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HANKEL DETERMINANT FOR GENERAL SUBCLASS OF BI-UNIVALENT FUNCTIONS DEFINED BY A GENERALIZED DIFFERENTIAL OPERATOR

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Abstract

A generalized differential operator has been applied in this paper to investigate a general subclass of the function class $\mathrm{R}^m(\alpha, \beta, \lambda)$ of bi-univalent functions defined in the open unit disc. Making use of the Hankel determinant we obtain upper bounds for the second Hankel determinant $\mathrm{H}_2(2)$ of this class.

1. Introduction

Let $\mathcal{A}$ denotes the class of function of the form,

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

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which are analytic in the open unit disc $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Further, by $\mathcal{S}$ we will show the family of all function in $\mathcal{A}$ which are univalent in $\mathbb{U}$.

It is well known that every function f $\in \mathcal{S}$ has an inverse f^{-1}, defined by

$$\begin{aligned} f^{-1}(f(z)) &= z \qquad (z \in \mathbb{U}) \quad \text{and} \\ f(f^{-1}(w)) &= w \qquad \left(|w| < r_0(f); r_0(f) \geq \tfrac{1}{4}\right) \end{aligned} \tag{1.2}$$

where,

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \ldots$$

A function f $\in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f(z) and f^{-1} (z) are univalent in $\mathbb{U}$. Let σ denotes the class of bi-univalent function in $\mathbb{U}$ given by (1.1). For a brief history and interesting examples of functions which are in (or which are not in) the class σ, together with various other properties of the bi-univalent function class σ one can refer the work of Srivastava et al. [25] and references therein. In fact, the study of coefficient problems involving bi-univalent functions was reviewed recently by Srivastava et al. [25]. Various subclasses of the bi-univalent function class σ were introduced and non sharp estimates on the first two coefficient $|a_2|$ and $|a_3|$ in the Taylor-Maclaurin series expansion (**??**) were found in several recent investigations. The aforecited all these papers on the subject were actually motivated by the pioneering work of Srivastava et al. [25]. However, the problem to find the coefficient bounds on $|a_n|$ (n = 3,4,...) for functions f $\in \sigma$ is still an open problem.

Some of the important and well investigated subclasses of the univalent function class $\mathcal{S}$ include (for example) the class $\mathcal{S}^*(\beta)$ of starlike functions of order β in $\mathbb{U}$ and the class $\mathcal{K}(\beta)$ of convex functions of order β in $\mathbb{U}$. By definition, we have

$$\begin{aligned} \mathcal{S}^*(\beta) &:= \left\{ f : f \in \mathcal{A} \quad and \quad \Re\left(\frac{zf'(z)}{f(z)}\right) > \beta; \quad z \in \mathbb{U}; \quad 0 \leq \beta < 1 \right\} \quad \text{and} \\ \mathcal{K}(\beta) &:= \left\{ f : f \in \mathcal{A} \quad and \quad \Re\left(1 + \frac{zf'(z)}{f(z)}\right) > \beta; \quad z \in \mathbb{U}; \quad 0 \leq \beta < 1 \right\} \end{aligned} \tag{1.3}$$

For $0 \leq \beta < 1$, a function f $\in \sigma$ is in the class $\mathcal{S}^*_\sigma(\beta)$ of bi-starlike function of order β or $\mathcal{K}_{\sigma,\beta}$ of bi-convex function of order β if both f and f^{-1} are respectively starlike or convex functions of order β.

