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ISOMORPHISM OF UNIDIRECTED PLANE GRAPHS USING MARCOV CHAIN

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Abstract

The graph isomorphism problem for undirected plane graphs is to determine whether two given undirected plane graphs are isomorphic or not. The motivation behind the isomorphism of undirected plane graphs is Marcov chain. In this paper, develop and use probabilistic finite automaton approach for graph invariant [1, 4] called the probability propagation matrix for isomorphism of undirected plane graphs having six vertices by using the Marcov chain. It may be extended for 'n'-number of vertices where 'n' is a positive integer.

1. Introduction

The graph isomorphism problem is to determine if there exists a one-to-one correspondence between the vertices of two graphs G_1 and G_2 that preserves the adjacency of

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vertices. Two undirected plane graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there exists a one-to-one mapping π from V_1 and V_2 such that $(x, y) \in E_1)$ if and only if $(\pi(x), \pi(y)) \in E_2$. The problem of graph isomorphism arises many fields such as chemistry, switching theory, information retrieval, and linguistics [3,5].

Theoretically, it is always possible to determine whether or not two undirected plane graphs G_1 and G_2 are isomorphic by keeping G_1 fixed and reordering vertices of G_2 to check if their adjacency matrices become identical. This process may require all $(n)!$ reordering and comparisons, $1n$' being the number of vertices. Such an inefficient procedure, in which the running time grows factorially with 'n', is of limited use of practical problems [3].

In the present paper, develop and use probabilistic finite approach by using graph invariant that is probability propagation matrix which extracts information from degrees and adjacent conditions of vertices simultaneously. After some operations on the matrix, this kind of information about vertices is propagated to their adjacent vertices. Since probability propagation matrices can be computed using elementary operations of vectors and matrices only, therefore this property can be practically used in parallel computing.

2. Basics about the Probabilistic Finite Automaton Approach

Assume that all plane graphs $G = (V, E)$ are directed and strongly connected. If G is not directed, it can direct each edge $(u, v) \in E$ into two directed edges (u, v) and (v, u) to make G directed. If G is not strongly connected, it can add a new vertex w into v and for each vertex $v \in V$, add two directed edges (w, v) and (v, w) into E to make G strongly connected. For a plane graph $G = (V, E)$with $V = \{1, 2, \cdots, n\}$, let A be its adjacency matrix such that $A[i, h]$ is 1 iff $(i, j) \in E$, where $A[i, j]$ is the (i, j)-entry of A. Let d_i be the out degree of vertex i [3].

2.1 Probability Distribution Matrix (PDM)

The PDM B of plane graph G is defined as, for $1 \leq i, j \leq n, B[i, j] = \frac{A[i,j]}{d}$.

The B is used as follows. Let vertex $i \in V$ have $d_i = h$ and be adjacent to vertices $i_1, i_2, \cdots, i_h$, where the direction is from i to $i_k, 1 \leq k \leq h$. Suppose that moving from vertex i to the next vertex $i_k, 1 \leq k \leq h$, is random and equally likely. Then, the probability of moving from vertex i to vertex i_k is $\frac{1}{h}$ for each $1 \leq\leq h$. Therefore the iterative power $B^{(k)}$ of B represents the probability of moving from one vertex to another after k random walks. $B^k[i,j]$ is the probability of moving from vertex i to vertex j after k random walks [10].

2.2 Probabilistic Finite Automaton (PFA)

A PFA U is a 4-tuple (Q, Σ, M, Γ) with the following:

(i) $Q = \{1, 2, 3, \cdots, \}$ is a finite set of states.

(ii) Γ is the input alphabet.

(iii) M is a transition function from Σ into $(n \times n)$-dimensional matrices such that $M(\sigma)[i,j]$ is the probability that U moves from state i to state j after reading a symbol $\sigma \in \Sigma$. $M(\sigma)$ is stochastic [8] that is for any i, $\sum\limits_{j=1}^{n} M(\sigma)[i,j] = 1$.

Extend the domain of M from Σ to Σ^* as $M(\epsilon) = I_n$ and $M(x\sigma) = M(x)M(\sigma)$ for $x \in \Sigma^*$ and $\sigma \in \Sigma$, where ϵ is the empty string and I_n is the $(n \times n)$-dimensional identity matrix. Then, $M(x)[i,j]$ is the probability that U will move from state i to state j after reading string x.

(iv) $\Gamma = [\gamma_1, \gamma_2, \cdots, \gamma_n]$ with $\gamma_i \geq 0$ and $\sum\limits_{i=1}^{n} \Gamma_i = 1$.

