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FIXED POINT THEOREMS IN NORMAL CONE METRIC SPACE

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Abstract

In this paper, we proved some fixed point theorems in complete normal cone metric spaces, which are the generalization of some existing results in the literature.

1. Introduction

There exist a number of generalizations of metric spaces, and one of them is the cone metric spaces. The notion of cone metric space is initiated by Huang and Zhang [2] and also they discussed some properties of the convergence of sequences and proved the fixed point theorems of a contraction mappings cone metric spaces.

Many authors have studied the existence and uniqueness of strict fixed points for single valued mappings and multivalued mappings in metric spaces [1, 5, 6, 10]. In this paper discuss existence and unique fixed point in complete normal cone metric spaces, which are the generalization of some existing Contraction principle.

Key Words : *Normal cone, Cone metric space, Fixed point.*

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Definition 1.1 : A subset P of E is called a cone if and only if:

1. P is closed, nonempty and $P \neq 0$
2. $ax + by \in P$ for all $x, y \in P$ and nonnegative real numbers a, b
3. $P \cap P^{-} = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We will write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while x, y will stand for $y - x \in intP$, where $intP$ denotes the interior of P. The cone P is called normal if there is a number $K > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$ for all $x, y \in E$. The least positive number satisfying the above is called the normal constant. The cone P is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}$ is sequence such that $x_1 \leq x_2 \leq \cdots \leq x_n \cdots \leq y$ for some $y \in E$, then there is $x \in E$ such that $\|x_n - x\| \to 0$ as $n \to 0$. Equivalently the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone. Suppose E is a Banach space, P is a cone in E with $intP \neq 0$ and $\leq$ is partial ordering with respect to P.

Example 1 : Let $K > 1$ be given. Consider the real vector space with

$$E = \{ax + b : a, b \in \mathbb{R}; x \in [1 - \frac{1}{k}, 1]\}$$

with supremum norm and the cone

$$P = \{ax + b : a \geq 0, b \leq 0\}$$

in E. The cone P is regular and so normal.

Definition 1.2 : Suppose that E is a real Banach space, then P is a cone in E with $intP \neq \emptyset$, and $\leq$ is partial ordering with respect to P. Let X be a nonempty set, a function $d : X \times X \to E$ is called a cone metric on X if it satisfies the following conditions with

1. $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$ $\forall x, y \in X$,
2. $d(x, y) = d(y, x)$, $\forall x, y \in X$,

3. $d(x,y) \leq d(x,z) + d(z,y)$, $\forall x,y,z \in X$,

Then (X,d) is called a cone metric space *(CMS)*.

Example 2 : Let $E = \mathbb{R}^2$

$$P = \{(x,y) : x, y \geq 0\}$$

$X = \mathbb{R}$ and $d : X \times X \to E$ such that

$$d(x,y) = (|x-y|, \alpha|x-y|)$$

where $\alpha \geq 0$ is a constant. Then (X,d) is a cone metric space.

Definition 1.3 : Let (X,d) be a CMS and $\{x_n\}_{n\geq 0}$ be a sequence in X. Then $\{x_n\}_{n\geq 0}$ converges to x in X whenever for every $c \in E$ with $0 \ll c$, there is a natural number $N \in N$ such that $d(x_n, x) \ll c$ for all $n \geq N$. It is denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.

Definition 1.4 : Let (X,d) be a CMS and $\{x_n\}_{n\geq 0}$ be a sequence in X. $\{x_n\}_{n\geq 0}$ is a Cauchy sequence whenever for every $c \in E$ with $0 \ll c$, there is a natural number $N \in \mathbb{N}$, such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.

Lemma 1.5 : Let (X,d) be a cone metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to x and $\{x_n\}$ converges to y, then $x = y$. That is the limit of $\{x_n\}$ is unique.

Definition 1.6 : Let (X,d) be a cone metric space, if every Cauchy sequence is convergent in X, then X is called a complete cone metric space.

Lemma 1.7 : Let (X,d) be a cone metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0$ $(n, m \to \infty)$.

2. Main Result

Theorem 2.1 : Let (X,d) be a complete cone metric space and P be a normal cone with normal constant K. Suppose the mapping $T : X \to X$ satisfies the following conditions:

$$d(Tx,Ty) \leq \Big(\frac{d(x,Tx) + d(y,Ty)}{d(x,Tx) + d(y,Ty) + l}\Big) d(x,y) \tag{1}$$

for all $x, y \in X$, where $l \geq 1$. Then

(i) T has unique fixed point in X.

(ii) $T^n x'$ converges to a fixed point, for all $x' \in X$.

Proof : (i) Let $x_0 \in X$ be arbitrary and choose a sequence $\{x_n\}$ such that $x_{n+1}{=}Tx_n$.

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\ &\leq \Big(\frac{d(x_n, Tx_n) + d(x_{n-1}, Tx_{n-1})}{d(x_n, Tx_n) + d(x_{n-1}, Tx_{n-1}) + l}\Big) d(x_n, x_{n-1}) \\ &\leq \Big(\frac{d(x_n, x_{n+1}) + d(x_{n-1}, x_n)}{d(x_n, x_{n+1}) + d(x_{n-1}, x_n) + l}\Big) d(x_n, x_{n-1}) \end{aligned}$$

Take

$$\lambda_n = \frac{d(x_n, x_{n+1}) + d(x_{n-1}, x_n)}{d(x_n, x_{n+1}) + d(x_{n-1}, x_n) + l},$$

we have

$$\begin{aligned} d(x_{n+1}, x_n) &\leq \lambda_n d(x_n, x_{n-1}) \\ &\leq (\lambda_n \lambda_{n-1}) d(x_{n-1}, x_{n-2}) \\ &\vdots \\ &\leq (\lambda_n \lambda_{n-1} \cdots \lambda_1) d(x_1, x_0). \end{aligned}$$

Observe that (λ_n) is non increasing, with positive terms. So, $\lambda_1 ... \lambda_n \leq \lambda_1^n$ and $\lambda_1^n \to 0$. It follows that

$$\lim_{n\to\infty} (\lambda_1 \lambda_2 \cdots \lambda_n) = 0.$$

Thus, it is verified that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$$

Now for all $m, n \in \mathbb{N}$ and $m > n$ we have

$$\begin{aligned} d(x_m, x_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\leq [(\lambda_n\lambda_{n-1}\cdots\lambda_1) + (\lambda_{n+1}\lambda_n\cdots\lambda_1) + \cdots + (\lambda_{m-1}\lambda_{m-2}\cdots\lambda_1)]d(x_1, x_0) \\ &= \sum_{k=n}^{m-1}(\lambda_k\lambda_{k-1}\cdots\lambda_1)d(x_1, x_0) \\ \|d(x_m, x_n)\| &\leq K\|\sum_{k=n}^{m-1}(\lambda_k\lambda_{k-1}\cdots\lambda_1)d(x_1, x_0)\| \\ \|d(x_m, x_n)\| &\leq K\sum_{k=n}^{m-1}(\lambda_k\lambda_{k-1}\cdots\lambda_1)\|d(x_1, x_0)\| \\ \|d(x_m, x_n)\| &\leq K\sum_{k=n}^{m-1} a_k\|d(x_1, x_0)\|, \end{aligned}$$

where $a_k = (\lambda_k\lambda_{k-1}\cdots\lambda_1)$ and K is normal constant of P.
Now $\lim\limits_{k\to\infty} \frac{a_{k+1}}{a_k} < 1$ and $\sum\limits_{k=1}^{\infty} a_k$ is finite , and $\sum\limits_{k=n}^{m-1}(\lambda_k\lambda_{k-1}\cdots\lambda_1) \to 0,$ as $m, n \to \infty$.
Hence $\{a_k\}$ is convergent by D'Alembert's ratio test, Therefore $\{x_n\}$ is a Cauchy sequence. There is $x' \in X$ such that $x_n \to x'$ as $n \to \infty$.

$$\begin{aligned} d(Tx', x') &\leq d(Tx', Tx_n) + d(Tx_n, x') \\ &\leq \Big(\frac{d(x', Tx') + d(x_n, Tx_n)}{d(x', Tx') + d(x_n, Tx_n) + l}\Big)d(x_n, x') + d(Tx_n, x') \\ &\leq \Big(\frac{d(x', Tx') + d(x_n, x_{n+1})}{d(x', Tx') + d(x_n, x_{n+1}) + l}\Big)d(x_n, x') + d(x_{n+1}, x') \\ d(Tx', x') &\leq 0 \quad \text{as} \quad n \to \infty \end{aligned}$$

Therefore $\|d(x', Tx')\| = 0$. Thus, $Tx' = x'$.

Uniqueness

Suppose x' and y' are two fixed points of T.

$$\begin{aligned} d(x', y') &= d(Tx', Ty') \\ &\leq \Big(\frac{d(x', Tx') + d(y', Ty')}{d(x', Tx') + d(y', Ty') + l}\Big)d(x', y') \\ &\leq 0 \end{aligned}$$

Therefore $\|d(x', y')\| = 0$. Thus $x' = y'$.

Hence x' is an unique fixed point of T.

(ii) Now

$$d(T^n x', x') = d(T^{n-1}(Tx'), x') = d(T^{n-1}x', x') = d(T^{n-2}(Tx'), x') \cdots = d(Tx', x') = 0$$

Hence $T^n x'$ converges to a fixed point, for all $x' \in X$. □

Corollary 2.2 : Let (X, d) be a complete cone metric space and P be a normal cone with normal constant K. Suppose the mapping $T : X \to X$ satisfies the following conditions:

$$d(Tx, Ty) \leq \Big(\frac{d(x, Tx) + d(y, Ty)}{d(x, Tx) + d(y, Ty) + 1}\Big) d(x, y) \tag{2}$$

for all $x, y \in X$. Then

(i) T has unique fixed point in X.

(ii) $T^n x'$ converges to a fixed point, for all $x' \in X$.

Proof : The proof of the corollary immediate by taking $l = 1$ in the above theorem. □

Theorem 2.3 : Let (X, d) be a complete metric space and let T be a mapping from X into itself. Suppose that T satisfies the following condition:

$$d(Tx, Ty) \leq \Big(\frac{d(y, Ty)}{d(x, Tx) + d(y, Ty) + l}\Big) d(x, y) \tag{3}$$

for all $x, y \in X$, where $l \geq 1$. Then

(i) T has unique fixed point in X.

(ii) $T^n x'$ converges to a fixed point, for all $x' \in X$.

Proof : (i) Let $x_0 \in X$ be arbitrary and choose a sequence $\{x_n\}$ such that x_{n+1}=Tx_n. We have

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\ &\leq \Big(\frac{d(x_{n-1}, Tx_{n-1})}{d(x_n, x_{n+1}) + d(x_{n-1}, x_n) + l}\Big) d(x_n, x_{n-1}) \\ &\leq \Big(\frac{d(x_{n-1}, x_n)}{d(x_n, x_{n+1}) + d(x_{n-1}, x_n) + l}\Big) d(x_n, x_{n-1}) \\ &\leq \Big(\frac{d(x_{n-1}, x_n)}{d(x_n, x_{n+1}) + d(x_{n-1}, x_n) + l}\Big) d(x_n, x_{n-1}). \end{aligned}$$

Take

$$\lambda_n = \frac{d(x_{n-1}, x_n)}{d(x_n, x_{n+1}) + d(x_{n-1}, x_n) + l},$$

we have

$$\begin{aligned} d(x_{n+1}, x_n) &\leq \lambda_n d(x_n, x_{n-1}) \\ &\leq (\lambda_n \lambda_{n-1}) d(x_{n-1}, x_{n-2}) \\ &\vdots \\ &\leq (\lambda_n \lambda_{n-1} \cdots \lambda_1) d(x_1, x_0). \end{aligned}$$

Observe that $\{\lambda_n\}$ is non increasing, with positive terms.
So, $\lambda_1 ... \lambda_n \leq \lambda_1^n$ and $\lambda_1^n \to 0$. It follows that

$$\lim_{n \to \infty} (\lambda_1 \lambda_2 \cdots \lambda_n) = 0.$$

Thus, it is verified that

$$\lim_{n \to \infty} d(x_{n+1}, x_n) = 0.$$

Now for all $m, n \in \mathbb{N}$ we have

$$\begin{aligned} d(x_m, x_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\leq [(\lambda_n \lambda_{n-1} \cdots \lambda_1) + (\lambda_{n+1} \lambda_n \cdots \lambda_1) + \cdots + (\lambda_{m-1} \lambda_{m-2} \cdots \lambda_1)] d(x_1, x_0) \\ &= \sum_{k=n}^{m-1} (\lambda_k \lambda_{k-1} \cdots \lambda_1) d(x_1, x_0) \\ \|d(x_m, x_n)\| &\leq K \| \sum_{k=n}^{m-1} (\lambda_k \lambda_{k-1} \cdots \lambda_1) d(x_1, x_0) \| \\ \|d(x_m, x_n)\| &\leq K \sum_{k=n}^{m-1} a_k \|d(x_1, x_0)\| \end{aligned}$$

where $a_k = \lambda_k \lambda_{k-1} \cdots \lambda_1$ and K is normal constant of P.
Now $\lim_{k \to \infty} \frac{a_{k+1}}{a_k} < 1$ and $\sum_{k=1}^{\infty} a_k$ is finite , and $\sum_{k=n}^{m-1} (\lambda_k \lambda_{k-1} \cdots \lambda_1) \to 0,$ as $m, \to \infty$.
Hence $\{a_k\}$ is convergent by D'Alembert's ratio test, Therefore $\{x_n\}$ is a Cauchy se-

quence. There is $x' \in X$ such that $x_n \to x'$

$$\begin{aligned} d(Tx', x') &\leq d(Tx', Tx_n) + d(Tx_n, x') \\ &\leq \Big(\frac{d(x_n, Tx_n)}{d(x', Tx') + d(x_n, Tx_n) + l}\Big) d(x_n, x') + d(Tx_n, x') \\ &\leq \Big(\frac{d(x_n, x_{n+1})}{d(x', Tx') + d(x_n, x_{n+1}) + l}\Big) d(x_n, x') + d(x_{n+1}, x') \\ d(Tx', x') &\leq 0 \quad \text{as} \quad n \to \infty \end{aligned}$$

Therefore $\|d(x', Tx')\| = 0$. Thus, $Tx' = x'$.

Uniqueness

Suppose x' and y' are two fixed points of T.

$$\begin{aligned} d(x', y') &= d(Tx', Ty') \\ &\leq \Big(\frac{d(y', Ty')}{d(x', Tx') + d(y', Ty') + l}\Big) d(x', y') \\ &\leq 0 \end{aligned}$$

Therefore $\|d(x', y')\| = 0$. Thus $x' = y'$.

Hence x' is an unique fixed point of T.

(ii) Now

$$d(T^n x', x') = d(T^{n-1}(Tx'), x') = d(T^{n-1}x', x') = d(T^{n-2}(Tx'), x') \cdots = d(Tx', x') = 0$$

Hence $T^n x'$ converges to a fixed point, for all $x' \in X$. □

Corolary 2.4 : Let (X, d) be a complete metric space and let T be a mapping from X into itself. Suppose that T satisfies the following condition:

$$d(Tx, Ty) \leq \Big(\frac{d(y, Ty)}{d(x, Tx) + d(y, Ty) + 1}\Big) d(x, y) \tag{4}$$

for all $x, y \in X$. Then

(i) T has unique fixed point in X.

(ii) $T^n x'$ converges to a fixed point, for all $x' \in X$.

Proof : The proof of the corollary immediate by taking $l = 1$ in the above theorem. □

Theorem 2.5 : Let (X, d) be a complete cone metric space and P be a normal cone with normal constant K. Suppose the mapping $T : X \to X$ satisfies the following conditions:

$$d(Tx,Ty) \leq \Big(\frac{d(x,Ty)+d(y,Tx)}{d(x,Tx)+d(y,Ty)+l}\Big)(d(x,Tx)+d(y,Ty)) \tag{5}$$

for all $x, y \in X$, where $l \geq 1$. Then

(i) T has unique fixed point in X.

(ii) $T^n x'$ converges to a fixed point, for all $x' \in X$.

Proof :(i) Let $x_0 \in X$ be arbitrary and choose a sequence $\{x_n\}$ such that x_{n+1}=Tx_n.

$$\begin{aligned} d(x_n,x_{n+1}) &= d(Tx_n,Tx_{n-1}) \\ &\leq \Big(\frac{d(x_n,Tx_{n-1})+d(x_{n-1},Tx_n)}{d(x_n,Tx_n)+d(x_{n-1},Tx_{n-1})+l}\Big)(d(x_n,Tx_n)+d(x_{n-1},Tx_{n-1})) \\ &\leq \Big(\frac{d(x_n,x_n)+d(x_{n-1},x_{n+1})}{d(x_n,x_{n+1})+d(x_{n-1},x_n)+l}\Big)(d(x_n,x_{n+1})+d(x_n,x_{n-1})) \\ &\leq \Big(\frac{d(x_{n-1},x_{n+1})}{d(x_n,x_{n+1})+d(x_{n-1},x_n)+l})(d(x_n,x_{n+1})+d(x_n,x_{n-1})) \\ &\leq \Big(\frac{d(x_{n-1},x_n)+d(x_n,x_{n+1})}{d(x_n,x_{n+1})+d(x_{n-1},x_n)+l}\Big)(d(x_n,x_{n+1})+d(x_n,x_{n-1})) \end{aligned}$$

Take

$$\lambda_n = \frac{d(x_{n-1},x_n)+d(x_n,x_{n+1})}{d(x_n,x_{n+1})+d(x_{n-1},x_n)+l},$$

we have

$$\begin{aligned} d(x_{n+1},x_n) &\leq \lambda_n(d(x_n,x_{n+1})+d(x_n,x_{n-1})) \\ (1-\lambda_n)d(x_{n+1},x_n) &\leq \lambda_n d(x_n,x_{n-1}) \\ d(x_{n+1},x_n) &\leq \frac{\lambda_n}{(1-\lambda_n)}d(x_n,x_{n-1}) \\ &\leq \frac{\lambda_n\lambda_{n-1}}{(1-\lambda_n)(1-\lambda_{n-1})}d(x_{n-1},x_{n-2}) \\ &\vdots \\ &\leq \frac{\lambda_n\lambda_{n-1}\cdots\lambda_1}{(1-\lambda_n)(1-\lambda_{n-1})\cdots(1-\lambda_1)}d(x_1,x_0). \\ &\leq \gamma_n d(x_1,x_0) \end{aligned}$$

where

$$\gamma_n = \frac{\lambda_n\lambda_{n-1}\cdots\lambda_1}{(1-\lambda_n)(1-\lambda_{n-1})\cdots(1-\lambda_1)}$$

Observe that $\{\lambda_n\}$ is non increasing, with positive terms. So, $\lambda_1 ... \lambda_n \leq \lambda_1^n$ and $\lambda_1^n \to 0$. It follows that

$$\lim_{n\to\infty}(\lambda_1\lambda_2\cdots\lambda_n) = 0.$$

Therefore

$$\lim_{n\to\infty}\gamma_n = 0$$

Thus, it is verified that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$$

Now for all $m, n \in \mathbb{N}$ we have

$$\begin{aligned} d(x_m, x_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\leq [\gamma_n + \gamma_{n+1} + \cdots + \gamma_{m-1}]d(x_1, x_0) \\ &\leq \sum_{k=n}^{m-1}\gamma_k d(x_1, x_0) \\ \|d(x_m, x_n)\| &\leq K\|\sum_{k=n}^{m-1}\gamma_k d(x_1, x_0)\| \\ \|d(x_m, x_n)\| &\leq K\sum_{k=n}^{m-1}\gamma_k\|d(x_1, x_0)\|, \end{aligned}$$

where $a_k = \gamma_k$ and K is normal constant of P.

Now $\lim\limits_{k\to\infty}\frac{a_{k+1}}{a_k} < 0$ and $\sum\limits_{k=1}^{\infty} a_k$ is finite. Since $\sum\limits_{k=n}^{m-1}\gamma_k$ is convergent by D'Alembert's ratio test, as $m \to \infty$.

Therefore $\{x_n\}$ is a Cauchy sequence. There is $x' \in X$ such that $x_n \to x'$ as $n \to \infty$.

$$\begin{aligned} d(Tx', x') &\leq d(Tx', Tx_n) + d(Tx_n, x') \\ &\leq \Big(\frac{d(x', Tx_n) + d(x_n, Tx')}{d(x', Tx_n) + d(x_n, Tx') + l}\Big)(d(x_n, x') + d(Tx_n, x')) \\ &\leq \Big(\frac{d(x', x_{n+1}) + d(x_n, Tx')}{d(x', x_{n+1}) + d(x_n, Tx') + l}\Big)(d(x_n, x') + d(x_{n+1}, x')) \\ d(Tx', x') &\leq 0 \quad \text{as} \quad n \to \infty \end{aligned}$$

Therefore $\|d(x', Tx')\| = 0$. Thus, $Tx' = x'$.

Uniqueness

Suppose x' and y' are two fixed points of T.

$$\begin{aligned} d(x',y') &= d(Tx',Ty') \\ &\leq \Big(\frac{d(x',Ty')+d(y',Tx')}{d(x',Tx')+d(y',Ty')+l}\Big)(d(x',Tx')+d(y',Ty')) \\ &\leq 0 \end{aligned}$$

Therefore $\|d(x',y')\| = 0$. Thus $x' = y'$.

Hence x' is an unique fixed point of T.

(ii) Now

$$d(T^n x',x') = d(T^{n-1}(Tx'),x') = d(T^{n-1}x',x') = d(T^{n-2}(Tx'),x') \cdots = d(Tx',x') = 0$$

Hence $T^n x'$ converges to a fixed point, for all $x' \in X$. □

Corollary 2.6 : Let (X,d) be a complete cone metric space and P be a normal cone with normal constant K. Suppose the mapping $T : X \to X$ satisfies the following conditions:

$$d(Tx,Ty) \leq \Big(\frac{d(x,Ty)+d(y,Tx)}{d(x,Tx)+d(y,Ty)+1}\Big)(d(x,Tx)+d(y,Ty)) \tag{6}$$

for all $x,y \in X$. Then

(i) T has unique fixed point in X.

(ii) $T^n x'$ converges to a fixed point, for all $x' \in X$.

Proof : The proof of the corollary immediate by taking $l = 1$ in the above theorem. □

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