y_A^1, Y_A^2 in F such that $\mu_{Y_A^1}(x_k) = \alpha_k$ and $\mu_{Y_A^2}(x_k) = \beta_k$ for $k = 1m2, \cdots, p$. Therefore,

$$\mu_{T_A^1}(x_k) \leq f_r(\mu_{X_A}(x_k)) \leq \mu_{Y_A^2}(x_k).$$

for all $k = 1, 2, \cdots, p$ for all X in S, i.e.,

$$\mu_{Y_A^1}(xk) \leq \mu_{Y_A}(x_k) \leq \mu_{Y_A^2}(x_k)$$

for all $k = 1, 2, \cdots, p$ and for all X in S. That is, the set

$$T = \{Y \in F/Y = f(X), \quad X \in S\}$$

is bounded in the same sense as that of S.
Now our interest lies in whether X_A^m and X_A^M are mapped to Y_A^1 and Y_A^2 by f.
If f and hence f_r is monotone increasing then for all x_k in U,

$$\mu_{X_A^m}(xk) \leq \mu_{X_A}(xk) \leq \mu_{X_A^M}(x_k) \text{ for all } X \text{ in } S$$

$$\Rightarrow f_r(\mu_{X_A^m}(x_k)) \leq f_r(\mu_{X_A}(x_k)) \leq f_r(\mu_{X_A^M}(x_k)) \text{ for all } X \text{ in } S$$

$$\Rightarrow \mu_{Y_A^m}(x_k) \leq \mu_{Y_A}(x_k) \leq \mu_{Y_A^M}(x_k) \text{ for all } Y \text{ in } T$$

i.e.,

$$Y_A^1 = Y_A^m, Y_A^2 = Y_A^M$$

and if f is monotone decreasing, then in a similar way we can show that for all x_k in U,

$$\mu_{X_A^m}(x_k) \leq \mu_{X_A}(x_k) \leq \mu_{X_A^M}(x_k) \text{ for all } X \text{ in } S$$

$$\Rightarrow \mu_{Y_A^m}(x_k) \geq \mu_{Y_A}(x_k) \geq \mu_{Y_A^M}(x_k)$$

for all Y in T i.e.,

$$Y_A^1 = Y_A^M$$

and

$$Y_A^2 = Y_A^m.$$

Hence proved the result.

Theorem 3.8 : Let f be continuous function of fuzzy-valued variable X on $S \subset F$ and S is bounded in the fuzzy sense defined here. Let X^M and X^m be the bounds of S. Then for any Y lying between Y^M and Y^m there exists X lying between X^M and X^m, where $Y^M = f(X^M)$ and $Y^m = f(X^m)$.

Proof : Given, X^M and X^m are bounds of the set S and $Y^M = f(X^M)$ and $Y^m = f(X^m)$. Let us consider an arbitrary Y^P such that Y^P lies between Y^M and Y^m in the sense that

$$\mu_{Y_A^P}(x_k) \in [\mu_{Y_A^m}(x_k), \mu_{Y_A^M}(x_k)]$$

for all x_k in U, i.e.,

$$\mu_{Y_A^P}(x_k) \in [f_r(\mu_{X_A^m}(x_k)), f + r(\mu_{X_A^M}(x_k))]$$

for all x_k in U. Since f_r is continuous on $[0,1]$, there exists $\alpha_k \in [\mu_{X_A^m}(x_k), \mu_{X_A^M}(x_k)]$ such that $f_r(\alpha_k) = \mu_{Y_A^P}(x_k), k = 1, 2, \cdots, p$. We construct X_A^P such that $\alpha_k = \mu_{X_A^P}(x_k)$, where $k = 1, 2, \cdots, p$. Therefore, such a X_A^P exists that

$$\mu_{Y_A^P}(x_k) = f_r(\mu_{X_A^P}(x_k))$$

for $k = 1, 2, \cdots, p$ and X^P lies between X^M and X^m in the sense that

$$\mu_{X_A^P}(x_k) \in (\mu_{X_A^m}(x_k), \mu_{X_A^M}(x_k)) \ \ \forall \ \ x_k \in U.$$

Hence proved the theorem.