Γ is called the initial state distribution vector with the i-th element denoting the probability of state i being the initial state, let Γ^i denote the row vector with $\gamma_i = 1$ and $\gamma_j = 0$ if $i \neq j$.

(v) The state distribution vector of U with initial state i and reading string x is $P_U^i(x) = \Gamma^i \cdot M(x)$.

2.3 Probability Propagation Matrix (PPM)

For U with initial state i reading string $x = \sigma_1 \cdot \sigma_2 \cdots \sigma_k$ the PPM $P_U^i[x]$ is a collection of state distribution vectors $P_U^i(\epsilon), P_U^i(\sigma_1), \cdots, P_U^i(\sigma_1, \sigma_2, \cdots, \sigma_k)$; that is

$$P_U^i[x] = \begin{bmatrix} P_U^i(\epsilon) \\ P_U^i(\sigma_1) \\ P_U^i(\sigma_1\sigma_2) \\ \vdots \\ P_U^i(\sigma_1\sigma_2\cdots\sigma_k) \end{bmatrix}$$

A digraph $G = (V, E)$ can be transformed to a $U = (Q, \Sigma, M, \Gamma)$ according to following rules. Each vector in V is viewed as a state in Q so that $Q = V$. Since edges in E are not labeled, it can label by using symbol a. Thus $\Sigma = \{a\}$. The transition function M is defined by $M(a) = B$, where B is the PDM of G.

3. Probabilistic Finite Automaton Approach

Theorem 3.1 : If P is the transition matrix of a Markov chain and $\vec{p}^{(m)}$ denotes the probability distribution vector after the first m steps, then $\vec{p}^{(m)} = \vec{p}^{(0)} P^m$, where $\vec{p}^{(0)}$ is the initial probability distribution vector [8].

Theorem 3.2 : If P denotes the transition matrix of a Markov chain in one step and $P^{(m)}$ denotes the m-th step transition matrix, then $P^{(m)} = P^m$, that is, the m-th step matrix is the m-th power of P [8].

Main Theorem 3.3 : Graph $G_1 = (V_1, E_1)$ is isomorphic to graph $G_2 = V_2, E_2)$ only if for any fixed i, there exists some $j, 1 \leq j \leq n$ such that P_1^i and P_2^i are isomorphic.

Proof : Since G_1 is isomorphic to G_2, there exists a bijective mapping π from V_1 to V_2 such that $(i, j) \in E_1$ iff $(\pi(i), \pi(j)) \in E_2$. Therefore $B_1[i, j] = B_2[\pi(i), \pi(j)]$.

Let U_1 and U_2 be the PFAs, corresponding to G_1 and G_2 respectively. It is obvious that state i of the U_1 mapping to state $\pi(i)$ of U_2 witnesses the equivalence of U_1 and U_2.

It can be shown that π is the isomorphic mapping of $P_1^i[a^{2n-1}]$ and $P_2^i[a^{2n-1}]$ by induction on the length of the input string x. Let $g_{s,t}$ and $h_{s,t}$ be the (s, t)-entry of $P_1^i[x]$ and $P_2^{\pi(i)}[x]$ respectively.

1. Basis of Induction : $|x| = 0$. Let the initial states of U_1 and U_2 be i and $\pi(i)$ respectively. Then, $P_1^i[\epsilon] = [g_{0,1}, g_{0,2}, \cdots, h_{0,n}]$ with $g_{0,i} = 1$ and $g_{0,t} = 0$ if $i \neq t$, and $P_2^{\pi(i)}[\epsilon] = [h_{0,1}, h_{0,2}, \cdots, h_{0,n}]$ with $h_{0,\pi(i)} = 1$ and $h_{0,t} = 0$ if $\pi(i) \neq t$. Thus, $P_1^i[\epsilon]$ is isomorphic to $P_2^{\pi(i)}[\epsilon]$ via permutation.

2. Induction Hypothesis : $|x|| = k - 1$. The hypothesis is

$$\forall \ \ 1 \leq j \leq n, \quad P_1^{i,j}[a^{k-1}] = P_2^{\pi(i),\pi(j)}[a^{k-1}].$$

   By the hypothesis, $g_{k-1,j} = h_{k-1,\pi(j)}$ for $1 \leq j \leq n$.

