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CERTAIN FAMILY OF ANALYTIC AND MULTIVALENT FUNCTION WITH NORMALIZED CONDITIONS AND ASSOCIATED WITH SUBORDINATION

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Abstract

There are many subclasses of multivalent functions. The objectives of this paper is to introduce new classes and we have attempted to obtain coefficient estimates, distortion theorem, radius of starlikeness and convexity, and other related results for the classes $\mathcal{M}(A, B, a, \delta, p)$ and $S^*(\alpha, \beta, \xi, \gamma, p)$.

1. Introduction

Let $A(p)$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p - \sum_{k=1+p}^{\infty} a_k z^k, \quad a_k \geq 0 \tag{1}$$

$p \in \mathbb{N} = \{1, 2, 3, \cdots\}$ which are p-valent in the unit disc $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ for $p \in \mathbb{N}$.

Key Words and Phrases : *Multivalent function, Coefficient estimate, Distortion theorem, Radius of star likeness, Subordinate.*

Definition 1.1 : A function $f(z) \in A(p)$ is said to be in the subclass $H(\propto)$ of starlike function if

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \propto, \quad z \in U, \quad 0 \leq \propto < 1.$$

Definition 1.2 : A function $f(z) \in A(p)$ is said to be in the subclass $G(\propto)$ of convex function if

$$Re\left(1 + \frac{zf'(z)}{f(z)}\right) > \propto, \quad z \in U.$$

Definition 1.3 : If f and g are regular in U, we say that f is subordinate to g, denoted by $f \prec g$ or $f(z) \prec g(z)$, if there exist a Schwarz function w, which is regular in U with $w(0) = 0$ and $|w(z)| < 1$, $z \in U$ such that $f(z) = g(w(z)), z \in U$. In particular if g is univalent in , we have the equivalence $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

Definition 1.4 : We say that a function $f(z) \in A(p)$ is in the class $\mathcal{M}(A, B, a, \delta, p)$ if it satisfy

$$1 + \frac{1}{a}\left[\frac{\frac{zf'(z)}{f(z)} - p}{\frac{zf'(z)}{f(z)} + p - 2\delta}\right] \prec \frac{1 + Az}{1 + Bz} \tag{1.4.1}$$

for $0 < Re(a),\ 0 < \delta \leq 1,\ \ -1 \leq B < A \leq 1.$

Definition 1.5 : We say that a function $f(z) \in A(p)$ is in the class $S^*(\alpha, \beta, \xi, \gamma, p)$ if and only if,

$$\left|\frac{\frac{zf'(z)}{f(z)} - p}{(p - \alpha)\xi + \alpha\gamma - \gamma\frac{zf'(z)}{f(z)}}\right| < \beta$$

for $0 \leq a < 1,\ 0 < \beta \leq 1,\ 0 \leq \gamma < \xi \leq 1.$

2. Coefficient Estimates

Theorem 1 : A function $f(z) = z^P - \sum_{k=1+p}^{\infty} a_k z^k,\ a_k \geq 0$ is in $\mathcal{M}(A, B, a, \delta, p)$ if and only if $\sum_{k=1+p}^{\infty} \{(k-p) + |a(A-B)(k+p-2\delta) - B(k-p)|\}a_k \leq 2|a|(A-B)(p-\delta).$

Proof : Suppose $f(z)$ is in $\mathcal{M}(A, B, a, \delta, p)$

$$p(z) = 1 + \frac{1}{a}\left[\frac{\frac{zf'(z)}{f(z)} - p}{\frac{zf'(z)}{f(z)} + p - 2\delta}\right] \prec \frac{1 + Az}{1 + Bz}$$

$$p(z) = \frac{1 + Aw(z)}{1 + Bw(z)}$$

$$\left| w(z) = \frac{p(z)-1}{A-Bp(z)} \right| < 1$$

$$\left| \frac{zf'(z)-pf(z)}{a(A-B)[zf'(z)+(p-2\delta)f(z)]-B[zf'(z)-pf(z)]} \right| < 1$$

$$\left| \frac{-\sum\limits_{k=1+p}^{\infty}(k-p)a_k z^k}{a(A-B)\left[2(p-\delta)z^p - \sum\limits_{k=1+p}^{\infty}(k+p-2\delta)a_k z^k\right] + B\left[\sum\limits_{k=1+p}^{\infty}(k-p)a_k z^k\right]} \right| < 1.$$

