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ON SUMMATION FORMULAE INVOLVING MULTIVARIABLE *I*-FUNCTIONS

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Abstract

The object of this paper is to derive a Summation formula involving the I-function of r-variables. As a corollary we derived a Summation formula involving the I-function of 2-variables from the main result. Summation formulae involving hyper geometric functions and a number of known results can be deduced from them. A Summation formula involving H-function of r-variables derived by Prasanna kumari [1, p.127] is also obtained by specializing the parameters in our main result.

1. Introduction

Notations used:

$(a_p) = {}_1(a_j)_p$ stands for $a_1, a_2, \cdots, a_p$

Key Words : *I-function of two and several complex variables, Multivariable H-functions, Integral transforms, H-function transforms.*

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${}_1(a_j;\alpha_j)_p$ stands for $(a_1;\alpha_1),(a_2;\alpha_2),\cdots,(a_p;\alpha_p)$
${}_1(a_j;\alpha_j,A_j)_p$ stands for $(a_1;\alpha_1,A_1),(a_2;\alpha_2,A_2),\cdots,(a_p;\alpha_p,A_p)$
${}_1(a_j;\alpha_j,A_j;1)_p$ stands for $(a_1;\alpha_1,A_1;1),(a_2;\alpha_2,A_2;1),\cdots,(a_p;\alpha_p,A_p;1)$
$(-d)_k=\frac{\Gamma(-d+k)}{k!}$.
The generalized Fox's H-function, namely I-function of r-variables introduced by Prathima, et al.[2, p.38] is defined and represented in the following manner:

$$I[z_1,\cdots,z_r]=I^{0,N:m_1,n_1;\cdots;m_r,n_r}_{P,Q:p_1,q_1;\cdots;p_r,q_r}$$
$$\left[\begin{array}{c|c} z_1 & {}_1(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_P:(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{p_1};\cdots;\ {}_1(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r} \\ \vdots & \\ z_r & {}_1(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_Q:\ {}_1(d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{q_1};\cdots;(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{q_r}\end{array}\right]$$
$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\theta_1(s_1)\cdots\theta_r(s_r)\phi(s_1,\cdots,s_r)z_1^{s_1}\cdots z_r^{s_r}ds_1\cdots ds_r, \tag{1.1}$$

where $\phi(s_1,\cdots,s_r)$ and $\theta_i(s_i), i=1,2,\cdots,r$ are given by

$$\phi(s_1,\cdots,s_r)=\frac{\prod\limits_{j=1}^{N}\Gamma^{A_j}\left(1-a_j+\sum\limits_{i=1}^{r}\alpha_j^{(i)}s_i\right)}{\prod\limits_{j=1}^{Q}\Gamma^{B_j}\left(1-b_j+\sum\limits_{i=1}^{r}\beta_j^{(i)}s_i\right)\prod\limits_{j=N+1}^{P}\Gamma^{A_j}\left(a_j-\sum\limits_{i=1}^{r}\alpha_j^{(i)}s_i\right)}, \tag{1.2}$$

$$\theta_i(s_i)=\frac{\prod\limits_{j=1}^{m_i}\Gamma^{D_j^{(i)}}(d_j^{(i)}-\delta_j^{(i)}s_i)\prod\limits_{j=1}^{n_i}\Gamma^{C_j^{(i)}}(1-c_j^{(i)}+\gamma_j^{(i)}s_i)}{\prod\limits_{j=m_i+1}^{q_i}\Gamma^{D_j^{(i)}}(1-d_j^{(i)}+\delta_j^{(i)}s_i)\prod\limits_{j=n_i+1}^{p_i}\Gamma^{C_j^{(i)}}(c_j^{(i)}-\gamma_j^{(i)}s_i)}. \tag{1.3}$$

