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# USING AVERAGED GAUSS-LEGENDRE RULE TO EVALUATE SOME REAL DEFINITE INTEGRALS IN ADAPTIVE ENVIRONMENT

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**Abstract**

An averaged Gauss-Legendre rule of precision 5 is constructed taking the average of Gauss-Legendre two point rule and anti-Gauss three point rule of same precision, i.e., precision 3. This can also be called a mixed quadrature rule as it is formed by combining two rules of same lower precision. Also an adaptive integration algorithm is designed taking this averaged Gauss-Legendre five point rule into account and the adaptive integration scheme has been applied to evaluate some real definite integrals in order to test its efficiency.

## 1. Introduction

Gaussian rules are one of the powerful and foremost quadrature rules in numerical

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Key Words : *Gauss-Legendre 2 point rule* ($R_{GL_2}(f)$), *Anti-Gauss 3 point rule* ($R_{aG_3}(f)$), *averaged Gauss-Legendre 5 point rule* ($R_{avgGL5}(f)$), *Adaptive integration scheme.*

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analysis. It has been proved that Gaussian rules are better in comparison to classical quadrature rules (Newton-Cotes). But this rule suffers from the drawback that as we increase the nodes the new rules do not keep the nodes of the previous one except at origin. So it becomes cumbersome to evaulate $f(x)$ each time for higher order formula. Keeping this difficulty in mind, Dirk P. Laurie (1996) [3] designed anti-Gauss quadrature followed by averaged Gaussian quadrature. Some mixed rules based on anti-Gauss rule have been developed by ([5], [4]). Let us discuss some of the basic features of anti-Gauss quadrature.

### 1.1 Anti-Gauss Quadrature Rule

- An $n+1$-point anti-Gauss rule has same degree of precision as that of $n$-point Gaussian rule, $i.e.$, precision $2n-1$.
- It integrates polynomial up to degree $2n+1$ with an error equal in magnitude but of opposite in sign to that of $n$-point Gaussian rule.
- Anti-Gaussian rules have positive weights.
- Nodes of the anti-Gaussian rules are interior and interlaced by those of corresponding Gaussian formula.

Let us consider the integral

$$I(f) = \int_{-1}^{1} f(x)dx \approx R_{aG_{n+1}}(f) \tag{1.1}$$

Furthermore, we can express (1.1) in the form

$$R_{aG_{n+1}}(f) = \sum_{i=0}^{n} w_i f(x_i) \tag{1.2}$$

where, $w_i$'s are weights and $x_i$'s are the distinct points (nodes) in the interval $[-1,1]$. The error associated with the anti-Gauss $n+1$-point rule is $I(f) - R_{aG_{n+1}}(f)$. The error is equal to the negative of the error associated with Gauss-Legendre $n$-point rule. $i.e.$,

$$I(f) - R_{aG_{n+1}}(f) = -\left(I(f) - R_{GL_n}(f)\right) \tag{1.3}$$

$$R_{aG_{n+1}}(f) = 2I(f) - R_{GL_n}(f) \tag{1.4}$$

### 1.2 Averaged Gaussian Quadrature Rule

An averaged Gauss-Legendre rule, proposed by Laurie [3] is a suboptimal extension of the Gaussian rule. It is constructed by averaging two quadratures, *i.e.*, Gaussian and anti-Gaussian, of same order. The speciality of this rule is, it is of precision $2n+1$ where as Gaussian and anti-Gaussian are of precision $2n-1$. The averaged Gaussian quadrature possesses the following characteristic features.

- A $2n+1$-point averaged Gaussian formula has degree of precision $2n+1$ and it integrates polynomials up to degree $2n+1$.
- Its error is the average of Gaussian rule and anti-Gaussian rule of same precision *i.e.*, precision $2n-1$.
- It always exists.
- Its nodes are real.
- At worst two nodes may be exterior.
- It has positive weights.