Since $Re(z) < |z|$. We obtain after choosing the values of z on real axis and letting $z \to 1$ we get

$$\sum_{k=1+p}^{\infty} \{(k-p) + |a(A-B)(k+p-2\delta) - B(k-p)|\} a_k \leq 2|a|(A-B)(p-\delta).$$

Theorem 2 : A function $f(z) = z^p - \sum\limits_{k=1+p}^{\infty} a_k z^k, a_k \geq 0$ is in $S^*(\alpha, \beta, \xi, \gamma, p)$ if and only if

$$\sum_{k=1+p}^{\infty} [k-p+\beta[(p-\alpha)\xi + \alpha\gamma - \gamma k]] a_k \leq \beta[(p-\alpha)\xi + \alpha\gamma - \gamma p].$$

Proof : Suppose $f(z)$ is in $S^*(\alpha, \beta, \xi, \gamma, p)$

$$\left| \frac{\frac{zf'(z)}{f(z)} - p}{(p-\alpha)\xi + \alpha\gamma - \gamma\frac{zf'(z)}{f(z)}} \right| < \beta$$

$$Re \left| \frac{\sum\limits_{k=1+p}^{\infty}(k-p)a_k z^k}{[(p-\alpha)\xi + \alpha\gamma - \gamma p]z^p + \sum\limits_{k=1+p}^{\infty}[\gamma k - (p-\alpha)\xi - \alpha\gamma]a_k z^k} \right| < \beta$$

We know that $|Re(z)| < |z|$

$$\mathbf{Re} \left| \frac{\sum\limits_{k=1+p}^{\infty}(k-p)a_k z^k}{[(p-\alpha)\xi + \alpha\gamma - \gamma p]z^p + \sum\limits_{k=1+p}^{\infty}[\gamma k - (p-\alpha)\xi - \alpha\gamma]a_k z^k} \right| < \beta.$$

We choose values of above expression and allowing $z \to 1$ through real values we obtain

$$\sum_{k=1+p}^{\infty}(k-p)a_k z^k \leq \beta[[(p-\alpha)\xi + \alpha\gamma - \gamma p]$$
$$+\sum_{k=1+p}^{\infty}[\gamma k - (p-\alpha)\xi - \alpha\gamma]a_k]$$
$$\sum_{k=1+p}^{\infty}[k-p+\beta[(p-\alpha)\xi + \alpha\gamma - \gamma k]a_k \leq \beta[(p-\alpha)\xi + \alpha\gamma - \gamma p].$$

Corollary 1.1 : If $f(z) \in \mathcal{M}(A, B, a, \delta, q)$ then

$$a_k \leq \frac{2|a|(A-B)(q-\delta)}{(k-q) + |a(A-B)(k+q-2\delta) - B(k-q)|}$$

and the equality holds for

$$f(z) = z^q - \frac{2|a|(A-B)(q-\delta)}{(k-q) + |a(A-B)(k+q-2\delta) - B(k-q)|} z^k.$$

3. Growth and Distortion Theorem

Theorem 3 : If $f(z) \in \mathcal{M}(A, B, a, \delta, p)$ then

$$|z|^p - |z|^{p+1}\frac{2|a|(A-B)(p-\delta)}{1 + |a(A-B)(1+2p-2\delta) - B|} \leq |f(z)|$$
$$\leq |z|^p + |z|^{p+1}\frac{2|a|(A-B)(p-\delta)}{1 + |a(A-B)(1+2p-2\delta) - B|}.$$

With equality hold for

$$f(z) = z^p - z^{p+1}\frac{2|a|(A-B)(p-\delta)}{1 + |a(A-B)(1+2p-2\delta) - B|}.$$

