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OSCILLATION THEOREMS FOR FRACTIONAL ORDER NEUTRAL DIFFERENTIAL EQUATIONS

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Abstract

The purpose of this paper is to study the oscillation of the fractional order neutral differential equation

$$D_t^\alpha \left[r(t)[D_t^\alpha(x(t)+p(t)x(\tau(t)))]^\gamma\right]+q(t)x^\beta(\sigma(t))=0,$$

where $D_t^\alpha(\cdot)$ is a modified Riemann-Liouville derivative. The obtained results are based on the new comparison theorems, which enable us to reduce the oscillatory problem of 2α-order fractional differential equation to the oscillation of the first order equation. The results are easily verified.

1. Introduction

In this article, we are concerned with the oscillation of solutions to the nonlinear fractional order neutral differential equation with the form

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$$D_t^\alpha[r(t)[D_t^\alpha(x(t)+p(t)x(\tau(t)))]^\gamma]+q(t)x^\beta(\sigma(t))=0, \tag{1}$$

where $D_t^\alpha(\cdot)$ denote the modified Riemann-Liouville derivative [35], with respect to the variable t, $q(t)\in C([t_0,+\infty))$, $D_t^\alpha r(t)\in C([t_0,+\infty))$, $D_t^{2\alpha}p(t)\in C([t_0,+\infty))$ and we define $z(t)=x(t)+p(t)x(\tau(t))$. Throughout this paper, we assume that the following conditions hold:

(A_1) γ,β are the ratios of two positive odd integers;

(A_2) $r(t)>0$, $q(t)>0$, $0\le p(t)\le p_0<\infty$;

(A_3) $\lim\limits_{t\to+\infty}\tau(t)=+\infty$, $\lim\limits_{t\to+\infty}\sigma(t)=+\infty$;

(A_4) $\tau'(t)\ge\tau_0>0$, $\tau\circ\sigma=\sigma\circ\tau$;

(A_5) $\frac{t}{\tau(t)}\ge l>0$.

A solution of the equation is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise it is nonoscillatory. Equation is said to be oscillatory if all its solutions are oscillatory.

Fractional differential equations are generalizations of classical differential equations to an arbitrary (non-integer) order. Fractional differential and integral equations have found many applications in various problems in science and engineering such as electro-thermoelasticity, electrochemistry of corrosion, electrode-electrolyte polarization, optics and signal processing, circuit systems, diffusion wave, heat conduction, fluid flow, probability and statistics, control theory of dynamical systems, and so on; see ([1, 2, 3, 4, 5, 6]). We referred to the monographs of fractional calculus and fractional differential equations on Kilbas et al. [9], Lakshmikantham et al. [10], Miller and Ross [7], Podlubny [8], Baleanu et al [11].

Nowadays, many articles have been investigated in some aspects of fractional differential equations, such as the existence and uniqueness of solutions, exact solutions and stability of solutions, the methods for explicit and numerical solutions; see ([12, 13, 14, 15, 16, 17]). The problem is to determineing the oscillation of solutions of various equations like ordinary and partial differential equations, difference equation, dynamics equation on time scales and fractional differential equations is an interesting area of research and more effort has been made to establish oscillation criteria for these equations; see ([18, 19, 20, 21, 22, 23, 24, 27]).

Recently, the research on fractional differential equation is very hot topic and only few publications paid attention in the oscillation of fractional differential equation; see ([25, 26, 27, 28, 29, 30, 31]).

In 2012, Grace *et al.* [27] initiated the oscillatory theory of fractional differential equations of the form

$$D_a^q x + f_1(t,x) = v(t) + f_2(t,x), \quad \lim_{t\to a^+} J_a^{1-q} x(t) = b_1,$$

under the conditions

$$x f_i(t,x) > 0 \quad \text{for} \quad i = 1, 2,\ x \neq 0,\ \text{and } t \geq a,$$

and

$$|f_1(t,x)| > p_1(t)|x|^\beta \quad \text{and} \quad |f_2(t,x)| > p_2(t)|x|^\gamma \quad \text{for} \quad x \neq 0,\ \text{and } t \geq a,$$

where D_a^q denotes the Riemann-Liouville differential operator of order q with $0 < q \leq 1$, and J_a^p is Rieman-Liouville fractional integral operator and and the functions f_1, f_2 and v are continuous. By the expression of solution and some inequalities, oscillation criteria are obtained for a class of nonlinear fractional differential equations. The results are also stated when the Riemann-Liouville differential operator is replaced by Caputo differential operator.