For integer n ≤ 1 and q ≤ 1, the q^{th} Hankel Determinant, defined as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q-2} & \cdots & a_{n+2q-2} \end{vmatrix} \quad (a_1 = 1) \tag{1.4}$$

The Hankel determinant plays an important role in the study of singularities [7]. This is also an important in the study of power series with integral coefficients [4,7]. The properties of the Hankel determinants can be found in [26]. It is interesting to note that,

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2 \qquad a_1 = 1 \tag{1.5}$$

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2 \tag{1.6}$$

The Hankel determinants $H_2(1) = a_3 - a_2^2$ and $H_2(2) = a_2a_4 - a_3^2$ are well known as Fekete-Szegö and second Hankel determinant functional respectively. Further Fekete and Szegö [8] introduced the generalized functional $a_3 - \delta a_2^2$, where δ is some real number. In 1969, Keogh and Merkes [14] discussed the Fekete-Szegö problem for the classes $\mathcal{S}^*$ and $\mathcal{K}$. Recently, several authors have investigated upper bounds for the Hankel determinant of functions belonging to various subclasses of univalent functions [1, 2, 13, 15, 16, 17, 22] and reference therein. On the other hand, Zaprawa [27, 28] extended the study on Fekete-Szegö problem to some classes of bi-univalent functions. Following Zaprawa [27, 28], the Fekete-Szegö problem for functions belonging to various subclasses of bi-univalent functions were considered in [3,11,19]. Very recentl, the upper bounds of $H_2(2)$ for the classes $\mathcal{S}^*_\sigma(\beta)$ and $\mathcal{K}_\sigma(\beta)$ were discussed by Deniz et al. [6].
The Hadamard product or convolution of the functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n Z^n \qquad and \qquad g(z) = z + \sum_{n=2}^{\infty} b_n Z^n \tag{1.7}$$

is given by

$$(f \times g)(z) = z + \sum_{n=2}^{\infty} a_n b_n Z^n, \qquad z \in \mathbb{U}.$$

Let $f \in \mathcal{A}$. We consider the following differential operator introduced by Răducanu and

Orhan [13]:

$$\begin{aligned} D^0_{\alpha\beta}f(z) &= f(z) \\ D^1_{\alpha\beta}f(z) &= D_{\alpha\beta}f(z) = \alpha\beta z^2 f''(z) + (\alpha-\beta)zf'(z) + (1-\alpha+\beta)f(z) \\ D^m_{\alpha\beta}f(z) &= D_{\alpha\beta}(D^{m-1}_{\alpha\beta}f(z)) \end{aligned} \tag{1.8}$$

where $0 \leq \beta \leq \alpha$ and m $\in \mathbb{N} := \{1, 2, \ldots\}$
If the function f is given by (1.7) then, from (1.8) we see that,

$$D^m_{\alpha\beta}f(z) = z + \sum_{n=2}^{\infty} A_n(\alpha, \beta, m)a_n z^n \tag{1.9}$$

where,

$$A_n(\alpha, \beta, m) = [1 + (\alpha\beta n + \alpha - \beta)(n-1)^m] \tag{1.10}$$

When $\alpha = 1$ and $\beta = 0$, we get Sălăgean differential operator [14]. When $\beta = 0$, we obtain the differential operator defined by Al-Oboudi [1].
From (1.9) it follows that $\mathrm{D}^m_{\alpha\beta}\mathrm{f(z)}$ can be written in terms of convolution as,

$$D^m_{\alpha\beta}f(z) = (f \times g)f(z) \tag{1.11}$$

where

$$g(z) = z + \sum_{n=2}^{\infty} A_n(\alpha, \beta, m)z^n \tag{1.12}$$

Definition 1.1 : If f $\in \mathcal{A}$ is said to be in class $\Re^m_\lambda(\alpha, \beta, \delta)$. If the following conditions are satisfied,

$$\Re\left\{(1-\alpha)\frac{D^m_{\alpha\beta}f(z)}{z} + \lambda\left[D^m_{\alpha\beta}f(z)\right]'\right\} > \delta \quad (0 \leq \delta < 1 \quad \lambda \geq 1, \quad z \in U) \tag{1.13}$$

$$\Re\left\{(1-\alpha)\frac{D^m_{\alpha\beta}g(w)}{z} + \lambda\left[D^m_{\alpha\beta}g(w)\right]'\right\} > \delta \quad (0 \leq \delta < 1 \quad \lambda \geq 1, \quad z \in U) \tag{1.14}$$

Next we state the following lemmas we shall use to establish the desired bounds in our study.