Proof ; $f(z) \in \mathcal{M}(A, B, a, \delta, p)$. Therefore

$$\sum_{k=1+p}^{\infty} a_k \leq \frac{2|a|(A-B)(p-\delta)}{(k-p) + |a(A-B)(k+p-2\delta) - B(k-p)|}$$
$$|f(z)| \geq |z|^p - \sum_{k=1+p}^{\infty}|a_k||z|^k \geq |z|^p - |z|^{p+1}\sum_{k=1+p}^{\infty}|a_k|$$
$$\geq |z|^p - |z|^{p+1}\frac{2|a|(A-B)(p-\delta)}{1 + |a(A-B)(1+2p-2\delta) - B|}.$$

Similarly

$$|f(z) \leq |z|^p + |z|^{p+1} \frac{2|a|(A-B)(p-\delta)}{1+|a(A-B)(1+2p-2\delta)-B|}$$

Therefore the result.

Theorem 4 : If $f(z) \in \mathcal{M}(A,B,a,\delta,p)$ then

$$p|z|^{p-1} - |z|^p \frac{2|a|(A-B)(p-\delta)(p+1)}{1+|a(A-B)(1+2p-2\delta)-B|}$$
$$|f'(z)|$$
$$\leq p|z|^{p-1} + |z|^p \frac{2|a|(A-B)(p-\delta)(p+1)}{1+|a(A-B)(1+2p-2\delta)-B|}.$$

Theorem 5 : If $f(z) \in S^*(\alpha,\beta,\xi,\gamma,p)$ then

$$|z|^p - \frac{\beta[(p-\alpha)\xi+\alpha\gamma]}{1-\gamma\beta+\beta[(p-\alpha)\xi+\alpha\gamma]}|z|^{p+1} \leq |f(z)| \leq |z|^p + \frac{\beta[(p-\alpha)\xi+\alpha\gamma]}{1-\gamma\beta+\beta[(p-\alpha)\xi+\alpha\gamma]}|z|^{p+1}.$$

Proof : $f(z) \in S^*(\alpha,\beta,\xi,\gamma,p)$ if and only if

$$\sum_{k=1+p}^{\infty} \left[k - \left[p - \frac{\beta[(p-\alpha)\xi+\alpha\gamma]}{1-\gamma\beta}\right]\right] a_k \leq \frac{\beta[(p-\alpha)\xi+\alpha\gamma]}{1-\gamma\beta}$$

$$\sum_{k=1+p}^{\infty} a_k(k-F) \leq p-F$$

where $F = p - \frac{\beta[(p-\alpha)\xi+\alpha\gamma]}{1-\gamma\beta}$.

But we have

$$(1+p-F)\sum_{1+p}^{\infty} a_k \leq \sum_{k=1+p}^{\infty} a_k(k-F) \leq p-F.$$

We obtain

$$|f(z)| \leq |z|^p + \sum_{1+p}^{\infty} a_k|z|^k \leq |z|^p + |z|^{p+1}\sum_{1+p}^{\infty} a_k \leq |z|^p + |z|^{p+1}\left(\frac{p-F}{1+p-F}\right).$$

Similarly

$$|f(z)| \geq |z|^p - \sum_{1+p}^{\infty} a_k|z|^k \geq |z|^{p+1} - |z|^{p+1}\sum_{1+p}^{\infty} a_k \geq |z|^p - |z|^{p+1}\left(\frac{p-F}{1+p-F}\right).$$

So we have

$$|z|^p - \left(\frac{p-F}{1+p-F}\right)|z|^{p+1} \leq |f(z)| \leq |z|^p + \left(\frac{p-F}{1+p-F}\right)|z|^{p+1}.$$

Therefore we have

$$|z|^p - \frac{\beta[(p-\alpha)\xi+\alpha\gamma]}{1-\gamma\beta+\beta[(p-\alpha)\xi+\alpha\gamma]}|z|^{p+1} \leq |f(z)| \leq |z|^p + \frac{\beta[(p-\alpha)\xi+\alpha\gamma]}{1-\gamma\beta+\beta[(p-\alpha)\xi+\alpha\gamma]}|z|^{p+1}.$$

Theorem 6 : If $f(z) \in S^*(\alpha,\beta,\xi,\gamma,p)$ then

$$p|z|^{p-1} - |z|^p \frac{(1+p)\beta[(p-\alpha)\xi+\alpha\gamma-\gamma p]}{1+\beta[(p-\alpha)\xi+\alpha\gamma-\gamma(1+p)]} \leq |f'(z)|$$
$$\leq p|z|^{p-1} + |z|^p \frac{(1+p)\beta[(p-\alpha)\xi+\alpha\gamma-\gamma p]}{1+\beta[(p-\alpha)\xi+\alpha\gamma-\gamma(1+p)]}.$$

4. Radius of Convexity

A function $f(z) \in A(p)$ is said to be close to convex of order $\propto$ $(0 \leq \propto < 1)$ if for all $Re\{f'(z)\} > \propto$ for all $z \in U$.

A function $f(z) \in A(p)$ is said to be starlike of order if $\propto$ $(0 \leq \propto < 1)$ if $Re\left\{\frac{zf'(z)}{f(z)}\right\} > \propto$ for $z \in U$.

A function $f(z) \in A(p)$ is said to be convex of order $\propto$ $(0 \leq \propto < 1)$ if and only if $zf'(z)$ is starlike of order $\propto$, that is

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \alpha \text{ for all } z \in U.$$

Theorem 7 : If $f(z) \in \mathcal{M}(A,B,a,\delta,p)$, then f is close to convex of order $\propto$ in $|z| < r_1(A,B,a,\delta,p)$ where

$$r_1(A,B,a,\delta,p) = \inf_k \left(\left(\frac{(p-\propto)\{(k-p)+|a(A-B)(k+p-2\delta)-B(k-p)|\}}{2k|a|(A-B)(p-\delta)}\right)^{\frac{1}{k-p}}\right)$$

Proof : It is sufficient to show that $\left|\frac{f'(z)}{z^{p-1}} - p\right| < p-\propto$

$$\left|\frac{f'(z)}{z^{p-1}} - p\right| \leq \sum_{k=1+p}^{\infty} k|a_k||z|^{k-p} < p-\propto$$

we have

$$\sum_{k=1+p}^{\infty} \frac{(k-p)+|a(A-B)(k+p-2\delta)-B(k-p)|}{2|a|(A-B)(p-\delta)} a_k \leq 1$$

$$\frac{k|z|^{k-p}}{p-\propto} \leq \frac{(k-p)+|a(A-B)(k+p-2\delta)-B(k-p)|}{2|a|(A-B)(p-\delta)}.$$

Therefore

$$|z| \leq \left(\frac{(p-\alpha)\{(k-p)+|a(A-B)(k+p-2\delta)-B(k-p)|\}}{2k|a|(A-B)(p-\delta)} \right)^{\frac{1}{k-p}},$$

$(p \neq k, p, k \in \mathbb{N})$, which complete the proof.

Theorem 8 : If $f(z) \in \mathcal{M}(A, B, a, \delta, p)$, then f is starlike of order $\propto$ in $|z| < r_2(A, B, a, \delta, p)$ where

$$r_2(A, B, a, \delta, p) = \inf_k \left(\left(\left(\frac{p-\propto}{k-\propto} \right) \left(\frac{(k-p)+|a(A-B)(k+p-2\delta)-B(k-p)|}{2|a|(A-B)(p-\delta)} \right) \right)^{\frac{1}{k-p}} \right)$$

Theorem 9 : If $f(z) \in \mathcal{M}(A, B, a, \delta, p)$, then f is convex of order $\propto$ in $|z| < r_3(A, B, a, \delta, p)$ where

$$r_3(A, B, a, \delta, p) = \inf_k \left(\left(\left(\frac{p(p-\propto)}{k(k-\propto)} \right) \left(\frac{(k-p)+|a(A-B)(k+p-2\delta)-B(k-p)|}{2|a|(A-B)(p-\delta)} \right) \right)^{\frac{1}{k-p}} \right).$$

