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SOME FIXED POINT THEOREMS IN PARAMETRIC $b-$METRIC SPACE

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Abstract

In this paper we proved the some fixed point theorems in Parametric $b-$Metric spaces.

1. Introduction and Preliminaries

The concept of a $b-$metric space was introduced by Czerwik in [7] and many fixed point results for single and multi-valued mappings are proved by many authors in the setting of b-metric spaces. Alghamdi, et al. [2] proved some fixed point and coupled fixed point theorems on b-metric-like spaces. Hussain et al. [9,10] introduced a new type of generalized metric space, called parametric b-metric space, as a generalization of both metric and b-metric spaces. The aim of this paper is to extend the Banach fixed-point

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point theorem to continuous mappings on complete parametric b-metric spaces in several senses. These results improve and generalize some important known results in existing literature.

2. Preliminaries

Definition 2.1 : Let X be a nonempty set, $s \geq 1$ be a real number and $\rho : X \times X \times (0, +\infty) \to [0, +\infty)$ be a function. We say ρ is a parametric b-metric on X if,

(1) $\rho(x, y, t) = 0$ for all $t > 0$ if and only if $x = y$,

(2) $\rho(x, y, t) = \rho(y, x, t)$ for all $t > 0$,

(3) $\rho(x, y, t) \leq s[\rho(x, z, t) + \rho(z, y, t)]$ for all $x, y, z \in X$ and all $t > 0$, where $s \geq 1$.

and one says the pair (X, ρ) is a parametric metric space with parameter $s \geq 1$. Obviously, for $s = 1$, parametric b-metric reduces to parametric metric.

Definition 2.2 : Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in a parametric b-metric space (X, ρ, s).

(1) $\{x_n\}_{n=1}^{\infty}$ is said to be convergent to $x \in X$, written as $\lim_{n\to\infty} x_n = x$, for all $t > 0$, if $\lim_{n\to\infty} \rho(x_n, x, t) = 0$.

(2) $\{x_n\}_{n=1}^{\infty}$ is said to be a Cauchy sequence in X if for all $t > 0$, if $\lim_{n,m\to\infty} \rho(x_n, x_m, t) = 0$.

(3) (X, ρ, s) is said to be complete if every Cauchy sequence is a convergent sequence.

Example 2.3 : Let $X = [0, +\infty)$ and $\rho : X \times X \times (0, +\infty) \to [0, +\infty)$ defined by $\rho(x, y, t) = t(x - y)^p$. Then ρ is a parametric b-metric with constant $s = 2^p$.

Definition 2.4 : Let (X, ρ, s) be a parametric b-metric space and the mapping $T : X \to X$ is a continuous mapping at x in X, if for any sequence $\{x_n\}_{n=1}^{\infty}$ in X such that $\lim_{n\to\infty} x_n = x$, then

$$\lim_{n\to\infty} Tx_n = Tx.$$

Lemma 2.5 : Let (X, ρ, s) be a b-metric space with the coefficient $s = 1$ and let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X, if $\{x_n\}_{n=1}^{\infty}$ converges to x and also $\{x_n\}_{n=1}^{\infty}$ converges to y, then $x = y$. That is, the limit of $\{x_n\}_{n=1}^{\infty}$ is unique.

Lemma 2.6 : Let (X, ρ, s) be a b-metric space with the coefficient $s = 1$ and let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X. If $\{x_n\}_{n=1}^{\infty}$ converges to x. Then

$$\frac{1}{s}\rho(x, y, t) \leq \lim_{n\to+\infty} \rho(x_n, y, t) \leq s\rho(x, y, t) \;\; \forall y \in X \;\text{ and all }\; t > 0.$$

Lemma 2.7 : Let (X, ρ, s) be a b-metric space with the coefficient $s = 1$ and let $\{x_k\}_{k=0}^{n} \subset X$. Then

$$\rho(x_n, x_0, t) \leq s\rho(x_0, x_1, t) + s^2\rho(x_2, x_3, t) + ... + s^{n-2}\rho(x_{n-2}, x_{n-1}, t) + s^{n-1}\rho(x_{n-1}, x_n, t).$$

Lemma 2.8 : Let (X, ρ, s) be a parametric metric space with the coefficient $s = 1$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of points of X such that

$$\rho(x_n, x_{n+1}, t) \preceq \lambda\rho(x_{n-1}, x_n, t) \;\text{ where }\; \lambda \in [0, \frac{1}{s}) \text{ and } n = 1, 2, ...$$

Then $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in (X, ρ, s).