Let X and Y be fuzzy-valued variables with values $X_1, X_2, \cdots, X_n$ and $Y_1, Y_2, \cdots, Y_n$ respectively. $X, Y, X_i, Y_i \in F$ for $i = 1, 2, \cdots, n$ where F bears usual meaning. Let h be the relevant aggregation function and $f : F \to F$ be a function of fuzzy-valued variable such that $Y = f(X)$. $f : F \to F$ is defined to be a morphism of fuzzy-valued variable if for each x_k corresponding continuous function $f_r : [0,1] \to [0,1]$ be such that

$$f_r(\mu_{X_i}(x_k)) = \mu_{Y_i}(x_k)$$

for all $i = 1, 2, \cdots, n$.

Since values of X and Y are chosen by an agent according to the need of a situation, it may so happen that they are not equal in numbers. In such cases, we shall make them equal by considering pseudo values. These pseudo values are fuzzy sets for which every element has membership 0.

We now consider how continuity of a function of fuzzy-valued variable affects the fuzziness involved in the variable.

But first let us discuss about fuzziness of a fuzzy-valued variable. General class of measure of fuzziness for fuzzy-valued variable X can be defined as

$$f(X) = g_{X_i}(h_{x \in A}(d(\mu_{X_i}(x), C\mu_{X_i}(x))))$$

where $h_{x\in A} : R^+ \to R^+ \cup \{0\}$ and $g_{X_i} : R^+ \to R^+ \cup \{0\}$ are aggregation functions, d is a metric distance and C is fuzzy complement function. Since we are concentrating on the concept of fuzziness which is based on lack of distinction between a set and its complement, we measure the local distinction of X and $\acute{X}$ for the element $x_k \in U$ in terms of its constituent fuzzy sets X_i and $\acute{X}_i$ as $\sum\limits_{i=1}^{n} |2\mu_{X_i}(x_k) - 1|$.

Therefore, the lack of distinction between X and $\acute{X}$ is taken to be measured as

$$\sum_{k=1}^{p} \left(n - \sum_{i=1}^{n} |2\mu_{X_i}(x_k) - 1| \right).$$

Thus, we measure fuzziness involved in fuzzy-valued variable X as a function

$$\psi : F \to R^+ \cup \{0\}$$

where F is the set of all fuzzy sets defined over universal set $U = \{x_1, x_2, \cdots, x_p\}$ and we define

$$\psi(X) = \sum_{k=1}^{p} \left(n - \sum_{i=1}^{n} |2\mu_{X_i}(x_k) - 1| \right)$$

$$= np - \sum_i \sum_k |2\mu_{X_i}(x_k) - 1|. \quad (12)$$

Definition 3.9 : Let $\phi : F \to F$ be a morphism of fuzzy-valued variable X such that $Y = \phi(X)$. Let $\psi : F \to R^+ \cup \{0\}$ represents measure of fuzziness involved in X. We define a function $\phi_r^{Ext} : R^+ \cup \{0\} \Rightarrow R^+ \cup \{0\}$ such that

$$\psi(Y) = \phi_r^{Ext}(\psi(X)).$$

Theorem 3.10 : Let $\phi : F \to F$ be a continuous morphism of fuzzy-valued variable X. Let $\psi : F \to R^+ \cup \{0\}$ represents the measure of fuzziness. $\phi^{Ext} f_r : R^+ \cup \{0\} \to R^+ \cup \{0\}$ is continuous.

Proof : $F \to F$ is continuous morphism of fuzzy-valued variable with $Y = \phi(X)$. Let the values of fuzzy-valued variables X and Y be $X_1, X_2, \cdots, X_n$ and $Y_1, Y_2, \cdots, Y_n$ respectively. Therefore, for each x_k, $\exists\ \phi_r : [0,1] \to [0,1]$ such that

$$\phi_r(\mu_{X_i}(x_k)) = \mu_{Y_i}(x_k)$$

for all $i = 1, 2, \cdots, n$.