We can write

$$I(f) = \int_{-1}^{1} f(x)dx \approx R_{avgG_{2n+1}}(f) \tag{1.5}$$

Unlike other mixed quadrature rules using anti-Gauss quadrature, in averaged Gaussian rule we can get promptly a rule of precision 5 from the features of anti-Gauss quadrature without using any lengthy process of calculation. This is the primary advantage of averaged Gaussian rule. As we can see from equation (1.4)

$$2I(f) = R_{aG_{n+1}}(f) + R_{GL_n}(f) \tag{1.6}$$

or

$$I(f) = \frac{R_{aG_{n+1}}(f) + R_{GL_n}(f)}{2} \tag{1.7}$$

or

$$I(f) - R_{avgG_{2n+1}} = 0, \quad \text{for } f(x) = x^{2n+1} \tag{1.8}$$

i.e.,

$$E_{avgG_{2n+1}}(f) = 0, \quad \text{for } f(x) = x^{2n+1} \tag{1.9}$$

where

$$R_{avgG_{2n+1}}(f) = \frac{R_{aG_{n+1}}(f) + R_{GL_n}(f)}{2} = \text{ } 2n+1\text{-point averaged Gauss-Legendre rule}$$

and

$$E_{avgG_{2n+1}}(f) = \text{ Error incurred in the } 2n+1\text{-point averged Gaussian rule} \tag{1.10}$$

The primary aim of this paper is to implement averaged Gauss-Legendre rule in adaptive environment by fixing up a termination criterion and also to prove its efficiency not only in comparison to its constituent rules but also in comparison to some other mixed rules in the literature. Also we have compared the results with the results obtained by ([2], [1]) by evaluating some test integrals.

## 2. Construction of the Averaged Gaussian Quadrature Rule of Precision Five

We choose the Gauss-Legendre two point rule

$$I(f) = \int_{-1}^{1} f(x)dx \approx R_{GL_2}(f) = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) \tag{2.1}$$

and the anti-Gauss three point rule

$$I(f) = \int_{-1}^{1} f(x)dx \approx R_{aG_3}(f) = \frac{1}{13}\left[5f\left(-\sqrt{\frac{13}{15}}\right) + 16f(0) + 5f\left(\sqrt{\frac{13}{15}}\right)\right] \tag{2.2}$$

Let $E_{GL_2}(f)$ and $E_{aG_3}(f)$ denote the error terms of the rules (2.1) and (2.2) respectively. So

$$I(f) = R_{GL_2}(f) + E_{GL_2}(f) \tag{2.3}$$

$$I(f) = R_{aG_3}(f) + E_{aG_3}(f) \tag{2.4}$$

Now it is evident from [4] that the rules (2.1) and (2.2) are of precision 3.

Now adding the equations (2.3) and (2.4) and then dividing by 2 we get

$$I(f) = \frac{1}{2}\left[R_{GL_2}(f) + E_{GL_2}(f) + R_{aG_3}(f) + E_{aG_3}(f)\right] \tag{2.5}$$

or

$$I(f) = \frac{1}{2}\left[R_{GL_2}(f) + R_{aG_3}(f)\right] + \frac{1}{2}\left[E_{GL_2}(f) + E_{aG_3}(f)\right] \tag{2.6}$$

or

$$I(f) = R_{avgGL_5}(f) + E_{avgGL_5}(f) \tag{2.7}$$

where

$$R_{avgGL_5}(f) = \frac{1}{2}\left[R_{GL_2}(f) + R_{aG_3}(f)\right] \tag{2.8}$$

and

$$E_{avgGL_5}(f) = \frac{1}{2}\left[E_{GL_2}(f) + E_{aG_3}(f)\right] \tag{2.9}$$

This is the desired *averaged 5-point Gaussian quadrature rule of precision five* for approximate evaluation of $I(f)$. The truncation error generated in this approximation is given by

$$\begin{aligned} E_{avgGL_5}(f) &= \tfrac{1}{2}\left[E_{GL_2}(f) + E_{aG_3}(f)\right] \\ &= -\tfrac{8}{7!\times 675} f^{(vi)}(0) - \tfrac{8}{9!\times 125} f^{(viii)}(0) - \tfrac{17296}{11!\times 151875} f^{(x)}(0) - \cdots \end{aligned} \tag{2.10}$$

The rule (2.10) may be called as a mixed type rule as it is constructed from two different types of rules of the same precision ($i.e.$, precision 3).