3. Main Results

Theorem 3.1 : Let (X, ρ, s) be a complete parametric b-metric space and T a continuous mapping satisfying the following condition:

$$\rho(Tx, Ty, t) \geq \beta\frac{\rho(x, Tx, t)\rho(y, Ty, t)}{\rho(x, y, t) + \rho(x, Ty, t)\rho(y, Tx, t)} + \gamma\rho(x, y, t) - \alpha\rho(y, Tx, t)$$

for all $x, y \in X$, $x \neq y$, and for all $t > 0$, where $\alpha, \beta, \gamma \geq 0$ are real constants and $s\beta + \gamma > (1 + \alpha)s + s^2\alpha, \gamma > 1 + \alpha$. Then T has a unique fixed point in X.

Proof : Choose $x_0 \in X$ be arbitrary, to define the iterative sequence $\{x_n\}_{n\in N}$ as follows, $Tx_n = x_{n-1}$ for $n = 1, 2, 3, ...$ Taking $x = x_{n+1}$ and $y = x_{n+2}$ we obtain

$$\begin{aligned}\rho(Tx_{n+1}, Tx_{n+2}, t) \geq \beta&\frac{\rho(x_{n+1}, Tx_{n+1}, t)\rho(x_{n+2}, Tx_{n+2}, t)}{\rho(x_{n+1}, x_{n+2}, t) + \rho(x_{n+1}, Tx_{n+2}, t)\rho(x_{n+2}, Tx_{n+1}, t)} \\ &+\gamma\rho(x_{n+1}, x_{n+2}, t) - \alpha\rho(x_{n+2}, Tx_{n+1}t)\}.\end{aligned}$$

$$\rho(x_n,x_{n+1},t)\geq\beta\frac{\rho(x_{n+1},x_n,t)\rho(x_{n+2},x_{n+1},t)}{\rho(x_{n+1},x_{n+2},t)+\rho(x_{n+1},x_{n+1},t)\rho(x_{n+2},x_n,t)}+\gamma\rho(x_{n+1},x_{n+2},t)-\alpha\rho(x_{n+2},x_n,t).$$

$$\geq\beta\frac{\rho(x_{n+1},x_n,t)\rho(x_{n+1},x_{n+2},t)}{\rho(x_{n+1},x_{n+2},t)+\rho(x_{n+1},x_{n+1},t)\rho(x_{n+2},x_n,t)}+\gamma\rho(x_{n+1},x_{n+2},t)-\alpha\rho(x_{n+2},x_n,t)$$

$$\geq\beta\rho(x_n,x_{n+1},t)+\gamma\rho(x_{n+1},x_{n+2},t)-\alpha\rho(x_n,x_{n+2},t)$$

$$\geq\beta\rho(x_n,x_{n+1},t)+\gamma\rho(x_{n+1},x_{n+2},t)-\alpha s[\rho(x_n,x_{n+1},t)+\rho(x_{n+1},x_{n+2},t)]$$

$$\rho(x_n,x_{n+1},t)\geq\frac{(\gamma-s\alpha)}{(1+s\alpha-\beta)}\rho(x_{n+1},x_{n+2},t)$$

for all $t>0$. The last inequality gives

$$\rho(x_{n+1},x_{n+2},t)\leq\frac{1+s\alpha-\beta}{\gamma-s\alpha}\rho(x_n,x_{n+1},t)=k\rho(x_n,x_{n+1},t)$$

for all $t>0$, where $k=\frac{1+s\alpha-\beta}{\gamma-s\alpha}<\frac{1}{s}$.
Hence by induction, we obtain $\rho(x_{n+1},x_{n+2},t)\leq k^{n+1}\rho(x_0,x_1,t)$
By Lemma 2.8, $\{x_n\}_{n\in N}$ is a Cauchy sequence in X, But X is a complete parametric b-metric space; hence,$\{x_n\}_{n\in N}$ is converges. Call the limit $x^*\in X$. Then, $x_n\to x^*$ as $n\to+\infty$.
By continuity of T we have

$$Tx^*=T\left(\lim_{n\to\infty}x_n\right)=\lim_{n\to\infty}Tx_n=\lim_{n\to\infty}x_{n-1}=x^*$$

That is, $Tx^*=x^*$; thus, T has a fixed point in X.
Uniqueness:
Let y^* be another fixed point of T in X; then $Ty^*=y^*$ and $Tx^*=x^*$. Now,

$$\rho(Tx^*,Ty^*,t)\geq\beta\frac{\rho(x^*,Tx^*,t)\rho(y^*,Ty^*,t)}{\rho(x^*,y^*,t)+\rho(x^*,Ty^*,t)\rho(y^*,Tx^*,t)}+\gamma\rho(x^*,y^*,t)-\alpha\rho(y^*,Tx^*,t)$$

This implies that

$$\rho(x^*,y^*,t)\geq\gamma\rho(x^*,y^*,t)-\alpha\rho(x^*,y^*,t)$$
$$\geq(\gamma-\alpha)\rho(x^*,y^*,t)$$
$$\Rightarrow\rho(x^*,y^*,t)\leq\frac{1}{\gamma-\alpha}\rho(x^*,y^*,t)$$

This is true only when $\rho(x^*,y^*,t)=0$. So $x^*=y^*$,. Hence T has a unique fixed point in X. □

Theorem 3.2 : Let (X, ρ, s) be a complete parametric b-metric space and T a continuous mapping satisfying the following condition:

$$\rho(Tx, Ty, t) \geq \beta \frac{\rho(x, Tx, t)\rho(y, Ty, t)}{\rho(x, y, t)} + \gamma\rho(x, y, t) - \alpha\rho(y, Tx, t)$$

for all $x, y \in X$, $x \neq y$, and for all $t > 0$, where $\alpha, \beta, \gamma \geq 0$ are real constants and $s\beta + \gamma > (1 + \alpha)s + s^2\alpha, \gamma > 1 + \alpha$. Then T has a fixed point in X.

Proof : Choose $x_0 \in X$ be arbitrary, to define the iterative sequence $\{x_n\}_{n \in N}$ as follows, $Tx_n = x_{n-1}$ for $n = 1, 2, 3, ...$ Taking $x = x_{n+1}$ and $y = x_{n+2}$ we obtain

$$\begin{aligned}\rho(Tx_{n+1}, Tx_{n+2}, t) \geq \beta \frac{\rho(x_{n+1}, Tx_{n+1}, t)\rho(x_{n+2}, Tx_{n+2}, t)}{\rho(x_{n+1}, x_{n+2}, t)} &+ \gamma\rho(x_{n+1}, x_{n+2}, t) \\ &-\alpha\rho(x_{n+2}, Tx_{n+1}t)\}\end{aligned}$$

$$\begin{aligned}\rho(x_n, x_{n+1}, t) &\geq \beta \frac{\rho(x_{n+1}, x_n, t)\rho(x_{n+2}, x_{n+1}, t)}{\rho(x_{n+1}, x_{n+2}, t)} + \gamma\rho(x_{n+1}, x_{n+2}, t) - \alpha\rho(x_{n+2}, x_n, t) \\ &\geq \beta \frac{\rho(x_{n+1}, x_n, t)\rho(x_{n+1}, x_{n+2}, t)}{\rho(x_{n+1}, x_{n+2}, t)} + \gamma\rho(x_{n+1}, x_{n+2}, t) - \alpha\rho(x_{n+2}, x_n, t) \\ &\geq \beta\rho(x_n, x_{n+1}, t) + \gamma\rho(x_{n+1}, x_{n+2}, t) - \alpha\rho(x_n, x_{n+2}, t) \\ &\geq \beta\rho(x_n, x_{n+1}, t) + \gamma\rho(x_{n+1}, x_{n+2}, t) - \alpha s[\rho(x_n, x_{n+1}, t) + \rho(x_{n+1}, x_{n+2}, t)] \\ \rho(x_n, x_{n+1}, t) &\geq \frac{(\gamma - s\alpha)}{(1 + s\alpha - \beta)}\rho(x_{n+1}, x_{n+2}, t)\end{aligned}$$

for all $t > 0$. The last inequality gives

$$\rho(x_{n+1}, x_{n+2}, t) \leq \frac{1 + s\alpha - \beta}{\gamma - s\alpha}\rho(x_n, x_{n+1}, t) = k\rho(x_n, x_{n+1}, t)$$

for all $t > 0$, where $k = \frac{1+s\alpha-\beta}{\gamma-s\alpha} < \frac{1}{s}$. Hence by induction, we obtain

$$\rho(x_{n+1}, x_{n+2}, t) \leq k^{n+1}\rho(x_0, x_1, t)$$

By Lemma 2.8, $\{x_n\}_{n \in N}$ is a Cauchy sequence in X, But X is a complete parametric b-metric space; hence,$\{x_n\}_{n \in N}$ is converges. