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CONTROL CHARTS FOR THE GENERALIZED POISSON PROCESS WITH UNDER-DISPERSION

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Abstract

This paper studies control charts based on the generalized Poisson (GP) process in the case of Under-dispersion. The study of an affectation in the control charts based on GP distribution. The $CUSUM$ *chart* based on GP distribution; called the $\lambda_{GP}-CUSUM$ *chart*, $\lambda_{or}-CUSUM$ *chart* and exponentially Weighted Moving Average based on GP distribution ($EWMA_{GP}-chart$). These charts could be used to detect the mean of nonconformities (λ) shifts. The $\lambda_{GP}-CUSUM$ *chart*, $\lambda_{or}-CUSUM$ *chart* and $EWMA_{GP}-chart$ were compared with the $c-chart$ based on GP distribution ($c_{GP}-chart$). The average run length (ARL) is used to consider for the performance. The study results showed that when there was a shift in λ, all levels of parameters, the $EWMA_{GP}-chart$ has the best performance for all levels of the shift.

1. Introduction

The Poisson distribution based on assumption; the mean and variance are both equal (equi-dispersion) to λ, where λ is the mean of nonconformities in a unit sample.

Key Words : *Generalized Poisson distribution, Under-dispersion, Cumulative sum chart, Exponentially weighted moving average, Shewhart control chart of nonconformities.*

However, the suitability of the Poisson distribution does not according to the Poisson assumption. May be either finding of the variance is less than the mean (under-dispersion) or else the variance is greater than the mean (over-dispersion). In the quality control process, the researchers have tried to develop the Poisson distribution to fit this assumption. The generalized Poisson distribution (GPD) was developed by Consul and Jain [10]. They are finding the appropriate alternative for the Poisson process by using the Poisson distribution with mixed dispersion (φ). He et al.[1] developed the attributive chart with exact probabilities ($ACEP$) and the cumulative count of conforming items chart ($CCC - Chart$) to monitoring the parameters of the GPD process. The results shows that when using a single chart to monitor together with λ and φ, it is difficult to tell which parameter has shift.

The most common application is $c - chart$ to monitoring the Poisson process. When occurring in a dispersion situation, the $c - chart$ is an unsuitable control chart. Because it has tighter control limits, which leads to a large number of false alarm rates. The cumulative sum chart ($CUSUM$ $chart$) and the Exponentially Weighted Moving Average ($EWMA$ $chart$) are appropriate charts in a dispersion situation and efficient to monitor smaller shifts in the process, see Montgomery [2]. Page [3] was the researcher who first proposed the $CUSUM$ $chart$ and many authors followed, see Ewan [13] Gan [4], Lucus [6], and Woodall and Adams [14]. The $CUSUM$ $chart$ for the Poisson process proposed by Lucus [7]. He shows that the $CUSUM$ $chart$ was effective for detecting small shifts in the mean. He et al. [11] studied the CUSUM chart for the Zero Inflated Poisson (ZIP) process. They proposed both the $\omega - CUSUM$ $chart$ and $\lambda - CUSUM$ $chart$ based on the $CUSUM$ $chart$. These charts are used to detect individual parameter shifts. The $\omega - CUSUM$ $chart$ and $\lambda - CUSUM$ $chart$ are used for detecting single parameter ω and λ shifts respectively. Katemee and Mayureesawan [8] studied the $CUSUM$ $chart$ for the $Zero$ Inflated Generalized Poisson ($ZIGP$) process. We constructed the $CUSUM$ $chart$ for the $ZIGP$ process. The four charts are $\omega - CUSUM$ $chart$, $\lambda - CUSUM$ $chart$, $\varphi - CUSUM$ $chart$ and $T - CUSUM$ $chart$. We developed the $\omega - CUSUM$ $chart$, $\lambda - CUSUM$ $chart$ and $\varphi - CUSUM$ $chart$ to detecting in shift for individual parameters of ω, λ and φ, respectively. However, the $T - CUSUM$ $chart$ for detecting in shift of together in parameters ω, λ and φ. The results showed that the $\omega - CUSUM$ $chart$, $\lambda - CUSUM$ $chart$ and $\varphi - -CUSUM$ $chart$ are perform