## 3. Error Analysis

An asymptotic error estimate and error bound of the rule (2.6) are given in theorems (3.1) and (3.2) respectively.

**Theorem 3.1** : Let $f(x)$ be a continuously differentiable function in the closed interval $[-1, 1]$. Then the error $E_{avgGL_5}(f)$ associated with the rule $R_{avgGL_5}(f)$ is given by

$$|E_{avgGL_5}(f)| \approxeq \frac{8}{7! \times 675}\left|f^{(vi)}(0)\right|$$

**Proof** : Follows directly from (2.8). □

**Theorem 3.2** : The bound of the truncation error $E_{avgGL_5}(f)$=$I(f) - R_{avgGL_5}(f)$ is

$$|E_{avgGL_5}(f)| \leq \frac{M}{270}\left|\eta_2 - \eta_1\right|, \qquad \eta_1, \eta_2 \in [-1, 1]$$
$$\text{where} \quad M = \max_{-1 \leq x \leq 1}\left|f^{(v)}(x)\right|$$

**Proof** : We have

$$
\begin{aligned}
E_{aG_3}(f) &\approx -\frac{1}{135} f^{(iv)}(\eta_1), \qquad \eta_1 \in [-1,1] \\
E_{GL_2}(f) &\approx \frac{1}{135} f^{(iv)}(\eta_2), \qquad \eta_2 \in [-1,1] \\
\text{Hence} \quad E_{avgGL_5}(f) &= \frac{1}{2}\left[E_{aG_3}(f) + E_{GL_2}(f)\right] \\
&\approx \frac{1}{270}\left[f^{(iv)}(\eta_2) - f^{(iv)}(\eta_1)\right] \\
&= \frac{1}{270}\int_{\eta_1}^{\eta_2} f^{(v)}(x)dx \qquad (\text{assuming} \quad \eta_1 < \eta_2)
\end{aligned}
$$

$$
\text{So we obtain,} \quad |E_{avgGL_5}(f)| = \left|\frac{1}{270}\int_{\eta_1}^{\eta_2} f^{(v)}(x)dx\right| \le \frac{1}{270}\int_{\eta_1}^{\eta_2}\left|f^{(v)}(x)\right| dx
$$

$$
\text{So} \quad |E_{avgGL_5}(f)| \le \frac{M}{270}\left|\eta_2 - \eta_1\right|, \ \text{where } M = \max_{-1\le x\le 1}\left|f^{(v)}(x)\right|
$$

which gives only a theoretical error bound as $\eta_1$, $\eta_2$ are unknown points in $[-1,1]$. It shows that the error in the approximation will be less if the points $\eta_1$, $\eta_2$ are closed to each other. □

**Corollary 3.1** : The error bound for the truncation error $E_{avgGL_5}(f)$ is given by

$$
|E_{avgGL_5}(f)| \le \frac{M}{135}, \qquad \text{where} \quad M = \max_{-1\le x\le 1}\left|f^{(v)}(x)\right|
$$

**Proof** : we know from theorem (3.2) that

$$
\begin{aligned}
|E_{avgGL_5}(f)| &\le \frac{M}{270}\left|\eta_2 - \eta_1\right| \\
\text{where} \quad M &= \max_{-1\le x\le 1}\left|f^{(v)}(x)\right| \\
\text{choosing} \quad |\eta_1 - \eta_2| &\le 2 \\
\text{we get} \quad |E_{avgGL_5}(f)| &\le \frac{M}{135}
\end{aligned}
$$

□

## 4. Algorithm for Adaptive Quadrature Routine

Applying the constituent rules $R_{GL_2}(f)$, $R_{aG_3}(f)$ and the 5-point averaged Gaussian quadrature rule ($R_{avgGL_5}(f)$), one can evaluate real definite integrals of the type $\int_a^b f(x)dx$

in adaptive integration scheme. In the adaptive integration scheme, the desired accuracy is sought by progressively subdividing the interval of integration according to the computed behaviour of the integrand, and applying the same formula over each subinterval. The algorithm for adaptive integration scheme is outlined using the mixed quadrature rule $(R_{avgGL_5}(f))$ in the following four steps.