In 2012, Chen et al.[25] studied the oscillatory behavior of the following fractional differential equation

$$[r(t)(D_-^\alpha y)^\eta(t)]' - q(t) f\Big(\int_t^\infty (v-t)^{-\alpha} y(v)dv\Big) = 0 \quad \text{for} \quad t > 0,$$

where $D_-^\alpha y$ denotes the Liouville right-sided fractional derivative of order α with the form

$$(D_-^\alpha y)(t) := -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_t^\infty (u-t)^{-\alpha} y(v)dv \quad \text{for} \quad t \in \mathbb{R}_+; = (0, \infty).$$

By the Riccati transformation technique the authors obtained some sufficient conditions, which guarantee that every solution of the equation is oscillatory.

In 2013, Q. Feng *et al.* [26] concerned with the oscillation of solutions to the nonlinear forced fractional differential equation

$$D_t^\alpha[r(t)\psi(x(t))D_t^\alpha x(t)] + q(t)f(x(t)) = e(t),\ t \geq t_0 > 0,\, 0 < \alpha < 1,$$

where $D_t^\alpha(\cdot)$ denotes the modified Riemann-Liouville derivative. Based on a transformation of variables and properties of the modified Riemann-Liouville derivative, the fractional differential equation is transformed into a second-order ordinary differential equation. There by a generalized Riccati transformation, inequalities, and an integration average technique, they establish oscillation criteria for the fractional differential equation.

In 2015, Wang et al. [34] studied the oscillatory behavior of the following fractional differential equation is

$$D_t^\alpha\,[a(t)[D_t^\alpha(x(t)+p(t)x(\tau(t)))]] + q(t)x(\sigma(t)) = 0,$$

where $D_t^\alpha(\cdot)$ denotes the modified Riemann-Liouville derivative and they establish some new comparison theorems and then they use it to get some sufficient conditions for oscillations of all solutions in the equation.

This paper focuses on the fractional neutral differential equations involving a modified Riemann-Liouville derivative, which is given by Jumarie in [35]. The modified Riemann-Liouville derivative is defined as

$$D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-\xi)^{-\alpha}(f(\xi)-f(0))d\xi, & 0<\alpha<1; \\ (f^{(n)}(t))^{(\alpha-n)}, & n \leq \alpha < n+1, \quad n \geq 1. \end{cases}$$

and it has some properties that

$$D_t^\alpha t^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)}t^{r-\alpha}, \tag{2}$$

$$D_t^\alpha(f(t)g(t)) = g(t)D_t^\alpha f(t) + f(t)D_t^\alpha g(t), \tag{3}$$

$$D_t^\alpha f[g(t)] = f_g'[g(t)]D_t^\alpha g(t) = D_g^\alpha f[g(t)](g'(t))^\alpha \tag{4}$$

Regarding the integer case of our equation (1), B. Baculíková et al. [24], they have studied the sceond order nonlinear neutral differential equation

$$(a(t)[z'(t)]^\gamma)' + q(t)x^\beta(\sigma(t)) = 0$$

where $z(t) = x(t)+p(t)x(\tau(t))$. By comparison theorem, they established some sufficient conditions for the oscillation of equation. We also extend B. Baculíková and J. Džurina results to the fractional order differential equations.

However, to the best of our knowledge very little is known regarding the oscillatory behavior of fractional differential equations up to now. In this paper we will consider the oscillation of fractional order differential equation of (1). We organise this paper as follows. In section 2, we give a transformation of variables to the fractional differential equation, and provide a new transformation with delay then translate our fractional neutral differential equation to a second-order neutral differential equation. In section 3, we first establish some new comparison theorems and then use them to get some sufficient conditions for oscillation of all solutions of (1) and section 4 we present a example that apply the results established.

2. Preliminary Lemmas

For the sake of convenience, in this article, we denote:

$$\xi = y(t) = \frac{t^\alpha}{\Gamma(1+\alpha)}, \quad \xi_i = y(t_i) = \frac{t_i^\alpha}{\Gamma(1+\alpha)}, \quad i = 0, 1.$$
$$\widetilde{x}(\xi) = x(t), \quad \widetilde{r}(\xi) = r(t), \quad \widetilde{p}(\xi) = p(t), \quad \widetilde{q}(\xi) = q(t).$$

Towards to $\tau(t), \sigma(t)$ we have the next transformations.