Theorem 10 : If $f(z) \in S^*(\alpha, \beta, \xi, \gamma, p)$, then f is close to convex of order $\propto$ in $|z| < r_1(\alpha, \beta, \xi, \gamma, p)$ where

$$r_1(\alpha, \beta, \xi, \gamma, p) = \inf_k \left(\left(\frac{(p-\propto)\{k-p+\beta[(p-\alpha)\xi+\alpha\gamma-\gamma k]\}}{k\beta[(p-\alpha)\xi+\alpha\gamma-\gamma p]} \right)^{\frac{1}{k-p}} \right)$$

Proof : It is sufficient to show that $\left| \frac{f'(z)}{z^{p-1}} - p \right| < p-\propto$

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq \sum_{k=1+p}^{\infty} k|a_k||z|^{k-p} < p-\propto \tag{10.1}$$

we have

$$\sum_{k=1+p}^{\infty} \frac{k-p+\beta[(p-\alpha)\xi+\alpha\gamma-\gamma k]}{\beta[(p-\alpha)\xi+\alpha\gamma-\gamma p]} a_k \leq 1 \tag{10.2}$$

Observe that (10.1) is true if

$$\frac{k|z|^{k-p}}{p-\propto} \leq \frac{k-p+\beta[(p-\alpha)\xi+\alpha\gamma-\gamma k]}{\beta[(p-\alpha)\xi+\alpha\gamma-\gamma p]}.$$

Therefore

$$|z| \leq \left(\frac{(p-\propto)\{k-p+\beta[(p-\alpha)\xi+\alpha\gamma-\gamma k]\}}{k\beta[(p-\alpha)\xi+\alpha\gamma-\gamma p]} \right)^{\frac{1}{k-p}},$$

$(p \neq k, p, k \in \mathbb{N})$, which complete the proof.

Theorem 11 : If $f(z) \in S^*(\alpha,\beta,\xi,\gamma,p)$, then f is starlike of order $\propto$ in $|z| < r_2(\alpha,\beta,\xi,\gamma,p)$ where

$$r_2(\alpha,\beta,\xi,\gamma,p) = \inf_k \left(\left(\left(\frac{p-\propto}{k-\propto} \right) \left(\frac{k-p+\beta[(p-\alpha)\xi+\alpha\gamma-\gamma k]}{\beta[(p-\alpha)\xi+\alpha\gamma-\gamma p]} \right) \right)^{\frac{1}{k-p}} \right).$$

Theorem 12 : If $f(z) \in S^*(\alpha,\beta,\xi,\gamma,p)$, then f is convex of order $\propto$ in $|z| < r_3(\alpha,\beta,\xi,\gamma,p)$ where

$$r_3(\alpha,\beta,\xi,\gamma,p) = \inf_k \left(\left(\left(\frac{p-\propto}{k(k-\propto)} \right) \left(\frac{k-p+\beta[(p-\alpha)\xi+\alpha\gamma-\gamma k]}{\beta[(p-\alpha)\xi+\alpha\gamma-\gamma p]} \right) \right)^{\frac{1}{k-p}} \right).$$

5. Neighborhood Property

Let $f \in \mathcal{M}(A,B,a,\delta,p), \tau \geq 0$, then a (t,τ)-neighborhood of the function $f \in \mathcal{M}(A,B,a,\delta,p)$ is defined by

$$\aleph_\tau^t(f) = \left\{ g \in \mathcal{M}(A,B,a,\delta,p) : g(z) = z^p - \sum_{k=1+p}^{\infty} b_k z^k \text{ and } \sum_{k=1+p}^{\infty} k^{t+1}|a_k - b_k| \leq \tau \right\}$$

we have if $e(z) = z^p, p \in \mathbb{N}$, then

$$\aleph_\tau^t(e) = \left\{ g \in \mathcal{M}(A,B,a,\delta,p) : g(z) = z^p - \sum_{k=1+p}^{\infty} c_k z^k \text{ and } \sum_{k=1+p}^{\infty} k^{t+1}|c_k| \leq \tau \right\}$$