Input: Function $f : [a, b] \to \mathbb{R}$ and the prescribed tolerance $\varepsilon$.

Output: An approximation $Q(f)$ to the integral $I(f) = \int_a^b f(x)dx$ such that $|Q(f) - I(f)| \leq \varepsilon$.

Step-1: The mixed quadrature rule $(R_{avgGL_5}(f))$ is applied to approximate the integral $I(f) = \int_a^b f(x)dx$. The approximated value is denoted by $(R_{avgGL_5}[a, b])$.

Step-2: The interval of integration $[a, b]$ is divided into two equal pieces; $[a, c]$ and $[c, b]$. The mixed quadrature rule $(R_{avgGL_5}(f))$ is applied to approximate the integral $I_1(f) = \int_a^c f(x)dx$ and the approximated value is denoted by $(R_{avgGL_5}[a, c])$. Similarly, the mixed quadrature rule $(R_{avgGL_5}(f))$ is applied to approximate the integral $I_2(f) = \int_c^b f(x)dx$ and the approximated value is denoted by $(R_{avgGL_5}[c, b])$.

Step-3: $(R_{avgGL_5}[a, c])+(R_{avgGL_5}[c, b])$ is compared with $(R_{avgGL_5}[a, b])$ to estimate the error in $(R_{avgGL_5}[a, c]) + (R_{avgGL_5}[c, b])$.

Step-4: If $|\ \textit{estimated error}\ | \leq \frac{\varepsilon}{2}$ (termination criterion) then $(R_{avgGL_5}[a, c])+(R_{avgGL_5}[c, b])$ is accepted as an approximation to $I(f) = \int_a^b f(x)dx$. Otherwise the same procedure is applied to $[a, c]$ and $[c, b]$, allowing each piece a tolerance of $\frac{\varepsilon}{2}$. If the termination criterion is not satisfied on one or more of the subintervals, then those subintervals must be further subdivided and the entire process is repeated. When the process stops, the addition of all accepted values yields the desired approximate value $Q(f)$ of the integral $I(f)$ such that $|Q(f) - I(f)| \leq \varepsilon$.

**N.B** : In this algorithm we can use any quadrature rule to evaluate real definite integrals in adaptive integration scheme.

## 5. Numerical Verification

**Table 1** : Comparative study among the quadrature rules ($R_{S_3}(f)$, $R_{S_4}(f)$, $R_{GL_2}(f)$, $R_{aG_3}(f)$) for approximation of some real definite integrals without using adaptive integration scheme

| Integrals | Exact Value | Approximate Value $Q(f)$ | | | |
|---|---|---|---|---|---|
| | | $R_{S_3}(f)$ | $R_{S_4}(f)$ | $R_{GL_2}(f)$ | $R_{aG_3}(f)$ |
| $\int_{-1}^{1} e^x dx$ | 2.350402387 | 2.362053 | 2.355648 | 2.342696 | 2.358113 |
| $\int_0^1 e^{-x^2} dx$ | 0.7468241330 | 0.747180 | 0.746992 | 0.746594 | 0.747054 |
| $\int_0^1 e^{x^2} dx$ | 1.46265174 | 1.4757 | 1.4687 | 1.4541 | 1.4711 |
| $\int_1^3 \frac{\sin^2 x}{x} dx$ | 0.794825 | 0.7894 | 0.7926 | 0.7985 | 0.7911 |
| $\frac{2}{\sqrt{\pi}} \int_0^4 e^{-x^2} dx$ | 1 | 0.8073 | 0.8516 | 1.1046 | 0.9024 |
| $\int_0^1 e^{-x^3} dx$ | 0.807511182 | 0.8163 | 0.81118 | 0.8014 | 0.8135 |
| $\int_0^{2\pi} e^{-x} \cos x dx$ | 0.499066278634 | 0.8681 | 0.6239 | 0.2056 | 0.7855 |
| $\int_1^3 \frac{100}{x} \sin\left(\frac{10}{x}\right) dx$ | $-18.798296836787$ | $-84.179$ | $-57.023$ | 21.589 | $-59.973$ |
| $\int_0^1 \sqrt{x} \log x dx$ | $-0.4444444\cdots$ | $-0.3267$ | $-0.362004$ | $-0.4626$ | $-0.4285$ |
| $\int_0^1 \sqrt{x} \sin x dx$ | 0.3642219 | 0.3662 | 0.365359 | 0.3632 | 0.365236 |
| $\int_0^1 \sqrt{x} dx$ | $0.66666666\cdots$ | 0.638071 | 0.647692 | 0.673887 | 0.659834 |
| $\int_0^{\frac{5\pi}{4}} \frac{\cos 2x}{e^x} dx$ | 0.207881149 | 0.394651 | 0.200132 | 0.012295 | 0.400048 |