Lemma 2.1 : Assume $A \geq 0$, $B \geq 0$, $\beta \geq 1$. Then

$$(A+B)^\beta \leq 2^{\beta-1}(A^\beta + B^\beta). \tag{5}$$

Proof : We may assume that $0 < A < B$. Consider a function $g(u) = u^\beta$. Since $g'' > 0$ for $u > 0$, function $g(u)$ is convex, that is

$$g\left(\frac{A+B}{2}\right) \leq \frac{g(A)+g(B)}{2}$$

which implies (5). □

Lemma 2.2 : Assume $A \geq 0$, $B \geq 0$, $0 \leq \beta \leq 1$. Then

$$(A+B)^\beta \leq A^\beta + B^\beta. \tag{6}$$

Proof : If $A = 0$ or $B = 0$, then (6) holds. For $A \neq 0$, on setting $x = B/A$, Condition (6) takes the form $(1+x)^\beta \leq 1 + x^\beta$, which is for $x > 0$ evidently true. □

Lemma 2.3 [34] : Suppose $(A_4),(A_5)$ hold, we define the functions $\tilde{\xi}$, $\tilde{\sigma}$ as the following forms

$$\tilde{\xi} = y(\tau(y^{-1}(\xi))), \quad \tilde{\sigma} = y(\sigma(y^{-1}(\xi))),$$

then it satisfies

$$x(\tau(t)) = \tilde{x}(\tilde{\tau}(\xi)), \quad x(\sigma(t)) = \tilde{x}(\tilde{\sigma}(\xi));$$

and a new condition

$$(\mathbf{A}_4') : \tilde{\tau}'(\xi) \geq \tau_0 l^{1-\alpha} = \tilde{\tau}_0, \quad \tilde{\tau}\circ\tilde{\sigma} = \tilde{\sigma}\circ\tilde{\tau}.$$

Lemma 2.4 : If $x(t)$ is a eventually positive solution of (1), and a sufficient large t_1 such that

$$R(t) = \int_{t_1}^{t} r^{-1/\gamma}(s)ds \to +\infty \ \text{ as } \ t \to +\infty, \tag{7}$$

then the corresponding function $z(t) = x(t) + p(t)x(\tau(t))$ satisfies

$$z(t) > 0, \quad r(t)\Big(D_t^\alpha(z(t))\Big)^\gamma > 0, \quad D_t^\alpha\Big[r(t)(D_t^\alpha(z(t)))^\gamma\Big] < 0 \tag{8}$$

eventually.

Proof : Let $x(t) = \tilde{x}(\xi)$, where $\xi = \frac{t^\alpha}{\Gamma(1+\alpha)}$. Then (2) we get $D_t^\alpha \xi(t) = 1$, and furthermore by use of (4) and Lemma 2.3 we have

$$D_t^\alpha x(t) = D_t^\alpha \tilde{x}(\xi) = \tilde{x}'(\xi) D_t^\alpha \xi(t) = \tilde{x}'(\xi),$$

$$D_t^\alpha x(\tau(t)) = D_t^\alpha \tilde{x}(\tilde{\tau}(\xi)) = (\tilde{x}(\tilde{\tau}(t)))' D_t^\alpha \xi(t) = (\tilde{x}(\tilde{\tau}(t)))'.$$

Similarly we have

$$D_t^\alpha r(t) = \tilde{r}'(\xi), \quad D_t^\alpha p(t) = \tilde{p}'(\xi) \quad D_t^\alpha q(t) = \tilde{q}'(\xi)$$

and

$$D_t^\alpha x(\sigma(t)) = (\tilde{x}(\tilde{\sigma}(\xi)))'.$$

Then we get

$$D_t^\alpha z(t) = \Big(\tilde{x}(\xi) + \tilde{p}(\xi)\tilde{x}(\tilde{\tau}(\xi))\Big)'.$$

We define $\tilde{z}(\xi) = \tilde{x}(\xi) + \tilde{p}(\xi)\tilde{x}(\tilde{\tau}(\xi))$, then $D_t^\alpha z(t) = \tilde{z}'(\xi)$. So the equation (1) can be transformed into the following form

$$(\tilde{r}(\xi)(\tilde{z}'(\xi))^\gamma)' + \tilde{q}(\xi)\tilde{x}^\beta(\tilde{\sigma}(\xi)) = 0 \quad \xi \geq \xi_0 > 0 \tag{9}$$

Since $x(t)$ is an eventually positive solution of (1), $\tilde{x}(\xi)$ is an eventually positive solution of (9). Hence there exists $\xi_1 > \xi_0$ such that $\tilde{x}(\xi) > 0$ on $[\xi_1, \infty)$. Also we know $\tilde{z}(\xi) > 0$ on $[\xi_1, \infty)$. It follows from (9) that

$$(\tilde{r}(\xi)(\tilde{z}^{'}(\xi))^{\gamma})' = -\tilde{q}(\xi)\tilde{x}^{\beta}(\tilde{\sigma}(\xi)) < 0$$

holds eventually. Therefore, $((\tilde{r}(\xi)\tilde{z}^{'}(\xi))^{\gamma})'$ is decreasing and thus either $\tilde{z}'(\xi) < 0$ or $\tilde{z}'(\xi) > 0$ eventually. We claim $\tilde{z}'(\xi) > 0$. Otherwise if $\tilde{z}'(\xi) < 0$, then there exists a constant c such that

$$\tilde{r}(\xi)(\tilde{z}'(\xi))^{\gamma} < -c < 0$$
$$\tilde{z}'(\xi) \leq -\frac{c}{\tilde{r}^{\frac{1}{\gamma}}(\xi)} < 0.$$