Definition 5.1 : A function $f(z) = z^p - \sum\limits_{k=1+p}^{\infty} a_k z^k, a_k \geq 0$ is said to be in the class $\mathcal{M}^\pi(A,B,a,\delta,p)$ if there exist a function $g \in \mathcal{M}(A,B,a,\delta,p)$ such that

$$\left| \frac{f(z)}{g(z)} - 1 \right| < p - \pi, \quad z \in U, \quad 0 \leq \pi < 1.$$

Theorem 13 : If $g \in \mathcal{M}(A,B,a,\delta,p)$ and

$$\pi = p - \frac{\tau}{k^{t+1}} \left[\frac{(k-p) + |a(A-B)(k+p-2\delta) - B(k-p)|}{[(k-p) + |a(A-B)(k+p-2\delta) - B(k-p)|] - 2|a|(A-B)(p-\delta)} \right]$$

Then $\aleph_\tau^t(g) \subset \mathcal{M}^\pi(A,B,a,\delta,p)$.

Proof : Let $f \in \aleph_\tau^t(g)$. We have $\sum\limits_{k=1+q}^{\infty} k^{t+1}|a_k - b_k| \leq \tau$. This implies that

$$\sum_{k=1+q}^{\infty} |a_k - b_k| \leq \frac{\tau}{k^{t+1}} \quad g \in \mathcal{M}(A,B,a,\delta,q)$$

$$\sum_{k=1+p}^{\infty} b_k \leq \frac{2|a|(A-B)(p-\delta)}{(k-p)+|a(A-B)(k+p-2\delta)-B(k-p)|}$$

So that

$$\left|\frac{f(z)}{g(z)}-1\right| \leq \frac{\sum\limits_{k=1+p}^{\infty}|a_k-b_k|}{1-\sum\limits_{k=1+p}^{\infty} b_k}$$

$$\leq \frac{\tau}{k^{t+1}}\left[\frac{1}{1-\frac{2|a|(A-B)(p-\delta)}{(k-p)+|a(A-B)(k+p-2\delta)-B(k-p)|}}\right]$$

$$< \frac{\tau}{k^{t+1}}\left[\frac{(k-p)+|a(A-B)(k+p-2\delta)-B(k-p)|}{[(k-p)+|a(A-B)(k+p-2\delta)-B(k-p)|]-2|a|(A-B)(p-\delta)}\right]$$

$$= p-\pi.$$

Thus by above definition $f \in \mathcal{M}^{\pi}(A,B,a,\delta,p)$.

Thus $\aleph_{\tau}^{t}(g) \subset \mathcal{M}^{\pi}(A,B,a,\delta,q)$.

The recent investigation by Frasin and M. Darus [11], let $f \in S^{*}(\alpha,\beta,\xi,\gamma,p)$, $\tau \geq 0$ then a (t,τ)-neighborhood of the function $f \in S^{*}(\alpha,\beta,\xi,\gamma,p)$ is defined by

$$\aleph_{\tau}^{t}(f) = \left\{ g \in S^{*}(\alpha,\beta,\xi,\gamma,p) : g(z) = z^{p} - \sum_{k=1+p}^{\infty} b_k z^k \text{ and } \sum_{k=1+p}^{\infty} k^{t+1}|a_k-b_k| \leq \tau \right\}$$

we have if $e(z) = z^{p}, \in \mathbb{N}$, then

$$\aleph_{\tau}^{t}(e) = \left\{ g \in S^{*}(\alpha,\beta,\xi,\gamma,p) : g(z) = z^{p} - \sum_{k=1+p}^{\infty} c_k z^k \text{ and } \sum_{k=1+p}^{\infty} k^{t+1}|c_k| \leq \tau \right\}.$$