**Note** :

$R_{S_3}(f)$: Simpson's $\frac{1}{3}$ rule

$R_{S_4}(f)$: Simpson's $\frac{3}{8}$ rule

$R_{GL_2}(f)$: Gauss-Legendre two point rule

$R_{aG_3}(f)$: Anti-Gauss three point rule.

**Table 2** : Comparative study among the mixed quadrature rules and averaged Gauss-Legendre rule ($R_{S_3GL_2}(f)$, $R_{S_4GL_3}(f)$ and $R_{avgGL_5}(f)$) for approximation of some real definite integrals (given in Table-1) without using adaptive integration scheme

| Integrals | Exact Value | Approximate Value $Q(f)$ | | |
|---|---|---|---|---|
| | | $R_{S_3GL_2}(f)$ | $R_{S_4GL_2}(f)$ | $R_{avgGL_5}(f)$ |
| $\int_{-1}^{1} e^x dx$ | 2.350402387 | 2.350439 | 2.350467 | 2.35040491 |
| $\int_{0}^{1} e^{-x^2} dx$ | 0.7468241330 | 0.7468289 | 0.7468332 | 0.74682435 |
| $\int_{0}^{1} e^{x^2} dx$ | 1.46265174 | 1.462792 | 1.462895 | 1.462662 |
| **Integrals** | **Exact Value** | **Approximate Value** $Q(f)$ | | |
| | | $R_{S_3GL_2}(f)$ | $R_{S_4GL_2}(f)$ | $R_{avgGL_5}(f)$ |
| $\int_{1}^{3} \frac{\sin^2 x}{x} dx$ | 0.79482518 | 0.794916 | 0.794992 | 0.794830 |
| $\frac{2}{\sqrt{\pi}} \int_{0}^{4} e^{-x^2} dx$ | 1 | 0.9857 | 0.9528 | 1.0035 |
| $\int_{0}^{1} e^{-x^3} dx$ | 0.807511182 | 0.80739 | 0.8072904 | 0.807504 |
| $\int_{0}^{2\pi} e^{-x} \cos x dx$ | 0.499066278634 | 0.4706 | 0.4566 | 0.4956 |
| $\int_{1}^{3} \frac{100}{x} \sin\left(\frac{10}{x}\right) dx$ | $-18.798296836787$ | $-20.718$ | $-25.578$ | $-19.191$ |
| $\int_{0}^{1} \sqrt{x} \log x dx$ | $-0.4444444\cdots$ | $-0.4083$ | $-0.4022$ | $-0.4456$ |
| $\int_{0}^{1} \sqrt{x} \sin x dx$ | 0.3642219 | 0.364432 | 0.364504 | 0.364228 |
| $\int_{0}^{1} \sqrt{x} dx$ | $0.66666666\cdots$ | 0.6595608 | 0.6581704 | 0.6668607 |
| $\int_{0}^{\frac{5\pi}{4}} \frac{\cos 2x}{e^x} dx$ | 0.207881149 | 0.165237 | 0.124997 | 0.206172 |

**Note** :

$R_{S_3GL_2}(f)$: Mixed quadrature rule by Simpson's $\frac{1}{3}$ Gauss-Legendre two point rule

$R_{S_4GL_3}(f)$: Mixed quadrature rule by Simpson's $\frac{3}{8}$ Gauss-Legendre three point rule

$R_{avgGL_5}(f)$: Averaged Gaussian five point rule.