Again, due to continuity of ϕ, for any preassigned small $\epsilon' > 0$, $\exists \delta' > 0$ such that for all $i = 1, 2, \cdots, n$ and $k = 1, 2, \cdots, p$

$$|\mu_{X_i^1}(x_k) - \mu_{X_i^2}(x_k)| < \delta' \Rightarrow |\phi_r(\mu_{X_i^1}(x_k)) - \phi_r(\mu_{X_i^2}(x_k))| < \epsilon' \quad (13)$$

where X_i^1 and X_i^2, $i = 1, 2, \cdots, n$ are values assigned to two representations X^1 and X^2 of X respectively.

Now,

$$|\psi(X^1) - \psi(X^2)|$$

$$= \left| np - \sum_k \sum_i |2\mu_{X_i^1}(x_k) - 1| - np + \sum_k \sum_i |2\mu_{X_i^2}(x_k) - 1 \right|$$

$$= \left| \sum_k \sum_i [|2\mu_{X_i^2}(x_k) - 1| - |2\mu_{X_i^1}(x_k) - 1|] \right| \quad (14)$$

$$\leq \sum_k \sum_i |2\mu_{X_i^2}(x_k) - 2\mu_{X_i^1}(x_k)|$$

$$\leq 2np\delta'$$

by equation (13). Again,

$$\begin{aligned} &|\psi(Y^1) - \psi(Y^2)| \\ &= \left|\sum_k \sum_i [|2\phi_r(\mu_{X_i^2}(x_k)) - 1| - |2\phi_r(\mu_{X_i^1}(x_k)) - 1|]\right| \\ &\le \sum_k \sum_i \|2\phi_r(\mu_{X_i^2}(x_k)) - 2\phi_r(\mu_{X_i^1}(x_k))\| \\ &< 2np\epsilon' \end{aligned} \tag{15}$$

by equation (13).
Taking $2np\delta' = \delta$ and $2np\epsilon' = \epsilon$, we get, for continuous morphism $\phi : F \to F$, for any $\epsilon > 0, \exists \ \delta > 0$ such that

$$|\psi(Y^1) - \psi(Y^2)| < \epsilon \quad \text{whenever} \ \ |\psi(X^1) - \psi(X^2)| < \delta.$$

Therefore, the function $\phi_r^{Ext} : R^+ \to R^+$ is continuous.
The above result shows that the change in fuzziness in $Y \in F$ is small if the change of the same is small in $X \in F$ where Y is the image of X under a continuous morphism.

4. Examples

Example 4.1 : We take the example of "personalty" and "intelligence". As values of "personality (P)", we choose fuzzy sets "well-behavior (WB)" and "charm (C)".
For "intelligence (I)" we take "knowledge (K)" and "mental sharpness (M)".
Let,

$$U = \{x_1, x_2, x_3, x_4, x_5\}.$$

We consider three different ways (either by using different groups of knowledge engineers or by using different sets of questionnaire) to measure the fuzzy sets "well-behavior (WB)", "charm (C)", "knowledge (K)" and "mental sharpness (M)". Let, the following are results of one such measure.

Let,

$$WB = \{0.3/x_1 + 0.9/x_2 + 0.7/x_3 + 0.2/x_4 + 0.3/x_5\}$$

$$C = \{0.4/x_1 + 0.6/x_2 + 0.2/x_3 + 0.7/x_4 + 0.9/x_5\}$$

$$K = \{0.4/x_1 + 0.9/x_2 + 0.3/x_3 + 0.4/x_4 + 0.1/x_5\}$$

$$M = \{0.8/x_1 + 0.5/x_2 + 0.4/x_3 + 0.5/x_4 + 0.6/x_5\}.$$

We choose arithmetic mean as the aggregation operation and P_A^1 and I_A^1 as given below are the resulting "personality" and "intelligence" sets. (using equation (3.3))