Integrating from ξ_1 to ξ, one gets,

$$\tilde{z}(\xi) \leq \tilde{z}(\xi_1) - c\int_{\xi_1}^{\xi} \tilde{r}^{-\frac{1}{\gamma}}(s)ds = \tilde{z}(\xi_1) - c\int_{t_1}^{t} r^{-\frac{1}{\gamma}}(s)ds \to -\infty \quad as \quad t \to \infty$$

This is contradiction and we conclude that $\tilde{z}'(\xi) < 0$ and the proof is complete. □

3. Main Results

For our further references, let as denote

$$Q(\xi) = \min\{\tilde{q}(\xi), \tilde{q}(\tilde{\tau}(\xi))\}, \quad Q^*(\xi) = Q(\xi)\left(\int_{\xi_1}^{\tilde{\sigma}(\xi)} \tilde{r}^{-\frac{1}{\gamma}}(s)ds\right)^{\beta}, \tag{10}$$

Theorem 3.1 : Assume that the hypotheses $(\mathbf{A}_1)$-$(\mathbf{A}_4)$ and $(\mathbf{A}_4^{'})$ hold and $0 < \beta \leq 1$. If the first order neutral differential inequality

$$\left(u(t) + \frac{p_0^{\beta}}{\tilde{\tau}_0}u(\tilde{\tau}(t))\right)' + Q^*(t)u^{\beta/\gamma}(\tilde{\sigma}(t)) \leq 0 \tag{11}$$

where $\tilde{\tau}(t), \tilde{\sigma}(t)$ are defined in Lemma 2.3 has no positive solution, then (1) is oscillatory.

Proof : Assume to the contrary that there exists a non-oscillatory solution x of equation (1). Without loss of generality, we only consider the case when $x(t)$ is eventually positive, since the case when $x(t)$ is eventually negative is similar. Then, let $x(t) > 0$ on $[t_1, \infty)$. It is equivalent to $\tilde{x}(\xi) > 0$ on $[\xi_1, \infty)$. Then from $(\mathbf{A}_2)$ and $(\mathbf{A}_4^{'})$ the corresponding function $\tilde{x}(\xi)$ satisfies

$$\tilde{z}(\tilde{\sigma}(\xi)) = \tilde{x}(\tilde{\sigma}(\xi)) + p(\tilde{\sigma}(\xi))\tilde{x}(\tilde{\tau}(\tilde{\sigma}(\xi))). \tag{12}$$

Furthermore, using Lemma 2.2 and (12), we obtain

$$\begin{aligned}\tilde{z}^{\beta}(\tilde{\sigma}(\xi)) &= \Big(\tilde{x}(\tilde{\sigma}(\xi)) + p(\tilde{\sigma}(\xi))\tilde{x}(\tilde{\tau}(\tilde{\sigma}(\xi)))\Big)^{\beta}\\ &\le \Big(\tilde{x}(\tilde{\sigma}(\xi)) + p_0\tilde{x}(\tilde{\tau}(\tilde{\sigma}(\xi)))\Big)^{\beta}\\ &\le \tilde{x}^{\beta}(\tilde{\sigma}(\xi)) + p_0^{\beta}\tilde{x}^{\beta}(\tilde{\sigma}(\tilde{\tau}(\xi))). \qquad (13)\end{aligned}$$

On the other hand, it follows from (9)

$$\Big(\tilde{r}(\xi)[\tilde{z}'(\xi)]^{\gamma}\Big)' + \tilde{q}(\xi)\tilde{x}^{\beta}(\tilde{\sigma}(\xi)) = 0, \qquad (14)$$

which in view of $(\mathbf{A}_2)$ and $(\mathbf{A}_4^{'})$ yields

$$\begin{aligned}0 &= \frac{p_o^{\beta}}{\tilde{\tau}'(\xi)}\Big(\tilde{r}(\tilde{\tau}(\xi))[(\tilde{z}'(\tilde{\tau}(\xi)))]^{\gamma}\Big)' + p_o^{\beta}\tilde{q}(\tilde{\tau}(\xi))\tilde{x}^{\beta}(\tilde{\sigma}(\tilde{\tau}(\xi)))\\ &\ge \frac{p_o^{\beta}}{\tilde{\tau}_0}\Big(\tilde{r}(\tilde{\tau}(\xi))[(\tilde{z}'(\tilde{\tau}(\xi))]^{\gamma}\Big)' + p_o^{\beta}\tilde{q}(\tilde{\tau}(\xi))\tilde{x}^{\beta}(\tilde{\sigma}(\tilde{\tau}(\xi))) \qquad (15)\end{aligned}$$