**Table 3** : Comparative study among the quadrature rules ($R_{S_3}(f)$, $R_{S_4}(f)$, $R_{GL_2}(f)$, $R_{aG_3}(f)$) for approximation of some real definite integrals using adaptive integration scheme

| Integrals | Approximate Value $Q(f)$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | $R_{S_3}(f)$ | # Steps | $R_{S_4}(f)$ | # Steps | $R_{GL_2}(f)$ | # Steps | $R_{aG_3}(f)$ | # Steps |
| $\int_0^1 e^{x^2} dx$ | 1.462656321 | 7 | 1.46265378 | 7 | 1.462648697 | 7 | 1.462654794 | 7 |
| $\int_1^3 \frac{\sin^2 x}{x} dx$ | 0.79482463 | 7 | 0.79482493 | 7 | 0.79482554 | 7 | 0.79482481 | 7 |
| $\frac{2}{\sqrt{\pi}} \int_0^4 e^{-x^2} dx$ | 1.00000181 | 19 | 1.000000179 | 13 | 1.00000349 | 15 | 0.99999647 | 15 |
| $\int_0^1 e^{-x^3} dx$ | 0.80751235 | 7 | 0.807511701 | 7 | 0.8075104 | 7 | 0.80751196 | 7 |
| $\int_0^{2\pi} e^{-x} \cos x dx$ | 0.49906351 | 15 | 0.49906504 | 15 | 0.49906812 | 15 | 0.49906443 | 15 |
| $\int_1^3 \frac{100}{x} \sin\left(\frac{10}{x}\right) dx$ | −18.79829444 | 49 | −18.79829026 | 45 | −18.79829839 | 49 | −18.79829523 | 49 |
| $\int_0^1 \sqrt{x} \log x dx$ | −0.4444125 | 17 | −0.4444242 | 17 | −0.4444629 | 15 | −0.44442703 | 15 |
| $\int_0^1 \sqrt{x} \sin x dx$ | 0.36422523 | 7 | 0.36422962 | 5 | 0.36421433 | 5 | 0.364229601 | 5 |
| $\int_0^1 \sqrt{x} dx$ | 0.6666434 | 13 | 0.666652007 | 13 | 0.6666831 | 11 | 0.6666509 | 11 |
| $\int_0^{\frac{5\pi}{4}} \frac{\cos 2x}{e^x} dx$ | 0.2078814173 | 19 | 0.20787598 | 15 | 0.207880968 | 19 | 0.207881329 | 19 |

**Note** : Here the prescribed tolerance $\varepsilon$=0.0001.

**Table 4** : Comparative study among the mixed quadrature rules and averaged Gaussian rule ($R_{S_3GL_2}(f)$, $R_{S_4GL_3}(f)$ and $R_{avgGL_5}(f)$) for approximation of some real definite integrals (given in Table-3) using adaptive integration scheme

| Integrals | Approximate Value $Q(f)$ | | | | | |
|---|---|---|---|---|---|---|
| | $R_{S_3GL_2}(f)$ | **#Steps** | $R_{S_4GL_2}(f)$ | **#Steps** | $R_{avgGL_5}(f)$ | **#Steps** |
| $\int_0^1 e^{x^2} dx$ | 1.4626518007 | 3 | 1.4626518431 | 3 | 1.462651963 | 1 |
| $\int_1^3 \frac{\sin^2 x}{x} dx$ | 0.79482519 | 3 | 0.794825202 | 3 | 0.79482523 | 1 |
| $\frac{2}{\sqrt{\pi}} \int_0^4 e^{-x^2} dx$ | 0.999999984488 | 7 | 0.999999984481 | 7 | 1.0000000111 | 5 |
| $\int_0^1 e^{-x^3} dx$ | 0.8075111708 | 3 | 0.807511162 | 3 | 0.8075111237 | 1 |
| $\int_0^{2\pi} e^{-x} \cos x dx$ | 0.49906601 | 7 | 0.499065801 | 7 | 0.499066314 | 5 |