for detecting in shift of individual parameters of ω, λ and φ, respectively. However, the $T-CUSUM$ *chart* is perform for both to detecting in shift of individual parameters λ and to detecting in shift of combined parameters ω, λ and φ. Gan [5] introduced in the $EWMA$ *chart* for monitoring the λ of the Poisson process. He shows that the $EWMA$ *chart* was perform for detecting shift in λ. Katemee and Yasuwan [9] developed a new cumulative sum chart ($CUSUM$ *chart*) based on the generalized Poisson (GP) process; called the $\lambda_{GP}-CUSUM$ *chart*. They founded when there was a shift in λ, all levels of parameters, the $\lambda_{GP}-CUSUM$ *chart* and Exponentially Weighted Moving Average ($EWMA$ *chart*) based on GP distribution are performed for all levels of the shift.
This research proposed control charts based on the generalized Poisson (GP) process in the case of Under-dispersion. The study of the influence of the $CUSUM$ *chart* based on GP distribution; also called the $\lambda_{GP}-CUSUM$ *chart*, $\lambda_{or}-CUSUM$ *chart* and Exponentially Weighted Moving Average ($EWMA$ *chart*) based on GP distribution. The control charts are used to detect changes in individual parameters of the mean of nonconformities (λ). The average run length (ARL) is used considered to measure for the performance. These studied control charts are compared with the $c-chart$ based on GP distribution ($cGP-chart$).

2. Materials and Methods

Generalized Poisson (GP) distribution

The probability function is given as (Consul and Jain [10]):

$$P(Y=y)=\exp(-(\lambda+y(\varphi-1)))\frac{\lambda(\lambda+y(\varphi-1))^{y-1}}{\varphi^y y!}, \quad y=0,1,2,\cdots \tag{1}$$

where:

$Y=$ the random variables of nonconformities in a sample unit,
$\lambda=$ the mean of nonconformities in a sample unit based on the GP distribution,
$\varphi=$ the over dispersion for GP distribution and

$$E(Y)=\lambda \text{ and‘ } V(Y)=\varphi^2\lambda. \tag{2}$$

Shewhart control chart of nonconformities based on GP distribution ($c_{GP}-chart$)

The $c-chart$ based on GP distribution called $c_{GP}-chart$. The upper control limit (UCL) of the $c_{GP}-chart$ called H_c where $H_c$is $c+L\sqrt{c}$. Define c is the mean number of nonconformities and L is the coefficient of the control limit of the $c-chart$. The $c_{GP}-chart$ will signal when any observation of nonconformities (y_i) is more than H_c.

Cumulative Sum Chart based on a GP distribution ($\lambda_{GP}-CUSUM\ chart$)

The $CUSUM\ chart$ based on a GP distribution for detecting shifts in parameter λ called $\lambda_{GP}-CUSUM\ chart$. The cumulative sum statistics are constructed based on a log-likelihood ratio for plotting the $\lambda_{GP}-CUSUM\ chart$ (R_i), defined as:

$$R_i=\max(0,R_{i-1}+W_i),\quad i=1,2,\cdots \tag{3}$$

The head start value of the cumulative sum statistics $(R_0)=0$ and W_i is the loglikelihood ratio of GP distribution for a shift in parameter $\lambda(\lambda_1)$, defined as:

$$W_i=W(y_i)=\lambda_0-\lambda_1+ln\left(\frac{\lambda_1}{\lambda_0}\right)+(y_i-1)ln\left(\frac{(\lambda_1+y_i(\varphi_0-1))}{(\lambda_0+y_i(\varphi_0-1))}\right),\ \ y_i=0,1,2,\cdots \tag{4}$$

where:

y_i is the observation of y taken at the time i,

λ_0 is the in-control value of the mean number of nonconformities for GP distribution,

λ_1 is out-of-control values of the mean number of nonconformities for GP distribution,

φ_0 is the in-control value of over dispersion for GP distribution.

The $\lambda_{GP}-CUSUM\ chart$ will signal in the process when $R_i>H_\lambda$, where H_λ is the UCL of the $\lambda_{GP}-CUSUM\ chart$.

Cumulative Sum chart based on GP distribution ($\lambda_{or}-CUSUM\ chart$)

The $\lambda-GP-CUSUM\ chart$ is the same as a $CUSUM\ chart$ for detecting shifts in parameter λ. The cumulative sum statistics for plotting on the $\lambda_{or}-CUSUM\ chart$ (L_i) are defined as:

$$L_i=\max(0,y_i-k+L_{i-1}),\quad i=1,2,\cdots \tag{5}$$

The head start value of the cumulative sum statistics $(L_0)=0$ where:

y_i is the observation of y taken at the time i,

k is the reference value.