Lemma 1.1 [20] : If the function p $\in \mathcal{P}$ is given by the series,

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots \tag{1.15}$$

then the following sharp estimate holds:

$$|c_k| \leq 1 \qquad k = 1, 2, \ldots \tag{1.16}$$

Lemma 1.2 [10] : If the function p $\in \mathcal{P}$ is given by the series (1.15), then

$$\begin{aligned} 2c_2 &= c_1^2 + x(4 - c_1^2) \\ 4c_3 &= c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2(1 - |x|^2)z \end{aligned} \tag{1.17}$$

For some x,z with $|x| \leq 1$ and $|z| \leq 1$.

2. Main Result

Theorem 2.1 : Let f given by (1) be in the class $\Re_\lambda^m(\alpha, \beta, \delta)$ and $0 \leq \delta < 1$, then

$$|a_2 a_4 - a_3^2| \leq \begin{cases} \frac{4(1-\delta)^2}{(1+\lambda)A_4}\left[\frac{4(1-\delta)^2}{[1+\lambda]^3 A_2^3} + \frac{1}{1+3\lambda}\right] \in \left[0, 1 - \sqrt{\frac{(1+\lambda)^3 A_2^3}{8(1+3\lambda)A_4}}\right] \\ \frac{g(1+\lambda)^2(1-\delta)^2 A_2^2}{2(1+3\lambda)A_4\left[(1+\lambda)^3 A_2^3 - 2(1-\delta)^2(1+3\lambda)A_4\right]} \in \left[1 - \sqrt{\frac{(1+\lambda)^3 A_2^3}{8(1+3\lambda)A_4}}\right] \end{cases}$$

Proof : Since f $\in \Re_\lambda^m(\alpha, \beta, \delta)$ there exist two functions p(z) and q(w) $\in \mathcal{P}$ satisfying the condition og lemma (2.1) such that,

$$\frac{(1-\lambda)D_{\alpha\beta}^m f(z)}{z} + \lambda(D_{\lambda\beta}^m f(z))' = \delta + (1-\delta)p(z) \tag{2.1}$$

$$\frac{(1-\lambda)D_{\alpha\beta}^m g(w)}{w} + \lambda(D_{\lambda\beta}^m g(w))' = \delta + (1-\delta)q(w) \tag{2.2}$$

$$\frac{(1-\lambda)\left[z + \sum\limits_{n=2}^{\infty} A_n(\alpha, \beta, m) a_n z^n\right]}{z} + \lambda\left(1 + \sum_{n=2}^{\infty} n A_n(\alpha, \beta, m) a_n z^{n-1}\right) = \delta + (1-\delta)p(z) \tag{2.3}$$

$$(1-\lambda)\left[\sum_{n=2}^{\infty} A_n(\alpha, \beta, m) a_n z^n\right] + \lambda\left(1 + \sum_{n=2}^{\infty} n A_n(\alpha, \beta, m) a_n z^{n-1}\right) = \delta + (1-\delta)p(z) \tag{2.4}$$

Where $A_n(\alpha, \beta, m) = \left[1 + (2\beta n + \lambda - \beta)(n-1)\right]^m$

Equating the coefficient of (2.2) and (2.4) we get,

$$(1+\lambda)A_2 a_2 = (1-\delta)p_1 \tag{2.5}$$

$$(1+2\lambda)A_3 a_3 = (1-\delta)p_2 \tag{2.6}$$

$$(1+3\lambda)A_4a_4 = (1-\delta)p_3 \tag{2.7}$$

$$-(1+\lambda)A_2a_z = (1-\delta)q_1 \tag{2.8}$$

$$(1+2\lambda)A_3(2a_z^2 - a_3) = (1-\delta)q_2 \tag{2.9}$$

$$-(1+3\lambda)A_4(5a_z^2 - 5a_2a_3 + a_4) = (1-\delta)q_3 \tag{2.10}$$

From (2.5) and (2.8) we obtain,

$$p_1 = -q_1 \quad \text{and} \quad a_2 = \frac{(1-\delta)}{(1+\lambda)A_2}p_1 \tag{2.11}$$