Also $z_i\neq 0$ $(i=1,\cdots,r),\omega=\sqrt{-1},m_j,n_j,p_j,q_j$ $(j=1,\cdots,r)$, N,P,Q are non-negative intergers such that $0\le N\le P$, $Q\ge 0$, $0\le m_j\le q_j$, $0\le n_j\le p_j$ $(j=1,2,\cdots,r)$ (not all zero simultaneously). $\alpha_j^{(i)}$ $(j=1,2,\cdots,P,i=1,2,\cdots,r)$, $\beta_j^{(i)}$ $(j=1,2,\cdots,Q,i=1,2,\cdots,r)$, $\gamma_j^{(i)}$ $(j=1,2,\cdots,p_i,i=1,2,\cdots,r)$, and $\delta_j^{(i)}$ $(j=1,2,\cdots,q_i$, $i=1,2,\cdots,r)$ are positive numbers, a_j $(j=1,2,\cdots,P)$, b_j $(j=1,2,\cdots,Q)$, $c_j^{(i)}$ $(j=1,2,\cdots,p_i,i=1,2,\cdots,r)$ and $d_j^{(i)}$ $(j=1,2,\cdots,q_i,i=1,2,\cdots,r)$ are complex numbers. The exponents A_j $(j=1,2,\cdots,P)$, B_j $(j=1,2,\cdots,Q)$, $C_j^{(i)}$ $(j=1,2,\cdots,p_i,i=1,2,\cdots,r)$ and $D_j^{(i)}$ $(j=1,2,\cdot,q_i,i=1,2,\cdots,r)$ of various gamma functions may take non integer values. The I-function of r variables is analytic if

$$\Psi_i=\sum_{j=1}^{P}A_j\alpha_j^{(i)}-\sum_{j=1}^{Q}B_j\beta_j^{(i)}+\sum_{j=1}^{p_i}C_j^{(i)}\gamma_j^{(i)}-\sum_{j=1}^{q_i}D_j^{(i)}\delta_j^{(i)}\le 0,\ i=1,2,\cdots,r.$$

The integral (1.1) converges absolutely if $|arg(z_i)| < \frac{1}{2}\Delta_i\pi,\ i = 1, 2, \cdots, r$ where

$$\begin{aligned}\Delta_i &= -\sum_{j=n+1}^{P} A_j\alpha_j^{(i)} - \sum_{j=1}^{Q} B_j\beta_j^{(i)} + \sum_{j=1}^{m_i} D_j^{(i)}\delta_j^{(i)} - \sum_{j=m_i+1}^{q_i} D_j^{(i)}\delta_j^{(i)} \\ &\quad + \sum_{j=1}^{n_i} C_j^{(i)}\gamma_j^{(i)} - \sum_{j=n_i+1}^{p_i} C_j^{(i)}\gamma_j^{(i)} > 0.\end{aligned}$$

Prasanna kumari [1, p. 121]

$$\sum_{r=0}^{\infty} \frac{(-d)_r\Gamma(-1+r+s)}{k!\Gamma(c+r)} = \frac{\Gamma(-1+s)\Gamma(1+d+c-s)}{\Gamma(c+d)\Gamma(1+c-s)}, \quad d > 0, s > 1, c > s-1. \quad (1.4)$$

2. Summation Formulae Involving Multivariable I-Functions

Main Result :

$$\sum_{k=0}^{\infty} \frac{(-d)_k}{k!} I^{0,N+1:m_1,n_1;\cdots;m_r,n_r}_{P+1,Q+1:p_1,q_1;\cdots;p_r,q_r}\left[\begin{array}{c|l} z_1 & (2-k;\sigma_1,\sigma_2,\cdots,\sigma_r;1),\ {}_1(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_P : \\ & {}_1(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_Q, (1-k;\rho_1,\rho_2,\cdots,\rho_r;1) : \\ \vdots & {}_1(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{p_1};\cdots;(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r} \\ z_r & {}_1(d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{q_1};\cdots;(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{q_r} \end{array}\right]$$