Then combining (14) and (15), we get

$$\begin{aligned}\Big(\tilde{r}(\xi)[\tilde{z}'(\xi)]^{\gamma}\Big)' &+ \frac{p_o^{\beta}}{\tilde{\tau}_0}\Big(\tilde{r}(\tilde{\tau}(\xi))[\tilde{z}'(\tilde{\tau}(\xi))]^{\gamma}\Big)'\\ &+ \tilde{q}(\xi)\tilde{x}^{\beta}(\tilde{\sigma}(\xi)) + p_o^{\beta}\tilde{q}(\tilde{\tau}(\xi))\tilde{x}^{\beta}(\tilde{\sigma}(\tilde{\tau}(\xi))) \le 0\\ \Big(\tilde{r}(\xi)[\tilde{z}'(\xi)]^{\gamma}\Big)' &+ \frac{p_o^{\beta}}{\tilde{\tau}_0}\Big(\tilde{r}(\tilde{\tau}(\xi))[\tilde{z}'(\tilde{\tau}(\xi))]^{\gamma}\Big)'\\ &+ Q(\xi)\Big[\tilde{x}^{\beta}(\tilde{\sigma}(\xi)) + p_0^{\beta}\tilde{x}^{\beta}(\tilde{\sigma}(\tilde{\tau}(\xi)))\Big] \le 0, \qquad (16)\end{aligned}$$

where $Q(\xi)$ is defined in (10). Using (16) and (13), we have

$$\Big(\tilde{r}(\xi)[\tilde{z}'(\xi)]^{\gamma} + \frac{p_o^{\beta}}{\tilde{\tau}_0}\tilde{r}(\tilde{\tau}(\xi))[\tilde{z}'(\tilde{\tau}(\xi))]^{\gamma}\Big)' + Q(\xi)\tilde{z}^{\beta}(\tilde{\sigma}(\xi)) \le 0. \qquad (17)$$

Now we denote $u(\xi) = \tilde{r}(t)(\tilde{z}'(\xi))^{\gamma}$. From Lemma 2.4 we get $u(\xi) > 0$ eventually. Also we have

$$\tilde{z}(\xi) \ge \int_{\xi_1}^{\xi} \frac{[\tilde{r}(s)(\tilde{z}'(s))^{\gamma}]^{1/\gamma}}{\tilde{r}^{1/\gamma}(s)}ds \ge u^{1/\gamma}(\xi)\int_{\xi_1}^{\xi}\tilde{r}^{-1/\gamma}(s)ds. \qquad (18)$$

Therefore, using (18) in (17), we see that u is a positive solution of

$$\left(u(\xi) + \frac{p_0^{\beta}}{\tilde{\tau}_0}u(\tilde{\tau}(\xi))\right)' + Q(\xi)\Big(u^{1/\gamma}(\xi)\int_{\xi_1}^{\xi}\tilde{r}^{-1/\gamma}(s)ds\Big)^{\beta} \le 0$$

$$\left(u(\xi)+\frac{p_0^{\beta}}{\tilde{\tau}_0}u(\tilde{\tau}(\xi))\right)'+Q^*(\xi)u^{\beta/\gamma}(\tilde{\sigma}(\xi))\leq 0, \tag{19}$$

which is contradiction of (11) and the proof is complete. □

Next, by using the conclusion of Theorem 3.1, we will deduce oscillatory problem of our equation into the problem of first-order nonlinear delay differential equations, and establish some new oscillatory criteria for equation (1). We shall discuss both cases when τ is a delayed or advanced argument.

Theorem 3.2 : Assume that $0<\beta\leq 1$, $\tau(t)\geq t$ and $\sigma(t)\leq t$ is increasing. If the first order delay differential equation

$$w'(\xi)+\frac{\tilde{\tau}_0^{\beta/\gamma}}{(\tilde{\tau}_0+p_0^{\beta})^{\beta/\gamma}}Q^*(\xi)w^{\beta/\gamma}(\tilde{\sigma}(\xi))=0 \tag{20}$$

is oscillatory, the equation (1) is oscillatory.