3. Induction step : $|x| = k$, since

$$\begin{aligned} P_1^i(a^k) &= [g_{k,1}, g_{k,2}, \cdots, g_{k,n}] = P_1^i(a^{k-1}), B_1 \\ &= [g_{k-1,1}, g_{k-1,2}, \cdots, g_{k-1,n}], B_1. \end{aligned}$$

Therefore, for $1 \leq j \leq n$

$$\begin{aligned} g_{k,j} &= \sum_{h=1}^{n} g_{k-1,h} \cdot B_1(h,j) \\ &= \sum_{h=1}^{n} h_{k-1,\pi(h)} \cdot B_2(\pi(h),\pi(j)) = h_{k,\pi(j)}. \end{aligned}$$

Therefore, $P_1^i = (a^k) = P_2^{\pi(i)} = (a^k)$ and, thus $P_1^i[a^k] = P_2^{\pi(i)}[a^k]$.

Corollary 3.4 : For a fixed i, if $P_1^i[a^{2n-1}]$ is not isomorphic to $P_2^j[a^{2n-1}]$ for any $j, 1 \leq j \leq n$ then graphs G_1 and G_2 are not isomorphic.

Corollary 3.5 : If $P_1^i[a^{2n-1}]$ and $P_2^j[a^{2n-1}]$ are isomorphic via a permutation $\pi(i) = j$, then π is a possible isomorphic mapping for graphs G_1 and G_2.

4. Illustrative and Motivational Example

Consider an undirected graph $G_1 = (V_1, E_1)$ given by adjacency matrix M_1.

$$M_1 = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Since G_1 is connected undirected plane graph, transfer it to a strongly digraph by directing each edge $(u,v) \in E_1$ into two directed edges (u,v) and (v,u) and corresponding PFA is U_1.

The PDM B_1 of $G_1 = V_1, E_1$ is

$$\begin{bmatrix} 0 & 0.25 & 0 & 0.25 & 0.25 & 0.25 \\ 0.33 & 0 & 0.33 & 0 & 0 & 0.33 \\ 0 & 0.33 & 0 & 0.33 & 0 & 0.33 \\ 0.25 & 0 & 0.25 & 0 & 0.25 & 0.25 \\ 0.5 & 0 & 0 & 0.5 & 0 & 0 \\ 0.25 & 0.25 & 0.25 & 0.25 & 0 & 0 \end{bmatrix}$$

Let state 2 of U_1 is selected as the initial state. PPM corresponding U_1 is

$$P_1^2[a^3] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0.33 & 0 & 0.33 & 0 & 0 & 0.33 \\ 0.08 & 0.27 & 0.08 & 0.27 & 0.08 & 0.19 \\ 0.25 & 0.09 & 0.21 & 0.13 & 0.09 & 0.21 \end{bmatrix}.$$

Consider another undirected graph $G_2 = (V_2, E_2)$ given by adjacency matrix M_2.

$$M_2 = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Since G_2 is connected undirected plane graph, transfer it to a strongly digraph by directing each edge $(u, v) \in E_2$ into two directed edges $u, v)$ and (v, u) and corresponding PFA is U_2.

The PDM B_2 of $G_2 = (V_2, E_2)$ is

$$\begin{bmatrix} 0 & 0.25 & 0.25 & 0 & 0.25 & 0.25 \\ 0.33 & 0 & 0.33 & 0 & 0.33 & 0 \\ 0.33 & 0.33 & 0 & 0 & 0 & 0.33 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0.25 & 0.25 & 0 & 0.25 & 0 & 0.25 \\ 0.25 & 0 & 0.25 & 0.25 & 0.25 & 0 \end{bmatrix}$$

Let state 2 of U_2 is selected as the initial state. PPM corresponding U_2 is

$$P_2^2[a^3] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0.33 & 0 & 0.33 & 0 & 0 & 0.33 \\ 0.19 & 0.27 & 0.08 & 0.08 & 0.08 & 0.27 \\ 0.21 & 0.09 & 0.21 & 0.09 & 0.25 & 0.13 \end{bmatrix}.$$

It is found that $P_1^2[a^3]$ is isomorphic to $P_2^2[a^3]$ via the permutation

$$\pi = \begin{pmatrix} 1 & 2 & 34 & 5 & 6 \\ 5 & 2 & 36 & 4 & 1 \end{pmatrix}.$$

The permutation π witnesses that G_1 is isomorphic to G_2 since

$$A_1[i, j] = A_2[\pi(i), \pi(j)], \quad 1 \leq i, \ j \leq n.$$

5. Conclusion

In this paper, graph invariant that is probability propagation matrix for graph isomorphism if undirected plane graphs have been used. This invariant consists simultaneously of information about the adjacency conditions and degree of vertices. Using this property, an efficient heuristic algorithm for the graph isomorphism problem can be designed. This property may be suitable for parallel computation. It can be also applicable to testing of labeled graphs.

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