The $\lambda_{or}-CUSUM\ chart$ will signal in the process when $L_i>H_{or}$, where H_{or} is the UCL of the $\lambda_{or}-CUSUM\ chart$.

Exponentially Weighted Moving Average chart based on GP distribution ($EWMA_{GP} - chart$)

The $EWMA{-}chart$ based on GP distribution called $EWMAGP{-}chart$. The $EWMA$ statistics for plotting on the $EWMA_{GP} - chart$ (Z_i) are defined as (Roberts [12]):

$$Z_i = \xi y_i + (1_\xi) Z_{i-1}, \quad i = 1, 2, \cdots \tag{6}$$

The head start value of the $EWMA$ statistics $(Z_0) = \lambda_0$ where:

ξ is a constant that determinations must satisfy $0 < \xi \leq 1$,

y_i is the observation of y taken at time i,

λ_0 is the in-control value of the mean number of nonconformities for GP distribution.

The $EWMAGP - chart$ will signal in the process when $Z_i > H_{EWMA}$, where H_{EWMA} is the UCL of the $EWMAGP - chart$.

3. Simulation Results

Study of the situations where the process has a mean number of nonconformities of $(\lambda_0) = 1, 2$ and 3. The under dispersions are $(\varphi_0) = 0.8$ and 0.9. The out-of-control values of the mean number of nonconformities are $\lambda_1 = \lambda_0 + \rho$, where the mean shifts are $(\rho) = 2, 3, 4, 5, 6, 7$ and 8. The criteria for evaluating the performance of the control charts is the average run length (ARL). The research process has the following steps:

1. The R program is used to simulate the number of nonconforming items for a GP distribution where the parameters are $(n, \lambda_0, \varphi_0)$.

2. The value of the upper control limit with $ARL_0 = 370$ matching for all of the charts. The value of H_λ for $\lambda_{GP} - CUSUM\ chart$, the value of H_{or} for $\lambda_{or} - CUSUM\ chart$, the value of H_{EWMA} for $EWMAGP - chart$ and the value of H_c for $c_{GP} - chart$. Specification of the shift sizes of λ_1 that are interested in for the cumulative sum charts to have fast detection depends on pre-existing experience concerning the process being monitored. The average upper control limit is based on 100,000 replications that are in each level of the parameters.

3. Calculation of the log-likelihood ratios for the cumulative sum statistics are plotted in the chart. The $\lambda_{GP} - CUSUM\ chart$, calculates W_i value from (4) for R_i value from (3). The $\lambda_{or} - CUSUM\ chart$, calculates L_i value from (5). The

$EWMAGP - chart$ calculates Z_i value from (6). The number of simulations of nonconformance (y_i) are used for the $c_{GP} - chart$.

4. Examining the $CUSUM$ and $EWMA$ statistics with a control limit for each chart finds the run length (RL). The $\lambda_{GP} - CUSUM\ chart$, examines R_i value with H_λ value. The $\lambda_{or} - CUSUM\ chart$, examines L_i value with H_{or} value. The $EWMAGP - chart$, examines Z_i value with H_{EWMA} value. The $c_{GP} - chart$, examines y_i value with H_c value. Consider examining the R_i, L_i, Z_i and y_i that are out-ofcontrol points. If there are points outside the control limit, then they will be stored in the observations before a point indicates an out-of-control for run length (RL) calculation. If they are at i statistics indicates an out-of-control, then $RL = i - 1$.

5. Steps 3 to 4 were then repeated for 100,000 replications to compute the average run length (ARL) for each of the charts.

6. Comparison of the performance of control charts gives a low ARL. That means the control charts are efficient.

7. The study is complete when transform during for all parameters value.

4. Results

The summary of efficient control charts is shown for all levels of parameters and all levels of shifts. Table 1 defines the levels for the individual parameter shifts when $\lambda_0 = 1$.
Table 2 shows the upper control limit of $\lambda_{or} - CUSUM\ chart$ (H_{or}), $\lambda_{GP} - CUSUM\ chart$ (H_λ) and $EWMA_{GP} - chart$ (H_{EWMA}) were matching with the $ARL_0 = 370$. However, the $c_{GP} - chart$ (H_c) does not match with the required ARL_0. Therefore, this chart was not discussed. It can be seen that, as all values of λ_0, the values of $H_{or}, H_\lambda, H_{EWMA}$ and H_c are higher when the values of λ_0 are higher.
Table 3 and Fig. 1 show the ARL_1 of $\lambda_{GP} - CUSUM\ chart$, $\lambda_{or} - CUSUM\ chart$ and $EWMA_{GP} - chart$ for shift in parameter λ. The results found that for all the λ_0, φ_0 and the level of a shift, the $EWMA_{GP} - chart$ returned similar low values of ARL_1. That is, $EWMA_{GP} - chart$ performs as control charts because it can detect shifts quickly.