| Integrals | Approximate Value $Q(f)$ | | | | | |
|---|---|---|---|---|---|---|
| | $R_{S_3GL_2}(f)$ | **#Steps** | $R_{S_4GL_2}(f)$ | **#Steps** | $R_{avgGL_5}(f)$ | **#Steps** |
| $\int_1^3 \frac{100}{x}\sin\left(\frac{10}{x}\right)dx$ | −18.79829768 | 15 | −18.79829834 | 15 | − 18.798298357 | 9 |
| $\int_0^1 \sqrt{x}\log x dx$ | −0.44442598 | 15 | −0.4444226 | 15 | −0.44447106 | 7 |
| $\int_0^1 \sqrt{x}\sin x dx$ | 0.36422856 | 3 | 0.364230857 | 3 | 0.364223157 | 1 |
| $\int_0^1 \sqrt{x}dx$ | 0.66665278 | 11 | 0.66665005 | 11 | 0.66669091 | 3 |
| $\int_0^{\frac{5\pi}{4}} \frac{\cos 2x}{e^x}dx$ | 0.2078813092 | 7 | 0.2078814336 | 7 | 0.207881816 | 3 |

**Note** : Here the prescribed tolerance $\varepsilon$=0.0001.

#Steps: Number of steps.

## 6. Conclusion

From tables 1 and 2 we figured out that

1. the values of the integrals determined by averaged Gauss-Legendre five point rule ($R_{avgGL_5}(f)$) approach to the exact values of the test integrals minimizing the gap at a greater extent in comparison to those in case of its constituent rules $R_{GL_2}(f)$, $R_{aG_3}(f)$ and also in comparison to the rules $R_{S_3}(f)$, $R_{S_4}(f)$ with out using adaptive integration scheme.

2. also the mixed quadrature rules, $R_{S_3GL_2}(f)$, $R_{S_4GL_2}(f)$ [1, 5], that have been developed previously are somehow less effective than averaged Gaussian five point rule $R_{avgGL_5}(f)$ in non-adaptive environment.

   Tables 3 and 4 spelled out that

3. imposing adaptive quadrature routine to evaluate the integrals using averaged Gauss-Legendre five point rule ($R_{avgGL_5}(f)$) is much more emphatic than determining the integrals by applying the rules $R_{S_3}(f)$, $R_{S_4}(f)$, $R_{GL_2}(f)$, $R_{aG_3}(f)$ so far the number of steps is concerned.

4. supplementing to the above remark, we have also drawn another advantageous remark that the averaged Gauss-Legendre five point rule $R_{avgGL_5}(f)$ is much more

competent and gave away encouraging results than those in case of other established mixed quadrature rules $R_{S_3GL_2}(f)$, $R_{S_4GL_2}(f)$ so far the number of steps is concerned using adaptive integration scheme.

A strong observation follows here, not only in comparison to some classical rules at a same level of precision but also averaged Gauss-Legendre five point rule $(R_{avgGL_5}(f))$ is dominating over some of the mixed quadrature rules ([2], [1]) developed earlier, both in adaptive and non adaptive integration scheme which is the basic notion behind this paper.

## References

[1] Das R. N. and Hota M. K., On mixed quadrature rules for numerical integration of real definite integrals, International Journal of Computational Science and Mathematics, International Research Publication House, 2(3) (2010), 109-121.

[2] Das R. N. and Pradhan, G., A mixed quadrature rule for approximate evaluation of real definite integrals, Int. J. Math. Educ. Sci. Technol., 27(2) (1996), 279-283.

[3] Laurie D. P., Anti Gaussian quadrature formulas, Math. Comp., A. M. S., 65 (1996), 739-747.

[4] Patra P., Das D. and Dash R. B., Approximation of singular integrals by a mixed quadrature of anti-Gauss and Steffensen's quadrature rules in the adaptive environment, Advances in Theoretical and Applied Mathematics, 11(1) (2016), 79-92.

[5] Singh B. P. and Dash R. B., Forming two mixed quadrature rules using an anti-Gaussian quadrature rule, IJMSEA, 9(4) (2015), 27-38.