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# RATIONAL APPROXIMATION OF GREGORY SERIES

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**Abstract**

In this paper we give a rational approximation to Gregory series, by applying a correction function to the series. The introduction of correction function certainly improves the value of sum of the series and gives a better approximation to it. We also show that the correction function follows an infinite continued fraction.

## 1. Introduction

Commenting on the Lilavati rule for finding the value of circumference of a circle from its diameter, the commentator Sankara refers to several important enunciations from the works of earlier and contemporary mathematicians and gives a detailed exposition of various results contained in them. Sankara also refers to various infinite series for computing the circumference from the diameter. One such series attributed to illustrious mathematician Madhava of $14^{\text{th}}$ century is

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Key Words : *Correction function, Error function, Remainder term, Gregory series, Rational approximation, Infinite continued fraction, Successive convergents.*

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$$C = \frac{4d}{1} - \frac{4d}{2} + \frac{4d}{5} \cdots \pm \frac{4d}{2n-1} \mp \frac{4d\left(\frac{2n}{2}\right)}{(2n)^2+1},$$

where + or − indicates that $n$ is odd or even and $C$ is the circumference of a circle of diameter $d$, or more specifically,

$$C = \frac{4d}{1} - \frac{4d}{3} + \frac{4d}{5} - \cdots + (-1)^{n-1}\frac{4d}{2n-1} + (-1)^n \frac{4d(2n)/2}{(2n)^2+1}$$

The remainder term $(-1)^n 4dG_n$ where $G_n = \frac{(2n)/2}{(2n)^2+1}$ has been augmented to the series for $C$ by Madhava to get a better approximation. The introduction of the remainder term definitely improves the value of $C$ and is very effective in giving a better approximation for it. Sankara has provided two other forms of the multiplier $G_n$ denoted by $G'_n$ and $G''_n$ where $G'_n = \frac{1}{4n}$ and $G''_n = \frac{n^2+1}{[4(n^2+1)+1]n}$ of which $G''_n$ is found to be more accurate correction function.

## 2. Rational Approximation of Gregory Series

The Gregory series is convergent and converges to $\frac{\pi}{4}$. Thus

$$\frac{\pi}{4} = 1 \cdot \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + (-1)^{n-1}\frac{1}{2n-1} + \cdots$$

If $R_n$ denotes the **remainder term** after n terms of the series , then $R_n = (-1)^n G_n$, where $G_n$ is the correction function after $n$ terms of the series.

**Theorem 1** : The Correction function after $n$ terms for Gregory Series is $G_n = \frac{1}{4n}$.

**Proof** : We have Gregory series is convergent and converges to $\frac{\pi}{4}$.

If $G_n$ denotes the correction function after n terms of the series, then it follows that $G_n + G_{n+1} = \frac{1}{2n+1}$.

The error function is $E_n = G_n + G_{n+1} - \frac{1}{2n+1}$.

We may choose $G_n$ in such a way that $|E_n|$ is a minimum function of $n$.

For a fixed $n$ and for any real number $r$, choose $G_n = \frac{1}{4n+2-r}$.

Then the error function $E_n(r)$ is a rational function of $r$.

i.e. where $E_n(r) = \frac{N_n(r)}{D_n(r)}$ where

$$D_n(r) = (4n+2-r)(4n+6-r)(2n+1) \approx 32n^2,$$

which is a maximum for large values of $n$.

$$N_n(r) = \begin{cases} (4n+2-r)(r-2)+2r), & r \neq 2 \\ 4, & r = 2. \end{cases}$$

Hence $|N_n(r)|$ is a minimum function of $n$ for $r = 2$.

So $|E_n(r)|$ is a minimum function of $n$ for $r = 2$.

Hence for $r = 2$, $G_n$ and $E_n$ are functions of a single variable $n$.

Hence the correction function for Gregory series is $G_n = \frac{1}{4n}$.

The corresponding error function is $|E_n| = \frac{4}{(32n^3+48n^2+16n)}$.

Hence the proof.

**Remark** : Clearly $G_n < \frac{1}{2n+1}$, absolute value of $(n+1)^{th}$ term.

**Theorem 2** : The correction functions of Gregory series follow an infinite continued fraction $\frac{1}{(4n)} + \frac{2^2}{(4n)} + \frac{4^2}{(4n)} + \frac{6^2}{(4n)} + \cdots$

**Proof** : In Theorem 1, we have showed that the correction function for Gregory series is $G_n = \frac{1}{4n}$.

The error function is $|E_n| = \frac{4}{(32n^3+48n^2+16n)}$.

Here onwards we shall denote $p = 2n+1$.

We rename the correction function as first order correction function and denote it by $G_n$ (1).

**The first order correction function is** $G_n(1) = \frac{1}{4n} = \frac{1}{2p-2}$ and the corresponding error function is $|E_n(1)| = \frac{4}{(4p^3-4p)} = \frac{k_1}{f_1(p)}$.

For further reducing error function we may add fractions of correction divisor to the correction divisor itself.

Choose the second order correction function as $G_n(2) = \frac{1}{(2p-2)+\frac{A_1}{(2p-2)}}$ where $A_1$ is any real number. Then it can be shown that $|E_n|$ will be a minimum function of $n$ when $A_1 = 4 = k_1$.