Proof : We assume that $x(t)$ is a positive solution of (1) eventually. Then it follows from the proof of Theorem 3.1 that $u(\xi)=\tilde{r}(t)(\tilde{z}'(\xi))^{\gamma}>0$ is decreasing eventually and it satisfies (11). We define

$$w(\xi)=u(\xi)+\frac{p_0^{\beta}}{\tilde{\tau}_0}u(\tilde{\tau}(\xi)). \tag{21}$$

Then

$$w(\xi)\quad\leq\quad u(\xi)\Big(1+\frac{p_0^{\beta}}{\tilde{\tau}_0}\Big),$$

Substituting this into (11), we see that $u(\xi)$ is a positive solution of the delay differential inequality

$$w'(\xi)+\frac{\tilde{\tau}_0^{\beta/\gamma}}{(\tilde{\tau}_0+p_0^{\beta})^{\beta/\gamma}}Q^*(\xi)w^{\beta/\gamma}(\tilde{\sigma}(\xi))\leq 0 \tag{22}$$

Then from Theorem 1 in [33], we know that the equation (20) also has a positive solution, which is a contradiction. The proof is complete. □

Theorem 3.3 : Assume that $0\leq\beta\leq 1$ and $\sigma(t)\leq\tau(t)\leq t$ is increasing. If the first order delay differential equation

$$w'(\xi)+\frac{\tilde{\tau}_0^{\beta/\gamma}}{(\tilde{\tau}_0+p_0^{\beta})^{\beta/\gamma}}Q^*(\xi)w^{\beta/\gamma}(\tilde{\tau}^{-1}(\tilde{\sigma}(\xi)))=0 \tag{23}$$

is oscillatory, the equation (1) is oscillatory.

Proof : We assume that $x(t)$ is a positive solution of (1) eventually. Then it follows from (21) that

$$\begin{aligned} w(\xi) &\leq u(\xi) + \frac{p_0^\beta}{\tilde{\tau}_0} u(\tilde{\tau}(\xi)) \\ w(\xi) &\leq u(\tilde{\tau}(\xi))\Big(1 + \frac{p_0^\beta}{\tilde{\tau}_0}\Big), \end{aligned}$$

or equivalently

$$w^{\beta/\gamma}(\tilde{\sigma}(\xi)) \geq \frac{\tilde{\tau}_0^{\beta/\gamma}}{(\tilde{\tau}_0 + p_0^\beta)^{\beta/\gamma}} w^{\beta/\gamma}(\tilde{\tau}^{-1}(\tilde{\sigma}(\xi))).$$

Using this in (11), we see that $u(\xi)$ is a positive solution of the delay differential inequality

$$w'(\xi) + \frac{\tilde{\tau}_0^{\beta/\gamma}}{(\tilde{\tau}_0 + p_0^\beta)^{\beta/\gamma}} Q^*(\xi) w^{\beta/\gamma}(\tilde{\tau}^{-1}(\tilde{\sigma}(\xi))) \leq 0.$$

Then from Theorem 1 in [33], we know that the equation (23) also has a positive solution, and a contradiction. The proof is complete. □

Next we will give some sufficient conditions such that equations (20) and (23) have only oscillatory solutions.

Lemma 3.4 : Let $\delta \in (0, 1]$ be a quotient of two positive integers. Assume that $e(\xi)$ is a positive continuous function on $(\xi_0, \infty]$. If

$$\liminf_{t\to\infty} \int_{\tilde{\sigma}(\xi)}^{\xi} e(s)ds > \frac{1}{e} \tag{24}$$

then the first-order delay differential equation

$$w'(\xi) + e(\xi) w^\delta(\tilde{\sigma}(\xi)) = 0 \tag{25}$$

is oscillatory.

Proof : From (24) we can get that

$$\int_{\xi_0}^{\infty} e(s)ds = +\infty. \tag{26}$$

Then assume to the contrary that there exists a positive solution $w(\xi)$ of equation (20) on $[\xi_1, \infty)$. Since $w(\xi)$ is decreasing, there exists

$$\lim_{\xi\to\infty} w(\xi) = k \geq 0.$$

If $k > 0$, then integrating (25) from ξ_1 to ξ, we have

$$w(\xi_1) \geq \int_{\xi_1}^{\xi} e(s)w^{\delta}(\tilde{\sigma}(s))ds \geq k^{\delta} \int_{\xi_1}^{\xi} e(s)ds \to +\infty \quad as \quad \xi \to +\infty \tag{27}$$

This is a contradiction. So we get that $\lim_{\xi \to +\infty} w(\xi) = 0$. And also $0 < w(\xi) < 1$, eventually. Therefore

$$w^{\delta}(\tilde{\sigma}(\xi)) \geq w(\tilde{\sigma}(\xi)).$$

Substituting this into (25), we deduce that $w(\xi)$ is a positive solution of the differential inequality

$$w'(\xi) + e(\xi)w(\tilde{\sigma}(\xi)) \leq 0, \tag{28}$$

But from the Theorem 2.1.1 in [32], the condition (24) yields that the equation (28) has no positive solution, which is a contradiction. The proof is complete. □

Applying Lemma 2.4 to (20) and (23), we obtain the following oscillation criteria of (1).

Corollary 3.5 : Let $0 < \beta \leq 1$, $\beta \leq \gamma$ and $\tau(t) \geq t$. If

$$\liminf_{\xi \to \infty} \int_{\tilde{\sigma}(\xi)}^{\xi} Q^*(s)ds > \frac{(\tilde{\tau}_0 + p_0^{\beta})^{\beta/\gamma}}{e\tilde{\tau}_0^{\beta/\gamma}}, \tag{29}$$

then (1) is oscillatory.

Corollary 3.6 : Let $0 < \beta \leq 1$, $\beta \leq \gamma$ and $\sigma(t) \leq \tau(t) \leq t$. If

$$\liminf_{\xi \to \infty} \int_{\tau^{-1}[\tilde{\sigma}(\xi)]}^{\xi} Q^*(s)ds > \frac{(\tilde{\tau}_0 + p_0^{\beta})^{\beta/\gamma}}{e\tilde{\tau}_0^{\beta/\gamma}}, \tag{30}$$

then (1) is oscillatory.

Theorem 3.7 : Assume that $\beta \geq 1$ and the assumptions $(A_1) - (A_5)$ holds. If the first-order delay neutral differential inequality

$$\left(u(t) + \frac{p_0^{\beta}}{\tilde{\tau}_0} u(\tilde{\tau}(t))\right)' + 2^{1-\beta} Q^*(t) u^{\beta/\gamma}(\tilde{\sigma}(t)) \leq 0 \tag{31}$$

has no positive solution, then (1) is oscillatory.

Proof : The result can be proved exactly as Theorem 3.1. We reply only the inequality (13) by

$$\begin{aligned} \tilde{z}^{\beta}(\tilde{\sigma}(\xi)) &= \Big(\tilde{x}(\tilde{\sigma}(\xi)) + p(\tilde{\sigma}(\xi))\tilde{x}(\tilde{\tau}(\tilde{\sigma}(\xi)))\Big)^{\beta} \\ &\leq \Big(\tilde{x}(\tilde{\sigma}(\xi)) + p_0\tilde{x}(\tilde{\tau}(\tilde{\sigma}(\xi)))\Big)^{\beta} \\ &\leq 2^{\beta-1}\Big[\tilde{x}^{\beta}(\tilde{\sigma}(\xi)) + p_0^{\beta}\tilde{x}^{\beta}(\tilde{\sigma}(\tilde{\tau}(\xi)))\Big], \end{aligned} \tag{32}$$

which follows from Lemma 2.1.

Theorem 3.8 : Assume that $\beta \geq 1$ and $\tau(t) \geq t$ is increasing. Then if the first-order delay differential equation

$$w'(\xi) + \frac{\tilde{\tau}_0^{\beta/\gamma}}{(\tilde{\tau}_0 + p_0^\beta)^{\beta/\gamma}} 2^{1-\beta} Q^*(\xi) w^{\beta/\gamma}(\tilde{\sigma}(\xi)) = 0 \tag{33}$$

is oscillatory, the equation (1) is oscillatory.

Proof : The Proof is similar to the proof of Theorem 3.2. □

Theorem 3.9 : Assume that $\beta \geq 1$ and $\sigma(t) \leq \tau(t) \leq t$ is increasing. Then if the first order neutral differential equation

$$w'(\xi) + \frac{\tilde{\tau}_0^{\beta/\gamma}}{(\tilde{\tau}_0 + p_0^\beta)^{\beta/\gamma}} 2^{1-\beta} Q^*(\xi) w^{\beta/\gamma}(\tilde{\tau}^{-1}(\tilde{\sigma}(\xi))) = 0 \tag{34}$$

is oscillatory, the equation (1) is oscillatory.

Proof : The Proof is similar to the proof of Theorem 3.3.

Combining Lemma 2.4 with Theorem 3.8 and Theorem 3.9, we archive the following oscillatory criteria for (1).

Corollary 3.10 : Let $\gamma \geq \beta \geq 1$ and $\tau(t) \geq t$. If

$$\liminf_{\xi \to \infty} \int_{\tilde{\sigma}(\xi)}^{\xi} Q^*(s) ds > 2^{\beta-1} \frac{(\tilde{\tau}_0 + p_0^\beta)^{\beta/\gamma}}{e \tilde{\tau}_0^{\beta/\gamma}}, \tag{35}$$

then (1) is oscillatory.