Subtracting (2.6) from (2.9) we have,

$$a_3 = \frac{(1-\delta)^2p_1^2}{(1+\lambda)^2A_2^2} + \frac{(1-\delta)(p_2-q_2)}{2(1+2\lambda)A_3} \tag{2.12}$$

Subtracting (2.7) from (2.10) we have,

$$a_4 = \frac{(1-\delta)(p_3-q-3)}{2(1+3\lambda)A_4} + \frac{5}{4}\frac{(1-\delta)^2p_1(p_2-q_2)}{(1+\lambda)(1+2\lambda)A_2A_3} \tag{2.13}$$

Then we can establish that,

$$\begin{aligned} |a_2a_4 - a_3^2| &= \left| \frac{(1-\delta)p_1}{(1+\lambda)A_2}\left(\frac{(1-\delta)(p_3-q_3)}{2(1+3\lambda)A_4} + \frac{5(1-\delta)^2p_1(p_2-q_2)}{4(1+\lambda)(1+2\lambda)A_2A_3}\right) - \right. \\ &\quad \left. \left[\frac{(1-\delta)^2p_1^2}{(1+\lambda)^2A_2^2} + \frac{(1-\delta)(p_2-q_2)}{2(1+2\lambda)A_3}\right]^2 \right| \\ &= \left| \frac{-(1-\delta)^4p_1^4}{(1+\lambda)^4A_2^4} + \frac{5(1-\delta)^2p_1^3(p_1-q_2)}{4(1+\lambda)(1+2\lambda)A_2^2A_3} - \frac{(1-\delta)^3p_1^2(p_2-q_2)}{(1-\lambda)^2(1+2\lambda)A_2^2A_3} + \right. \\ &\quad \left. \frac{(1-\delta)^2p_1(p_3-q_3)}{2(1+\lambda)(1+3\lambda)A_2a_4} - \frac{(1-\delta)^2(p_2-q_2)^2}{4(1+2\lambda)^2A_3^2} \right| \end{aligned} \tag{2.14}$$

According to Lemma,

$$p_2 = q_2 \begin{cases} 2p_2 = p_1^2 + x(4-p_1^2) \\ 2q_2 = q_1^2 + x(4-q_1^2) \end{cases} \tag{2.15}$$

and

$$p_3 - q_3 = \frac{p_1^3}{2} - p_1(4-p_1^2)x - \frac{p_1}{2}(4-p_1^2)x^2 \tag{2.16}$$

Then,

$$|a_2a_4 - a_3^2| = \left| \frac{-(1-\delta)^4}{(1+\lambda)^4 A_2^4} p_1^4 + \frac{(1+\delta)^2 p_1 \left[\frac{p_1^3}{2} - p_1(4-p_1^2)x - \frac{p_1}{2}(4-p_1^2)x_2\right]}{2(1+\lambda)(1+3\lambda)A_2A_4} \right|$$

$$= \left| \frac{-(1-\delta)^4}{(1+\lambda)^4 A_2^4} + \frac{(1-\delta)^2 p_1^4}{4(1+\lambda)(1+3\lambda)A_2A_4} - \frac{(1-\delta)^2 p_1^2(4-p_1^2)x}{2(1+\lambda)(1+3\lambda)A_2A_4} \right. \tag{2.17}$$

$$\left. - \frac{(1-\delta)^2 p_1^2(4-p_1^2)x^2}{4(1+\lambda)(1+3\lambda)A_2A_4} \right|$$

Since p $\in \mathcal{P}$, so $|p_1| \leq 2$, letting $|p_1| =$ p, we may assume without restriction that p $\in$ [0,2]. Then applying the triangle inequality on (2.17) with $\rho = |x| \leq 1$, we get,

$$|a_2a_4 - a_3^2| \leq \frac{(1-\delta)^4}{(1+\lambda)^4 A_2^4} p^4 + \frac{(1-\delta)^2}{2(1+\lambda)(1+3\lambda)A_2A_4} p^4 + \frac{(1+\delta)^2 p^2(4-p^2)\rho}{2(1+\lambda)(1+3\lambda)A_2A_4}$$

$$+ \frac{(1+\delta)^2 p^2(4-p^2)\rho^2}{4(1+\lambda)(1+3\lambda)A_2A_4} \tag{2.18}$$

$$= F(\rho)$$

Differentiating F(ρ), we obtain

$$F'(\rho) = \frac{(1-\delta)^2 p^2(4-p^2)}{2(1+\lambda)(1+3\lambda)A_2A_4} + \frac{(1-\delta)^2 p^2(4-p^2)\rho}{2(1+\lambda)(1+3\lambda)A_2A_4} \tag{2.19}$$

Furthermore, for F$'(\rho) > 0$ and $\rho > 0$, F is an increasing function and thus the upper bound for F(ρ) corresponds to $\rho = 1$.

$$F(\rho) \leq \frac{(1-\delta)^4 p^4}{(1+\lambda)^4 A_2^4} + \frac{(1-\delta)^2 p^4}{4(1+\lambda)(1+3\lambda)A_2A_4} + \frac{3(1-\delta)^2 p^2(4-p^2)}{4(1+\lambda)(1+3\lambda)A_2A_4}$$

$$\leq \frac{(1-\delta)^4 p^4}{(1+\lambda)^4 A_2^4} - \frac{1}{2} \frac{(1-\delta)^2 p^4}{(1+\lambda)(1+3\lambda)A_2A_4} \tag{2.20}$$

$$= G(p)$$

Assume that G(p) has a maximum value in an interior of p $\in$ [0,2] then,

$$G'(p) = \left[\frac{4(1-\delta)^4 p^3}{(1+\lambda)^4 A_2^4} - \frac{2(1-\delta)^2 p^3}{(1+\lambda)(1+3\lambda)(A_2A_4)} + \frac{6p(1-\delta)^2}{(1+\lambda)(1+3\lambda)A_2A_4} \right] \tag{2.21}$$

Then,

$$G'(p) = 0 = \begin{cases} p_{01} = 0 \\ p_{02} = \sqrt{\frac{3(1+\lambda)^3 A_2^3}{(1+\lambda)^3 A_2^3 - 2(1+3\lambda)(1-\delta)^2 A_4}} \end{cases} \tag{2.22}$$

Case 2.1 : When $\delta \in \left[0, 1-\sqrt{\frac{3(1+\lambda)^3 A_2^3}{8(1+3\lambda)A_4}}\right]$ we observe that $p_{02} > 2$ & G is an increasing function in interval [0,2], so the maximum value of G(p) occurs of p = 2. Thus we have,

$$G(2) = \left[\frac{4(1-\delta)^2}{(1+\lambda)^3 A_2^3} + \frac{1}{1+3\lambda}\right] \frac{4(1-\delta)^2}{(1+\lambda)A_4} \tag{2.23}$$

Case 2.2 : When $\delta \in \left[-\sqrt{\frac{3(1+\lambda)^3 A_2^3}{8(1+3\lambda)A_4}}, 1\right]$ we observe that $p_{02} < 2$, since $G''(p_{02}) < 0$, the maximum value of G(p) occurs $p = p_{02}$ then we have,

$$G(p_{02}) = \frac{9(1+\lambda)^2(1-\delta)^2 A_2^2}{2(1+3\lambda)A_4\left[(1+\lambda)^3 A_2^3 - 2(1-\delta)^2(1+3\lambda)A_4\right]} \tag{2.24}$$