$$= I^{0,N+2:m_1,n_1;\cdots;m_r,n_r}_{P+2,Q+1:p_1,q_1;\cdots;p_r,q_r}\left[\begin{array}{c|l} & (2;\sigma_1,\cdots,\sigma_r;1), (-d;\rho_1-\sigma_1,\cdots,\rho_r-\sigma_r;1), \\ z_1 & {}_1(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_P : \\ & {}_1(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_Q, (0;\rho_1-\sigma_1,\cdots,\rho_r-\sigma_r;1), \\ \vdots & (1-d;\rho_1,\cdots,\rho_r;1) : \\ z_r & {}_1(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{p_1};\cdots;(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r} \\ & {}_1(d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{q_1};\cdots;(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{q_r} \end{array}\right] \quad (2.1)$$

provided every z_i is complex, $\sigma_i > 0, \rho_i > 0, d > 0, U_i \leq 0, V_i > 0, V_i + \sigma_i - \rho_i > 0$ for $i = 1, 2, \cdots, r$, where

$$\sum_{j=1}^{P} A_j \alpha_j^{(i)} - \sum_{j=1}^{Q} B_j \beta_j^{(i)} + \sum_{j=1}^{p_i} C_j^{(i)} \gamma_j^{(i)} - \sum_{j=1}^{q_i} D_j^{(i)} \delta_j^{(i)} + \sigma_i - \rho_i \leq 0$$

$\Delta_i > 0$, $|arg(z_i)| < \frac{1}{2}\pi\Delta_i, i = 1, \cdots, r$ and

$$\Delta_i = -\sum_{j=N+1}^{P} A_j \alpha_j^{(i)} - \sum_{j=1}^{Q} B_j \beta_j^{(i)} + \sum_{j=1}^{m_i} D_j^{(i)} \delta_j^{(i)} - \sum_{j=m_i+1}^{q_i} D_j^{(i)} \delta_j^{(i)}$$
$$+ \sum_{j=1}^{n_i} C_j^{(i)} \gamma_j^{(i)} - \sum_{j=n_i+1}^{p_i} C_j^{(i)} \gamma_j^{(i)} - \rho_i \quad \text{for} \quad i = 1, 2, \cdots, r.$$

Moreover,

$$Re\left(-1 + \sigma_1 \frac{d_j^{(1)}}{\delta_j^{(1)}}\right) > 0 \quad \text{for} \quad j = 1, 2, \cdots, m_1,$$
$$Re\left(-1 + \sigma_2 \frac{d_j^{(2)}}{\delta_j^{(2)}}\right) > 0 \quad \text{for} \quad j = 1, 2, \cdots, m_2,$$
$$\vdots$$
$$Re\left(-1 + \sigma_r \frac{d_j^{(r)}}{\delta_j^{(r)}}\right) > 0 \quad \text{for} \quad j = 1, 2, \cdots, m_r,$$
$$Re\left(1 + d + (\rho_1 - \sigma_1) \frac{d_j^{(1)}}{\delta_j^{(1)}}\right) > 0 \quad \text{for} \quad j = 1, 2, \cdots, m_1,$$
$$Re\left(1 + d + (\rho_2 - \sigma_2) \frac{d_j^{(2)}}{\delta_j^{(2)}}\right) > 0 \quad \text{for} \quad j = 1, 2, \cdots, m_2,$$
$$\vdots$$
$$Re\left(1 + d + (\rho_r - \sigma_r) \frac{d_j^{(r)}}{\delta_j^{(r)}}\right) > 0 \quad \text{for} \quad j = 1, 2, \cdots, m_r.$$

Proof : Replace r by k , s by $(\sigma_1 s_1 + \cdots + \sigma_r s_r)$ and c by $(\rho_1 s_1 + \cdots + \rho_r s_r)$ in (1.4)

$$\sum_{r=0}^{\infty} \frac{(-d)_k \Gamma(-1 + k + \sigma_1 s_1 + \cdots + \sigma_r s_r)}{k! \Gamma(\rho_1 s_1 + \cdots \rho_r s_r + k)}$$
$$= \frac{\Gamma(-1 + \sigma_1 s_1 + \cdots \sigma_r s_r) \Gamma(1 + d + (\rho_1 - \sigma_1) s_1 + \cdots + (\rho_r - \sigma_r) s_r)}{\Gamma(\rho_1 s_1 + \cdots + \rho_r s_r + d) \Gamma(1 + (\rho_1 - \sigma_1) s_1 + \cdots + (\rho_r - \sigma_r) s_r)}. \tag{2.2}$$