Definition 5.2 : A function $f(z) = z^{p} - \sum\limits_{k=1+p}^{\infty} a_k z^k, a_k \geq 0$ is said to be in the class $S^{*,\pi}(\alpha,\beta,\xi,\gamma,p)$ if there exist a function $g \in S^{*}(\alpha,\beta,\xi,\gamma,p)$ such that

$$\left|\frac{f(z)}{g(z)}-1\right| < p-\pi, \quad z \in E, \quad 0 \leq \pi < 1.$$

Theorem 14 : If $g \in S^{*}(\alpha,\beta,\xi,\gamma,p)$ and

$$\pi = p - \frac{\tau}{k^{t+1}} \frac{k-p+\beta[(p-a)\xi+\alpha\gamma-\gamma k]}{k-p+\beta[(p-\alpha)\xi+\alpha\gamma-\gamma k]-\beta[(p-\alpha)\xi+\alpha\gamma-\gamma p]}.$$

Then $\aleph_{\tau}^{t}(g) \subset S^{*\pi}(\alpha,\beta,\xi,\gamma,p)$.

6. Integral Transform

The generalized Bernardi integral operator is defined by

$$K(z) = \frac{c+p}{z^c}\int_0^z x^{c-1} f(x)dx, \qquad (c > -1, z \in U) \tag{6.1.1}$$

$$K(z) = z^p - \sum_{k=1+p}^{\infty} \left(\frac{c+p}{c+k}\right) a_k z^k. \tag{6.1.2}$$

The integral operator $G(z)$ is defined by

$$G(z) = z^{p-1}\int_0^z \frac{f(x)}{x^p} dx \tag{6.1.3}$$

The Komatu integral operator is defined by

$$H(z) = \frac{(\theta+p)^d}{\Gamma(d) z^\theta}\int_0^z x^{\theta-1}\left(\log\frac{z}{x}\right)^{d-1} f(x)dx \tag{6.1.4}$$

$(d > 0, \theta > -p, z \in U)$.

The integral operator

$$L(z) = \begin{pmatrix} d+\theta+p-1 \\ \\ \theta+p-1 \end{pmatrix} \frac{d}{z^\theta}\int_0^z x^{\theta-1}\left(1-\frac{x}{z}\right)^{d-1} f(x)dx \tag{6.1.5}$$

$(d > 0, \theta > -p, z \in U)$.

We have from (6.1.4) and (6.1.5)

$$H(z) = z^p - \sum_{k=1+p}^{\infty} \left(\frac{\theta+p}{\theta+k}\right)^d a_k z^k \tag{6.1.6}$$

$$L(z) = z^p - \sum_{k=1+p}^{\infty} \frac{\Gamma(\theta+k)\Gamma(d+\theta+p)}{\Gamma(d+\theta+k)\Gamma(\theta+p)} a_k z^k. \tag{6.1.7}$$

Theorem 15 : Let $f \in \mathcal{M}(A, B, a, \delta, p)$ then $K(z) \in \mathcal{M}(A, B, a, \delta, p)$.

Proof : We have

$$K(z) = z^p - \sum_{k=1+p}^{\infty} \left(\frac{c+p}{c+k}\right) a_k z^k.$$

We need to prove that

$$\sum_{k=1+p}^{\infty} \frac{(k-p) + |a(A-B)(k+p-2\delta) - B(k-p)|}{2|a|(A-B)(p-\delta)} \left(\frac{c+p}{c+k}\right) a_k \le 1.$$

Since $f \in \mathcal{M}(A,B,a,\delta,q)$ then we have

$$\sum_{k=1+p}^{\infty} \frac{(k-p)+|a(A-B)(k+p-2\delta)-B(k-p)|}{2|a|(A-B)(p-\delta)} a_k \leq 1.$$

But $\left(\frac{c+q}{c+k}\right) < 1$ therefore Theorem 14 holds and the proof is over.

Theorem 16 : Let $f \in \mathcal{M}(A,B,a,\delta,p)$ then $K(z)$ is starlike of order σ, $0 \leq \sigma < 1$ in $|z| < r_1$, where

$$r_1 = \inf_k \left(\left(\left(\frac{p-\sigma}{k-\sigma} \right) \left(\frac{c+k}{c+q} \right) \left(\frac{(k-p)+|a(A-B)(k+p-2\delta)-B(k-p)|}{2|a|(A-B)(q-\delta)} \right) \right)^{\frac{1}{k-p}} \right).$$

Theorem 17 : Let $f \in S^*(\alpha,\beta,\xi,\gamma,p)$ then $H(z) \in S^*(\alpha,\beta,\xi,\gamma,p)$.