Table 1 : defines the levels for parameter shifts in λ when $\lambda_0 = 1$

Levels of Shifts	1	2	3	4	5	6	7
parameter shifts in λ ($\lambda_1 = \lambda_0 + \rho$)	$\lambda_1 = 3$	$\lambda_1 = 4$	$\lambda_1 = 5$	$\lambda_1 = 6$	$\lambda_1 = 7$	$\lambda_1 = 8$	$\lambda_1 = 9$

Table 2 : the upper control limit $H_\lambda, H_{or}, H_{EWMA}$ and H_c were matching with the desired in-control performance for all levels of the $\lambda_0, \varphi_0, \xi = 0.09$ and $k = 1, 2$ and 3

λ_0	1	1	2	2	3	3
φ_0	0.8	0.9	0.8	0.9	0.8	0.9
H_λ	5.7300	4.8000	8.9800	5.3500	9.0800	6.1000
ARL_0 for $\lambda_{GP} - CUSUM\ chart$	370.53	370.67	370.35	370.21	370.56	370.35
H_{or}	14.500	16.000	20.000	23.000	24.000	28.900
ARL_0 for $\lambda_{Or} - CUSUM\ chart$	370.45	370.42	370.10	370.46	370.12	370.68
H_{EWMA}	2.8000	2.1000	3.9000	2.8000	4.2900	3.2000
RL_0 for $EWMA_{GP} - chart$	370.34	370.21	370.45	370.46	370.23	370.17
H_c	3.0000	4.0000	5.0000	6.0000	7.0000	8.0000
ARL_0 for $c_{GP} - chart$	452.43	515.36	558.33	604.96	653.47	653.122

Table 3 the ARL_1 of $\lambda_{GP} - CUSUM\ chart$, $\lambda_{or} - CUSUM\ chart$ and $EWMA_{GP} - chart$ for shift in parameter λ

		$\varphi = 0.8$			$\varphi = 0.9$		
λ_0	ρ	$\lambda_{GP} - CUSUM\ chart$	$\lambda_{or} - CUSUM\ chart$	$EWMA_{GP} - chart$	$\lambda_{GP} - CUSUM\ chart$	$\lambda_{or} - CUSUM\ chart$	$EWMA_{GP} - chart$
1	2	4.350	8.055	3.247	3.778	8.919	1.836
	3	3.981	5.498	2.735	3.484	6.162	0.748
	4	3.016	4.188	1.436	2.616	4.779	0.624
	5	2.719	3.535	1.125	2.031	3.901	0.430
	6	2.136	3.001	0.522	1.732	3.285	0.148
	7	1.813	2.610	0.314	1.268	2.949	0.087
	8	1.103	2.354	0.132	10.45	2.608	0.079

λ_0	ρ	$\varphi = 0.8$			$\varphi = 0.9$		
		$\lambda_{GP}-$ $CUSUM$ $chart$	$\lambda_{or}-$ $CUSUM$ $chart$	$EWMA_{GP}-$ $chart$	$\lambda_{GP}-$ $CUSUM$ $chart$	$\lambda_{or}-$ $CUSUM$ $chart$	$EWMA_{GP}-$ $chart$
2	2	6.777	11.025	2.521	6.507	12.516	1.411
	3	5.246	7.463	1.427	4.228	8.532	0.593
	4	4.518	5.734	0.913	3.029	6.476	0.304
	5	3.217	4.695	0.521	2.435	5.317	0.170
	6	2.435	3.980	0.276	1.975	4.492	0.095
	7	1.816	3.529	0.189	1.697	3.926	0.045
	8	1.429	3.110	0.084	1.510	3.458	0.018
3	2	18.588	11.698	2.169	9.666	15.01	1.120
	3	11.708	8.917	0.933	6.268	10.22	0.410
	4	10.634	6.720	0.516	4.663	7.801	0.190
	5	7.363	5.483	0.435	3.680	6.378	0.094
	6	5.241	4.653	0.251	3.085	5.359	0.038
	7	3.240	4.082	0.164	2.650	4.692	0.012
	9	2.832	3.622	0.017	2.309	4.143	0.004