Multiply both sides of (2.2) by $\left(\frac{1}{2\pi\omega}\right)^r \theta_1(s_1),\cdots,\theta_r(s_r)\phi(s_1,\cdots,s_r)z_1^{s_1}\cdots z_r^{s_r}$ where $\omega=\sqrt{-1}$ and integrate over $L_1,L_2,\cdots,L_r$ to get

$$\begin{aligned}&\int_{L_1}\cdots\int_{L_r}\frac{1}{(2\pi\omega)^r}\theta_1(s_1)\cdots\theta_r(s_r)\phi(s_1,\cdots,s_r)\\&\times\sum_{k=0}^{\infty}\frac{(-d)_k\Gamma(-1+k+\sigma_1s_1+\cdots+\sigma_rs_r)}{k!\Gamma(\rho_1s_1+\cdots+\rho_rs_r+k)}z_1^{s_1}\cdots z_r^{s_r},ds_1\cdots ds_r\\&=\int_{L_1}\cdots\int_{L_r}\frac{1}{(2\pi\omega)^r}\theta_1(s_1)\cdots\theta_r(s_r)\phi(s_1,\cdots,s_r)\\&\times\frac{\Gamma(-1+\sigma_1s_1+\cdots+\sigma_rs_r)\Gamma(1+d+(\rho_1-\sigma_1)s_1+\cdots+(\rho_r-\sigma_r)s_r)}{\Gamma(\rho_1s_1+\cdots+\rho_rs_r+d)\Gamma(1+(\rho_1-\sigma_1)s_1+\cdots+(\rho_r-\sigma_r)s_r)}z_1^{s_1}\cdots z_r^{s_r}ds_1\cdots ds_r.\end{aligned}$$

Changing the order of summation and integration on left hand side

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{(-d)_k}{k!}\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\theta_1(s_1)\cdots\theta_r(s_r)\phi(s_1,\cdots,s_r)\\&\times\frac{\Gamma(-1+k+\sigma_1s_1+\cdots+\rho_rs_r+k)}{{}_1\Gamma(\rho_1s_1+\cdots+\rho_rs_r+k)}z_1^{s_1}\cdots z_r^{s_r}ds_1\cdots ds_r\\&=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\theta_1(s_1)\cdots\theta_r(s_r)\phi(s_1,\cdots,s_r)\\&\times\frac{\Gamma(-1+\sigma_1s_1+\cdots+\sigma_rs_r)\Gamma(1+d+(\rho_1-\sigma_1)s_1+\cdots+(\rho_r-\sigma_r)s_r)}{\Gamma(\rho_1s_1+\cdots+\rho_rs_r+d)\Gamma(1+(\rho_1-\sigma_1)s_1+\cdots+(\rho_r-\sigma_r)s_r)}z_1^{s_1}\cdots z_r^{s_r}ds_1\cdots ds_r\end{aligned}$$

from which (2.1) is obtained by using (1.1). The change of order of summation and integration is justified when the given conditions are satisfied because of the absolute convergence of the summation and the integrals involved.

Special Cases :

In (2.1), taking $r=2$, we get the summation formula for I-function of 2-variables, as given in the following Corollary :

Corollary :

$$\sum_{k=0}^{\infty}\frac{(-d)_k}{k!} I^{0,N+1:m_1,n_1;m_2,n_2}_{P+1,Q+1:p_1,q_1;p_2,q_2}\left[z_1, z_2 \left| \begin{array}{l} (2-k;\sigma_1,\sigma_2;1),\ {}_1(a_j;\alpha_j^{(1)},\alpha_j^{(2)};A_j)_P : \\ {}_1(b_j;\beta_j^{(1)},\beta_j^{(2)};B_j)_Q, (1-k;\rho_1,\rho_2;1) : \\ {}_1(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{p_1};\ {}_1(c_j^{(2)},\gamma_j^{(2)};C_j^{(2)})_{p_r} \\ {}_1(d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{q_1};\ {}_1(d_j^{(2)},\delta_j^{(2)};D_j^{(2)})_{q_r} \end{array} \right. \right]$$