**The second order correction function is** $G_n(2) = \frac{1}{(2p-2)+\frac{4}{(2p-2)}}$.

The error function is $|E_n(2)| = \frac{64}{\{16p^5+64p\}} = \frac{k_2}{f_2(p)}$.

Again for reducing error ,choose the third order correction function as

$$G_n(3) = \frac{1}{(2p-2)+\frac{4}{(2p-2)+\frac{A_2}{(2p-1)}}}.$$

It can be proved that $|E_n|$ is a minimum function of $n$ for $A_2 = 16 = \frac{k_2}{k_1}$.
**The third order correction function is** $G_n(3) = \frac{1}{(2p-2)+\frac{4}{(2p-2)+\frac{16}{(2p-2)}}}$.
The error function is $|E_n(3)| = \frac{2304}{\{64p^7+448p^5+1792p^3-2304p\}} = \frac{k_3}{f_3(p)}$.
Similarly the fourth order correction function is $G_n(4) = \frac{1}{(2p-2)+\frac{4}{(2p-2)+\frac{16}{(2p-2)+\frac{A_3}{(2p-2)}}}}$
where $A_3 = 36 = \frac{k_3}{k_2}$.
**The fourth order correction function is** $G_n(4) = \frac{1}{(2p-2)+\frac{1}{(2p-2)+\frac{16}{(2p-2)+\frac{36}{(2p-2)}}}}$.
In general **the $i^{th}$ order correction function is** $G_n(i) = \frac{1}{(2p-2)+\frac{4}{(2p-2)+\frac{16}{(2p-2)+\cdots+\frac{(2i-2)^2}{(2p-2)}}}}$.

Since $p = 2n + 1$, we have

$$G_n(i) = \frac{1}{4n+\frac{4}{4n+\frac{16}{4n+\frac{36}{4n+\cdots+\frac{(2iz)^2}{4n}}}}}.$$

Continuing this process we get the correction function follows an infinite continued fraction pattern as follows.

$$\frac{1}{(4n)+\frac{2^2}{(4n)+\frac{4^2}{(4n)+\frac{6^2}{(4n)+\frac{3^2}{(4n)+\cdots}}}}}.$$

**Corollary** : The $i^{th}$ order correction function for Gregory series is the $i^{th}$ successive convergent of the infinite continued fraction $\frac{1}{(4n)+}\frac{2^2}{(4n)+}\frac{4^2}{(4n)+}\frac{6^2}{(4n)+\cdots}$.
**Proof** : The $i^{th}$ order correction function is

$$G_n(i) = \frac{1}{4n+}\frac{2^2}{4n+}\frac{4^2}{4n+\cdots+}\frac{(2i-2)^2}{4n}.$$

Clearly $G_n(i)$ is the ith successive convergent of the infinite continued fraction
$\frac{1}{(4n)+}\frac{2^2}{(4n)+}\frac{4^2}{(4n)+}\frac{6^2}{(4n)+\cdots}$.
Hence the proof.

## 3. Correction Functions and Error Functions

The correction functions and the corresponding error functions are tabulated as follows.
For $p = 2n + 1$, we have $G_n = \frac{1}{2p-2}$

| Correction function $G_n$ | Error function $\|E_n\|$ |
|---|---|
| $\frac{1}{2p-2}$ | $\frac{4}{4p^2-4p}$ |
| $\frac{1}{(2p-2)+\frac{4}{(2p-2)}}$ | $\frac{64}{16p^6+64p}$ |
| $\frac{1}{(2p-2)+\frac{4}{(2p-2)+\frac{16}{(2p-2)}}}$ | $\frac{2304}{64p^7+448p^5+1792p^3-23-4p}$ |
| $\frac{1}{(2p-2)+\frac{4}{(2p-2)+\frac{16}{(2p-2)+\frac{36}{(2p-2)}}}}$ | $\frac{147456}{256p^9+6144p^7+4915p^5-16384p^3+147456p}$ |
| $\vdots$ | $\vdots$ |

## 4. Application

If $S_n$ denotes the sequence of partial sums of Gregory series and $G_n$ denotes the correction function after $n$ terms of the series, then the approximation of Gregory series while applying correction function is shown in the following table.

We have using a calculator $\pi = \mathbf{3.1415926536}$.

The following table shows the improvement in accuracy while applying correction function to the series.

| Number of terms ($n$) | $S_n$ | $S_n + (-1)^n G_n$ |
|---|---|---|
| 10 | **3**.0418396189 | **3.141**839619 |
| 100 | **3.1**315929035 | **3.141592**904 |
| 1000 | **3.14**059265384 | **3.141592654** |
| 10000 | **3.141**49265359 | **3.141592654** |
| 100000 | **3.1415**82653589 | **3.141592654** |

For $n = 10$, the approximation of series using successive convergents is shown below.

| Correction functions | $\pi$ | Accuracy |
|---|---|---|
| Without correction function | **3**.0418396189 | **1** |
| $G_n(1)$ | **3.141**839619 | **3** |
| $G_n(2)$ | **3.14159**0242 | **5** |
| $G_n(3)$ | **3.141592**705 | **6** |
| $G_n(4)$ | **3.1415926536** | **10** |

The table shows that as the order of correction function increases, the accuracy increases.

## 5. Conclusion

The introduction of correction function gives better approximation for the series and hence accuracy can be improved.

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