$$P_A^1 = \{0.35/x_1 + 0.75/x_2 + 0.45/x_3 + 0.45/x_4 + 0.6/x_5\}$$

$$I_A^1 = \{0.6/x_1 + 0.7/x_2 + 0.35/x_3 + 0.3/x_4 + 0.5/x_5\}.$$

Two other sets representing personality and intelligence are given by

$$P_A^2 = \{0.39/x_1 + 0.8/x_2 + 0.3/x_3 + 0.5/x_4 + 0.8/x_5\}$$

$$P_A^3 = \{0.5 = x_1 + 0.6/x_2 + 0.34/x_3 + 0.61/x_4 + 0.7/x_5\}$$

$$I_A^2 = \{0.5/x_1 + 0.75/x_2 + 0.6/x_3 + 0.42/x_4 + 0.63.x_5\}$$

$$I_A^3 = \{0.55/x_1 + 0.81/x_2 + 0.59/x_3 + 0.41/x_4 + 0.7/x_5\}$$

These are pivotal aggregates. For each $x_k, k = 1, 2, 3, 4, 5$ we construct $f_r : [0, 1] \to [0, 1]$ using Definition 3.3.2.

For $x_1 \in U$, we interpolate points $(0.6, 0.35)$, $(0.5, 0.39)$, $(0.55, 0.5)$, $(0, 0)$ and $(1, 1)$ to get f_r.

For $x_2 \in U$, f_r is obtained by interpolating $(0.7, 0.75)$, $(0.75, 0.8)$, $(0.81, 0.6)$, $(0, 0)$ and $(1, 1)$.

For $x + 3$, f_r is obtained from $(0.35, 0.45)$, $(0.6, 0.3)$, $(0.59, 0.34)$, $(0, 0)$ and $(1, 1)$.

Similarly, for x_4 and x_5 interpolating points are respectively

$$\{(0.3, 0.45), (0.42, 0.5), (0.41, 0.61), (0, 0), (1, 1)\} \text{ and}$$

$$\{(0.5, 0.6), (0.63, 0.8), (0.7, 0.7), (0, 0), (1, 1)\}.$$

Since, intuitively, personality and intelligence are directly proportional, in each case we take two additional points $(0, 0)$ and $(1, 1)$.

Now, if by using some other method, intelligence of these five people can be measured, their personalities can be obtained from f_r.

Example 4.2 : Let us consider "health (H)" as a function f of "nutrition (N)". "Health" has two values, "physique (P)" and "ailment (A)". "Nutrition" has two values "availability of square-meal(SqM)" and "disease-treatment (DT)".

In this example, we see how the method described in example 3.4.1 can be refined by considering the fuzziness of the fuzzy sets.
Let

$$U = \{x_1, x_2, x_3, x_4, x_5\}.$$

We consider arithmetic mean as the aggregation function. Let three different representations of H_A and N_A are obtained as

$$H_A^1 = \{0.4/x_1 + 0.5/x_2 + 0.55.x_3 + 0.52/x_4 + 0.55/x_5\},$$

$$H_A^2 = \{0.7/x_1 + 0.4/x_2 + 0.6/x_3 + 0.43/x_4 + 0.5/x_5\},$$

$$H_A^3 = \{0.35/x_1 + 0.39/x_2 + 0.71/x_3 + 0.64/x_4 + 0.6/x_5\},$$

$$N_A^1 = \{0.3/x_1 + 0.6/x_2 + 0.6/x_3 + 0.45/x_4 + 0.6/x_5\},$$

$$N_A^2 = \{0.4/x_1 + 0.7/x_2 + 0.63/x_3 + 0.55/x_4 + 0.57/x_5\} \text{ and}$$

$$N_A^3 = \{0.51/x_1 + 0.64/x_2 + 0.72/x_3 + 0.49/x_4 + 0.65/x_5\}.$$

We consider f to be a morphism. So in constructing f_r for each x_k, the constituent sets of "health" and "nutrition" will play major roles. Let

$$P^1 = \{0.3/x_1 + 0.6/x_2 + 0.8/x_3 + 0.7/x_4 + 0.9/x_5\}$$

and

$$A^1 = \{0.5/x_1 + 0.4/x_2 + 0.3/x_3 + 0.35/x_4 + 0.2/x_5\}$$

are the constituent sets of H_A^1. Similarly,

$$SqM^1 = \{0.4/x_1 + 0.5/x_2 + 0.7/x_3 + 0.6/x_4 + 0.8/x_5\}$$

and

$$DT^1 = \{0.2/x_1 + 0.7/x_2 + 0.5/x_3 + 0.3/x_4 + 0.4/x_5\}$$

are the values of N_A^1 . Similar sets are obtained for each of H_A^2, H_A^3, N_A^2 and N_A^3. Since f is a morphism, apart from relating H_A and N_A, f_r should relate these constituent sets also. Therefore, for x_1 we get points $(0, 0), (0.3, 0.4)$, $(0.4, 0.7)$, $(0.51, 0.35)$, $(0.4, 0.3)$, $(0.2, 0.5)$, $(0.6, 0.4)$, $(0.3, 0.7)$, $(0.5, 0.7)$, $(0.2, 0.5)$, $(1, 1)$ and interpolate them to find

f_r. Similarly, we can find f_r for each member of U. Now, we find the fuzziness of fuzzy-valued variable "health" and "nutrition". We get,

$$\psi(H_A^1) = 6.1, \psi(H_A^2) = 5.53, \psi(H_A^3) = 4.62.$$

Taking average of three we take fuzziness of "health" as $(H) = 5.41$. Similar method gives fuzziness of "nutrition" as $(N) = 6.67$. Here, fuzziness of "health" and "nutrition" are 'low' to the degree of 0.49 and 0.4 respectively.

The definition of fuzzy set "low" is based on the idea that fuzziness of "health" and "nutrition" is $10(= np)$ when maximum.

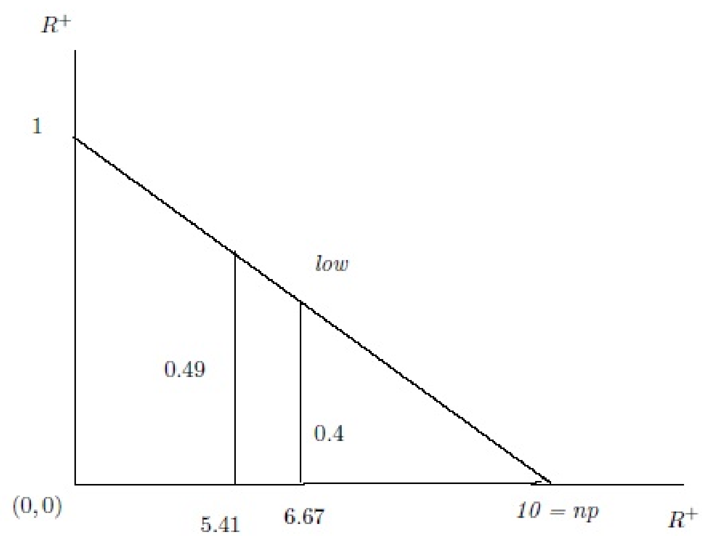

Fig 3.2: Fuzziness of H and N

If with given sets of data, it is observed that fuzziness of both "health" and "nutrition" are considerably low, the given data and the construction of f_r are acceptable. Then, for any given N_A, H_A can be calculated from f_r. If, on the other hand, H and N are fuzzy enough then some other aggregation function might be used so that H and N become more understandable.

The above method can be used in any fuzzy decision making situation where two fuzzy variables are related.

5. Conclusion

The present paper mainly deals with fuzzy function and its continuity property. It is shown that with certain necessary alterations, at least some properties of classical real continuous function hold good for fuzzy function too. Moreover, it is proved that fuzziness, measured as lack of distinction between a set and its complement, changes continuously under continuous morphism. That is to say, if two vague linguistic terms are related by some sort of fuzzy continuous function and if the vagueness of one such term is reduced by some information obtained say from an experiment or test or from any message then it is quite possible that the vagueness involved in the other term gets reduced. But the amount of reduction of fuzziness in both cases will be comparable.
In short, this paper is aimed at contributing a bit of information to the development of something that can be called fuzzy analysis.

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