$$= I^{0,N+2:m_1,n_1;m_2,n_2}_{P+2,Q+2:p_1,q_1;p_2,q_2}\left[z_1, z_2 \left| \begin{array}{l} (2;\sigma_1,\sigma_2;1), (-d;\rho_1-\sigma_1,\rho_2-\sigma_2;1), \\ {}_1(a_j;\alpha_j^{(1)},\alpha_j^{(2)};A_j)_P : \\ {}_1(b_j;\beta_j^{(1)},\beta_j^{(2)};B_j)_Q, (0;\rho_1-\sigma_1,\rho_2-\sigma_2;1), \\ (1-d;\rho_1,\rho_2;1) : \\ {}_1(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{p_1};\ {}_1(c_j^{(2)},\gamma_j^{(2)};C_j^{(2)})_{p_2} \\ {}_1(d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{q_1};\ {}_1(d_j^{(2)},\delta_j^{(2)};D_j^{(2)})_{q_2} \end{array} \right. \right] \qquad (2.3)$$

provided z_1, z_2 are complex, $\sigma_i > 0, \rho_i - \sigma_i > 0, d > 0, U_i \le 0, V_i > 0, V_i + \sigma_i - \rho_i > 0$ for $i = 1, 2$, where

$$\sum_{j=1}^{P} A_j\alpha_j^{(i)} - \sum_{j=1}^{Q} B_j\beta_j^{(i)} + \sum_{j=1}^{p_i} C_j^{(i)}\gamma_j^{(i)} - \sum_{j=1}^{q_i} D_j^{(i)}\delta_j^{(i)} + \sigma_i - \rho_i \le 0$$

$\Delta_i > 0$, $|arg(z_i)| < \frac{1}{2}\pi\Delta_i, i = 1, 2$ and

$$\Delta_i = -\sum_{j=N+1}^{P} A_j\alpha_j^{(i)} - \sum_{j=1}^{Q} B_j\beta_j^{(i)} + \sum_{j=1}^{m_i} D_j^{(i)}\delta_j^{(i)} - \sum_{j=m_i+1}^{q_i} D_j^{(i)}\delta_j^{(i)}$$
$$+\sum_{j=1}^{n_i} C_j^{(i)}\gamma_j^{(i)} - \sum_{j=n_i+1}^{p_i} C_j^{(i)}\gamma_j^{(i)} - \rho_i \quad \text{for} \quad i = 1, 2.$$

Moreover,

$$Re\left(-1+\sigma_1\frac{d_j^{(1)}}{\delta_j^{(1)}}\right)>0 \text{ for } j=1,2,\cdots,m_1,$$

$$Re\left(-1+\sigma_2\frac{d_j^{(2)}}{\delta_j^{(2)}}\right)>0 \text{ for } j=1,2,\cdots,m_2,$$

$$Re\left(1+d+(\rho_1-\sigma_1)\frac{d_j^{(1)}}{\delta_j^{(1)}}\right)>0 \text{ for } j=1,2,\cdots,m_1, \text{ and}$$

$$Re\left(1+d+(\rho_2-\sigma_2)\frac{d_j^{(2)}}{\delta_j^{(2)}}\right)>0 \text{ for } j=1,2,\cdots,m_2.$$

If all the exponents A_j $(j=1,2,\cdots,P), B_j$ $(j=1,2,\cdots,Q), C_j^{(i)}$ $(j=1,2,\cdots,p_i, i=1,2,\cdots,r)$ and $D_j^{(i)}$ $(j=1,2,\cdots,q_i, i=1,2,\cdots,r)$ in (2.1) are equal to unity, we obtain the Summation formula involving the H-function of r-complex variables as derived by Prasanna kumari [1, p.127].

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