Remark 2.1 : Putting m = 0, we get $A_2 = A_3 = A_4 = 1$, then we get Hankel determinant for $\Re_\lambda^o(\alpha, \beta, \delta)$ as $0 \leq \delta < 1$.

$$|a_2 a_4 - a_3^2| \leq \begin{cases} \frac{4(1-\delta)^2}{1+\lambda}\left[\frac{4(1-\delta)^2}{(1+\lambda)^3} + \frac{1}{1+3\lambda}\right], \delta \in \left[0, 1, -\frac{1}{2}\sqrt{\frac{(1+\lambda)^3}{2(1+3\lambda)}}\right] \\ \frac{9(1+\lambda)^2(1-\delta)^2}{2(1+3\lambda)\left[(1+\lambda)^3 - 2(1-\delta)^2(1+3\lambda)\right]}, \delta \in \left[1, -\frac{1}{2}\sqrt{\frac{(1+\lambda)^3}{2(1+3\lambda)}}, 1\right] \end{cases} \tag{2.25}$$

□

References

[1] Altinkaya Ş, Yalçin S., Construction of second Hankel determinant for a new subclass of bi-univalent functions, Turk J Math, doi: 10.3906/mat-1507-39.

[2] Altinkaya Ş, Yalçin S., Faber polynomial coe cient bounds for a subclass of bi-univalent functions, C R Acad Sci Paris Sr I, 353 (2015), 1075-1080.

[3] Brannan D. A., Clunie J. G., Aspects of contemporary complex analysis, Proceedings of the NATO Advanced Study Institute held at the University of Durham, Durham, July 120, 1979, Academic Press, New York, London, 1980.

[4] Brannan D. A., Taha T. S., On some classes of bi-univalent functions, in : S. M. Mazhar, A. Hamoui and N. S. Faour (Eds.), Math. Anal. and Appl., Kuwait; February 18.21, 1985, in: KFAS Proceedings Series, vol. 3, Pergamon Press, Elsevier Science Limited, Oxford, 1988, pp. 53.60. see also Studia Univ. Babeş-Bolyai Math. 1986; 31: 70.77.

[5] Bulut S., Faber polynomial coe cient estimates for a comprehensive subclass of analytic bi-univalent functions, C R Acad Sci Paris Sér I, 352 (2014), 479-484.

[6] Cantor D. G., Power series with integral coe cients, Bull Amer Math Soc., 69 (1963), 362-366.

[7] Çağlar M., Orhan H., Yağmur N., Coefficient bounds for new subclasses of bi-univalent functions, Filomat, 27 (2013), 1165-1171.

[8] Deniz E., Certain subclasses of bi-univalent functions satisfying subordinate conditions, J/ Class Anal., 2 (2013), 49-60.

[9] Deniz E., Çağlar M., Orhan H., Second Hankel determinant for bi-starlike and bi-convex functions of order β, Appl Math Comput., 271 (2015), 301-307.

[10] Duren P. L., Univalent Functions, Grundlehren der Mathematischen Wissenschaften, Vol. 259, Springer, New York, (1983).

[11] Eker S. S., Coefficient bounds for subclasses of m-fold symmetric bi-univalent functions, Turk J Math., 40 (2016), 641-646.

[12] Fekete M., Szegō G., Eine Bemerkung uber ungerade schlichte Funktionen., J London Math Soc., 8 (1933), 85-89.

[13] Frasin B. A., Aouf M. K., New subclasses of bi-univalent functions, Appl Math Lett., 24 (2011), 1569-1573.

[14] Grenander U., Szegō G., Toeplitz Forms and their Applications. California Monographs in Mathematical Sciences, University California Press, Berkeley, (1958).

[15] Hamidi S. G., Jahangiri J. M., Faber polynomial coefficient estimates for analytic bi-close-to-convex functions, C R Acad Sci Paris Sr I., 352 (2014), 17-20.

[16] Janteng A., Halim S. A., Darus M., Hankel Determinant for starlike and convex functions, Internat J Math Anal., 1 (2007), 619-625.

[17] Kanas S., Kim S-A., Sivasubramanian S., Verification of Brannan and Clunie's conjecture for certain subclasses of bi-univalent function, Ann Polon Math., 113 (2015), 295-304.

[18] Kedzierawski A. W., Some remarks on bi-univalent functions, Ann Univ Mariae Curie-Skodowska Sect A, 39 (1985), 77-81.

[19] Keogh F. R., Merkes E. P., A coefficient inequality for certain classes of analytic functions, Proc Amer Math Soc., 20 (1969), 8-12.

[20] Kumar S. S., Kumar V., Ravichandran V., Estimates for the initial coe cients of bi-univalent functions, Tamsui Oxford J Inform Math Sci., 29 (2013), 487-504.

[21] Lee S. K., Ravichandran V., Supramaniam S., Bounds for the second Hankel determinant of certain univalent functions., J. Inequal Appl., (2013), Article ID 281, 1-17.

[22] Lewin M., On a coefficient problem for bi-univalent functions, Proc Amer Math Soc., 18 (1967), 63-68.

[23] Netanyahu E., The minimal distance of the image boundary from the origin and the second coe cient of a univalent function in $|z| < 1$, Arch Rational Mech Anal., 32 (1969), 100-112.

[24] Noonan J. W., Thomas D. K., On the second Hankel determinant of areally mean p-valent functions, Trans Amer Math Soc., 223 (1976), 337-346.

[25] Orhan H., Deniz E., Raducanu D., The Fekete-Szegō problem for subclasses of analytic functions defined by a differential operator related to conic domains, Comput Math Appl., 59 (2010), 283-295.

[26] Orhan H., Magesh N., Balaji V. K., Initial coefficient bounds for a general class of bi-univalent functions, Filomat, 29 (2015), 1259-1267.

[27] Orhan H., Magesh N., Yamini J., Bounds for the second Hankel determinant of certain bi-univalent functions, Turk J Math., 40 (2016), 679-687.

[28] Pommerenke Ch., Univalent Functions, Vandenhoeck and Rupercht, Gttingen, (1975).

[29] Srivastava H. M., Mishra A. K., Gochhayat P., Certain subclasses of analytic and bi-univalent functions, Appl Math Lett., 23 (2010), 1188-1192.

[30] Srivastava H. M., Bulut S., Çağlar M., Yağmur N., Coefficient estimates for a general subclass of analytic and bi-univalent functions, Filomat, 27 (2013), 831-842.

[31] Srivastava H. M., Eker S. S., Ali R. M., Coefficient estimates for a certain class of analytic and bi-univalent functions, Filomat, 29 (2015), 1839-1845.

[32] Tan D. L., Coefficient estimates for bi-univalent functions, Chinese Ann Math Ser A ., 5 (1984), 559-568.

[33] Xu Q. H., Gui Y. C., Srivastava H. M., Coefficient estimates for a certain subclass of analytic and bi-univalent functions, Appl Math Lett., 25 (2012), 990-994.

[34] Xu Q. H., Xiao H. G., Srivastava H. M., A certain general subclass of analytic and bi-univalent functions and associated coefficient estimate problems, Appl Math Comput., 218 (2012), 11461-11465.

[35] Zaprawa P., On the Fekete-Szeg problem for classes of bi-univalent functions, Bull Belg Math Soc Simon Stevin, 21 (2014), 169-178.

[36] Zaprawa P., Estimates of initial coefficients for bi-univalent functions, Abstr Appl Anal (2014), Article ID 357480, 1-6.