Proof : We have

$$H(z) = z^p - \sum_{k=1+p}^{\infty} \left(\frac{\theta+p}{\theta+k} \right)^d a_k z^k.$$

We need to prove that

$$\sum_{k=1+p}^{\infty} \frac{k-p+\beta[(p-\alpha)\xi+\alpha\gamma-\gamma k]}{\beta[(p-\alpha)\xi+\alpha\gamma-\gamma p]} \left(\frac{\theta+p}{\theta+k} \right)^d a_k \leq 1.$$

Since $f \in S^*(\alpha,\beta,\xi,\gamma,p)$ then we have

$$\sum_{k=1+p}^{\infty} \frac{k-p+\beta[(p-\alpha)\xi+\alpha\gamma-\gamma k]}{\beta[(p-\alpha)\xi+\alpha\gamma-\gamma p]} a_k \leq 1.$$

But $\left(\frac{\theta+p}{\theta+k}\right)^d < 1$ therefore Theorem 17 holds and the proof is over.

Theorem 18 : Let $f \in S^*(\alpha,\beta,\xi,\gamma,p)$ then $L(z) \in S^*(\alpha,\beta,\xi,\gamma,p)$.

Proof :

$$L(z) = z - \sum_{k=1+p}^{\infty} \frac{\Gamma(\theta+k)\Gamma(d+\theta+p)}{\Gamma(d+\theta+k)\Gamma(\theta+p)} a_k z^k.$$

We need to prove that

$$\sum_{k=1+p}^{\infty} \frac{k-p+\beta[(p-\alpha)\xi+\alpha\gamma-\gamma k]}{\beta[(p-\alpha)\xi+\alpha\gamma-\gamma p]} \left(\frac{\theta+p}{\theta+k} \right)^d a_k \leq 1.$$

Since then from Theorem 1 we have

$$\sum_{k=1+p}^{\infty} \frac{k-p+\beta[(p-\alpha)\xi+\alpha\gamma-\gamma k]}{\beta[(p-\alpha)\xi+\alpha\gamma-\gamma p]} a_k \leq 1.$$

But $\frac{\Gamma(\theta+k)\Gamma(d+\theta+p)}{\Gamma(d+\theta+k)\Gamma(\theta+p)} < 1$ therefore Theorem 18 holds and the proof is over.

Theorem 19 ; Let $f \in S^*(\alpha, \beta, \xi, \gamma, p)$ then $H(z)$ is starlike of order σ, $0 \leq \sigma < 1$ in $|z| < r_1$, where

$$r_1 = \inf_k \left(\left(\left(\frac{p-\sigma}{k-\sigma} \right) \left(\frac{\theta+k}{\theta+p} \right)^d \left(\frac{k-p+\beta[(p-\alpha)\xi+\alpha\gamma-\gamma k]}{\beta[(p-\alpha)\xi+\alpha\gamma-\gamma p]} \right) \right)^{\frac{1}{k-p}} \right).$$

Theorem 20 : Let $f \in S^*(\alpha, \beta, \xi, \gamma, p)$ then $L(z)$ is starlike of order $0 \leq \sigma < 1$ in $|z| < r_2$, where

$$r_2 = \inf_k \left(\left(\left(\frac{p-\sigma}{k-\sigma} \right) \left(\frac{\Gamma(d+\theta+k)\Gamma(\theta+p)}{\Gamma(\theta+k)\Gamma(d+\theta+p)} \left(\frac{k-p+\beta[(p-\alpha)\xi+\alpha\gamma-\gamma k]}{\beta[(p-\alpha)\xi+\alpha\gamma-\gamma p]} \right) \right)^{\frac{1}{k-p}} \right) \right).$$

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