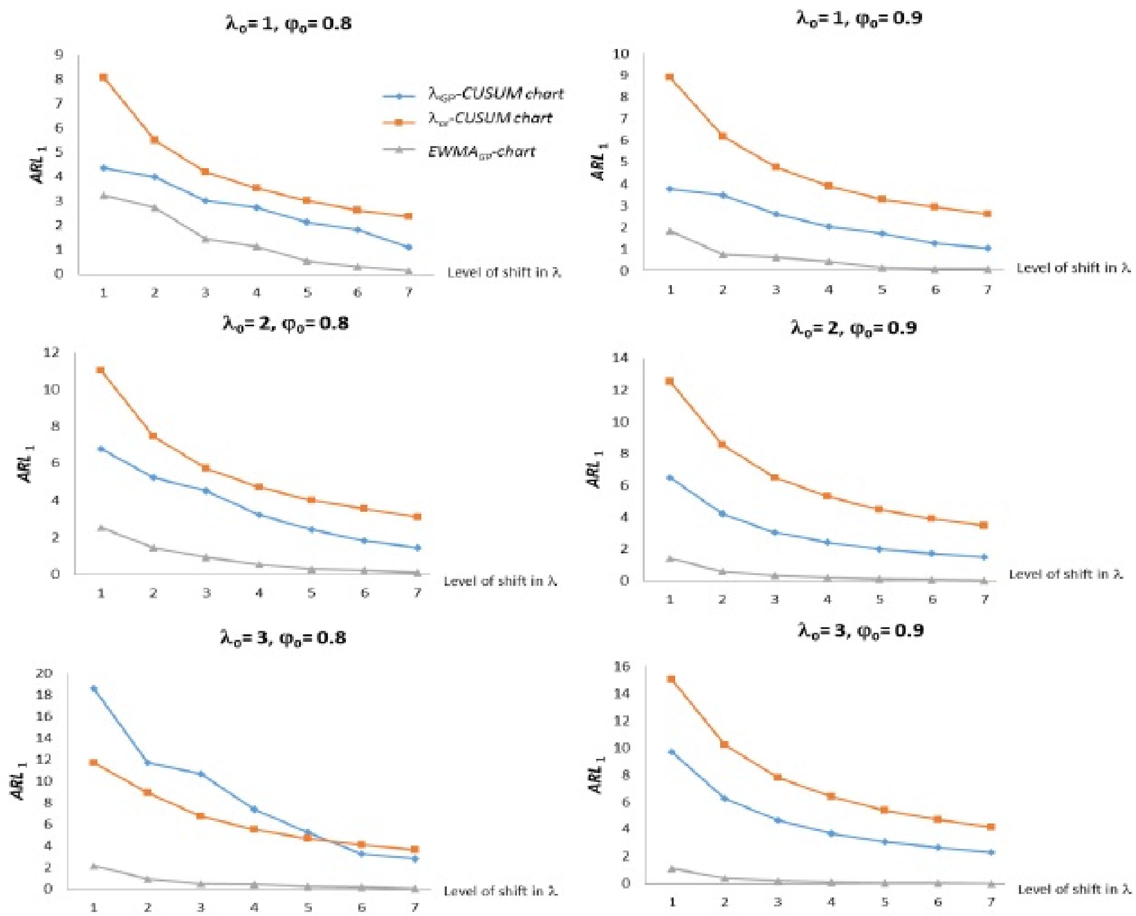

Figure 1 the ARL_1 of the $\lambda_{GP} - CUSUM\ chart$, $\lambda_{or} - CUSUM\ chart$ and $EWMA_{GP} - chart$ for the shifts in parameter λ

5. Conclusion

This paper studied the influence of the $CUSUM\ chart$, $EWMA-chart$ and $c-chart$ to detect changes for the GP process. These charts are used for detecting parameter shifts for parameter λ. The four charts are called the $\lambda_{GP} - CUSUM\ chart$, $\lambda_{or} - CUSUM\ chart$, $EWMA_{GP}-chart$ and $c_{GP}-chart$. The simulation of average run length (ARL) was considered for these charts. The results showed that, the $EWMA_{GP} - chart$ are preferential control charts for all values of parameters in processes.

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