Corollary 3.11 : Let $\gamma \geq \beta \geq 1$ and $\sigma(t) \leq \tau(t) \leq t$. If

$$\liminf_{\xi \to \infty} \int_{\tau^{-1}[\tilde{\sigma}(\xi)]}^{\xi} Q^*(s) ds > 2^{\beta-1} \frac{(\tilde{\tau}_0 + p_0^\beta)^{\beta/\gamma}}{e \tilde{\tau}_0^{\beta/\gamma}}, \tag{36}$$

then (1) is oscillatory.

4. Example

Consider fractional order differential equation

$$D_t^{\frac{1}{2}} \left(t^{\frac{3}{2}} \left[D_t^{\frac{1}{2}} \left[x(t) + \frac{1}{t} x(t+2) \right) \right]^3 \right) + t x^{\frac{5}{3}}(t/2) = 0, \qquad t \geq 1, \tag{37}$$

where D_t^α is a modified Riemann-Liouville derivative. In (37), we set $r(t) = t^{\frac{3}{2}}$, $p(t) = \frac{1}{t}$, $q(t) = t$, $\tau(t) = t + 2$, $\sigma(t) = \frac{t}{2}$, $\alpha = \frac{1}{2}$, $\gamma = 3$, $\beta = \frac{5}{3}$. Then using a variable substitution we have

$$\xi = y(t) = \frac{t^{1/2}}{\Gamma(\frac{3}{2})}, \quad y^{-1}(\xi) = \Gamma^2\Big(\frac{3}{2}\Big)\xi^2, \quad \xi_1 = m = \frac{1}{\Gamma(\frac{3}{2})}.$$

and we also have

$$\begin{aligned} \tilde{r}(\xi) &= r(y^{-1}(\xi)) = \Gamma^3\Big(\frac{3}{2}\Big)\xi^3, \\ \tilde{\sigma}(\xi) &= y(\sigma(y^{-1}(\xi))) = \frac{\Big(\Gamma^2\big(\frac{3}{2}\big)\xi^2\Big)^{\frac{1}{2}}}{2^{1/2}\Gamma(\frac{3}{2})} = \frac{\xi}{2^{1/2}} \\ \tilde{q}(\xi) &= q(y^{-1}(\xi)) = \Gamma^2\Big(\frac{3}{2}\Big)\xi^2. \end{aligned}$$

Easily we see the equation (37) satisfies $(A_1) - (A_4)$, furthermore we have $0 \le p(t) = \frac{1}{t} \le 1 = p_0$, $\tau_0 = (t+2)' = 1$, $\lim_{t\to\infty} \frac{t}{\tau(t)} = \frac{t}{t+2} = l = 1$, $\tilde{\tau}_0 = \tau_0 l^{1-\frac{1}{2}} = 1$. Here $\tilde{q}(\xi)$ is increasing and $\tau(t) > t$, $\tilde{\tau}(\xi) > \xi$ and we apply Corollary 3.10 to (37),

$$\begin{aligned} Q(\xi) &= \tilde{q}(\xi) = \Gamma^2\Big(\frac{3}{2}\Big)\xi^2, \\ \int_{\xi_1}^{\tilde{\sigma(\xi)}} r^{-1/\gamma}(s)\,ds &= \int_{\xi_1}^{2^{-1/2}\xi} \frac{1}{\Gamma(\frac{3}{2})s}\,ds = \frac{1}{\Gamma(\frac{3}{2})}[\ln 2^{-1/2}\xi - \ln\, m], \\ Q^*(\xi) &= Q(\xi)\left(\int_{\xi_1}^{\tilde{\sigma}(\xi)} \tilde{r}^{-\frac{1}{\gamma}}(s)\,ds\right)^{\beta} = \Gamma^{1/3}\Big(\frac{3}{2}\Big)\xi\Big([\ln\, 2^{-1/2}\xi - \ln\, m]\Big)^{5/3}. \end{aligned}$$

Then we get,

$$\begin{aligned} \liminf_{\xi\to\infty} \int_{\tilde{\sigma}(\xi)}^{\xi} Q^*(s)ds &= \liminf_{\xi\to\infty} \int_{2^{-1/2}\xi}^{\xi} \Gamma^{1/3}\Big(\frac{3}{2}\Big)s^2\Big([\ln\, 2^{-1/2}s - \ln\, m]\Big)^{5/3} ds \\ &= \infty > 2^{\beta-1}\frac{(\tilde{\tau}_0 + p_0^{\beta})^{\beta/\gamma}}{e\,\tilde{\tau}_0^{\beta/\gamma}} = \frac{2^{11/9}}{e}, \end{aligned}$$

which guaranties the oscillation of (37).

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