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MAPPINGS ON STRONGLY MAGIC SQUARES

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Abstract

A magic square is a square array of numbers where the rows, columns, diagonals and co-diagonals add up to the same number. The paper discuss about a well-known class of magic square; the strongly magic square. In this paper group homomorphisms and isomorphisms on strongly magic squares are discussed.

1. Introduction

The magic square is said to have been discovered in the third millennium B. C. by the Chinese Emperor Yu. According to tradition, the Emperor, while walking on the river bank, found a turtle with an odd diagram on its shell. The Emperor saw in the unusual pattern a numerical sequence. He called this pattern the "Lo Shu." His discovery was a magic square of the third order. Later, magic squares appeared in India, and then were

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known to the Arabs, who introduced them to the West. More research on the topic was done during the Renaissance by the mathematician Cornelius Agrippa (1486=1535) who constructed magic squares of orders 3 through 9 to represent various planets, the sun, and the moon. Another famous example of a magic square appeared in Albrecht Durer's engraving "Melancholia", or the Genius of the Industrial Science of Mathematics [6]. A normal magic square is a square array of consecutive numbers from where the rows, columns, diagonals and co-diagonals add up to the same number [1]. The constant sum is called magic constant or magic number. Along with the conditions of normal magic squares, strongly magic square will have a stronger property that the sum of the entries of the sub-squares taken without any gaps between the rows or columns is also the magic constant [2]. There are many recreational aspects of strongly magic squares. But, apart from the usual recreational aspects, it is found that these strongly magic squares possess advanced mathematical properties.

2. Notations and Mathematical Preliminaries

(A) Magic Square

A magic square of order n is an n^{th} order matrix $[a_{ij}]$ such that

$$\sum_{j=1}^{n} a_{ij} = \rho \quad \text{for} \quad i = 1, 2, \cdots, n \tag{1}$$

$$\sum_{j=1}^{n} a_{ji} = \rho \quad \text{for} \quad i = 1, 2, \cdots, n \tag{2}$$

$$\sum_{i=1}^{n} a_{ii} = \rho, \quad \sum_{i=1}^{n} a_{i,n-i+1} = \rho \tag{3}$$

Equation (1) represents the row sum, equation (2) represents the column sum, equation (3) represents the diagonal and co-diagonal sum and symbol ρ represents the magic constant. [3]

(B) Magic Constant

The constant ρ in the above definition is known as the magic constant or magic number. The magic constant of the magic square A is denoted as $\rho(A)$.

(C) Strongly magic square (SMS): Generic Definition

Let $[a_{ij}]$ be a matrix of order $n^2 \times n^2$, such that

$$\sum_{j=1}^{n^2} a_{ij} = \rho \quad \text{for} \;\; i = 1, 2, \cdots, n^2 \tag{4}$$

$$\sum_{j=1}^{n^2} a_{ji} = \rho \quad \text{for} \;\; i = 1, 2, \cdots, n^2 \tag{5}$$

$$\sum_{i=1}^{n^2} a_{ii} = \rho, \qquad \sum_{i=1}^{n^2} a_{i,n^2-i+1} = \rho \tag{6}$$

$$\sum_{i=0}^{n-1} \sum_{k=0}^{n-1} a_{i+k,j+l} = \rho \quad \text{for} \quad i, j = 1, 2, \cdots, n^2 \tag{7}$$

where the subscripts are congruent modulo n^2.

Equation (4) represents the row sum, equation (5) represents the column sum, equation (6) represents the diagonal and co-diagonal sum, equation (7) represents the $n \times n$ sub-square sum with no gaps in between the elements of rows or columns and is denoted as $M_{OC}^{(n)}$ or $M_{OR}^{(n)}$ and ρ is the magic constant.

(D) Group homomorphism

A mapping ϕ from a group $\langle G, * \rangle$ into a group $\langle G', *' \rangle$ is a homomorphism of G into G' if $\phi(a * b) = \phi(a) *' \phi(b)$ for all $a, b \in G$[4].

(E) Group Isomorphism

A one to one onto homomorphism ϕ from a group $\langle G, * \rangle$ into a group $\langle G', *' \rangle$ is defined as isomorphism [4].

(F) A One to One and Onto Mapping

A function $\phi : X \to Y$ is one to one if $\phi(x_1) = \phi(x_2)$ only when $x_1 = x_2$.

The function ϕ is onto of Y if the range of ϕ is Y. [4]

(G) Other Notations

1. SM_s denote the set of all strongly magic squares of order $n^2 \times n^2$.

2. $SM_{S(a)}$ denote the set of all strongly magic squares of the form $[a_{ij}]_{n^2 \times n^2}$ such that $a_{ij} = a$ for every $i, j = 1, 2, \cdots, n^2$. Here A is denoted as $[a]$, i.e. If $A \in SM_{S(a)}$ then $\rho(A) = n^2 a$.

3. $SM_{S(0)}$ denote the set of all strongly magic squares of order $n^2 \times n^2$ with magic constant 0, i.e. If $A \in SM_{S(0)}$, then $\rho(A) = 0$.

3. Propositions and Theorems

Proposition 1 : If A and B are two Strongly magic squares of order $n^2 \times n^2$ with $\rho(A) = a$ and $\rho(B) = b$, then $C = (\lambda + \mu)(A + B)$ is also a Strongly magic square with magic constant $(\lambda + \mu)(\rho(A) + \rho(B))$, for every $\lambda, \mu \in R$.

Proof : Let $A = [a_{ij}]_{n^2 \times n^2}$ and $B = [b_{ij}]_{n^2 \times n^2}$. Then

$$\begin{aligned} C &= (\lambda + \mu)(A + B) \\ &= (\lambda + \mu)[a_{ij} + b_{ij}] \\ &= [(\lambda + \mu)(a_{ij} + b_{ij})]. \end{aligned}$$

Sum of the i^{th} row elements of

$$\begin{aligned} C &= \sum_{j=1}^{n^2} c_{ij} \\ &= \sum_{j=1}^{n^2} ((\lambda + \mu)(a_{ij} + b_{ij})) \\ &= (\lambda + \mu) \left(\sum_{j=1}^{n^2} (a_{ij}) + \sum_{j=1}^{n^2} (b_{ij}) \right) \\ &= (\lambda + \mu)(a + b) \\ &= (\lambda + \mu)(\rho(A) + \rho(B)). \end{aligned}$$

A similar computation holds for column sum.

Main diagonal sum

$$\begin{aligned} \sum_{i=1}^{n^2} c_{ii} &= \sum_{i=1}^{n^2} ((\lambda + \mu)(a_{ii} + b_{ii})) \\ &= (\lambda + \mu) \left(\sum_{i=1}^{n^2} (a_{ii}) + \sum_{i=1}^{n^2} (b_{ii}) \right) \\ &= (\lambda + \mu)(a + b) \\ &= (\lambda + \mu)(\rho(A) + \rho(B)). \end{aligned}$$

A similar computation holds for co-diagonal sum.

The sum of the $n \times n$ sub squares $M_{kC}^{(n)}/M_{kR}^{(n)}$ is given by

$$\begin{aligned}\sum_{l=0}^{n-1}\sum_{k=0}^{n-1} C_{i+k,j+1} &= \sum_{l=0}^{n-1}\sum_{k=0}^{n-1}(\lambda+\mu)(a_{i+k,j+l}+b_{i+k,j+l}) \\ &= (\lambda+\mu)\left(\sum_{l=0}^{n-1}\sum_{k=0}^{n-1}(a_{i+k,j+l})+\sum_{l=0}^{n-1}\sum_{k=0}^{n-1}(b_{i+k,j+l})\right) \\ &= (\lambda+\mu)(a+b) \\ &= (\lambda+\mu)(\rho(A)+\rho(B)).\end{aligned}$$

From the above propositions the following results can be obtained.

Results :

If for every $\lambda, \mu \in R$ and $A, B \in SM_s$,

(1.1) $\lambda(A+B) \in SM_s$ with $\rho(\lambda(A+B)) = \lambda(\rho(A)+\rho(B))$.

Proof : By putting $\mu = 0$ in Proposition 1, result can be deduced.

(1.2) $(A+B) \in SM_s$ with $\rho(A+B) = \rho(A)+\rho(B)$.

Proof ; By putting $\lambda = 1$ in Result 1.1 this can be obtained.

Proposition 2 : The mapping $\phi : S_{M_s} \to R$ defined by $\phi(A) = \rho(A)$, $\forall\ A \in S_{M_s}$ is a group homomorphism.

Proof : Let $A, B \in S_{M_s}$, then

$$\begin{aligned}\phi(A+B) = \rho(A+B) &= \rho(A)+\rho(B) \quad \text{(By Result 1.2)} \\ &= \phi(A)+\phi(B).\end{aligned}$$

Proposition 3 : The mapping $\phi : SM_{S(a)} \to R$ defined by $\phi(A) = \rho(A)$, $\forall\ A \in SM_{S(a)}$ is a group homomorphism.

Proof : It can be easily verified since $SM_{S(a)} \subset S_{M_s}$.

Proposition 4 : The mapping $\phi : SM_{S(0)} \to R$ defined by $\phi(A) = \rho(A)$, $\forall\ A \in SM_{S(0)}$ is a group homomorphism.

Proof : It can be easily verified since $SM_{S(0)} \subset S_{M_s}$.

Theorem 5 : The mapping $\phi : SM_{S(a)} \to R$ defined by $\phi(A) = \rho(A)$, $\forall\ A \in SM_{S(a)}$ is a group isomorphism.

Proof : Let $A, B \in SM_{S(a)}$; $A = [a], B = [b]$ then $\rho(A) = n^2 a$ and $\rho(B) = n^2 b$.

(i) To show that ϕ is one to one

$$\begin{aligned} &\phi(A) = \phi(B) \\ &\Rightarrow \rho(A) = \rho(B) \\ &\Rightarrow n^2 a = n^2 b \\ &\Rightarrow a = b. \end{aligned}$$

(ii) To show that ϕ is onto

For every $a \in R$, there exists $A = \left[\frac{a}{n^2}\right] \in SM_{S(a)}$ such that $\rho(A) = a$.

Since ϕ is $1-1$ and onto and from Proposition 3, it can be deduced.

4. Conclusion

While magic squares are recreational in grade school, they may be treated somewhat more seriously in different linear algebra courses. The study of strongly magic squares is an emerging innovative area in which mathematical analysis can be done. Here some advanced properties regarding strongly magic squares are described.

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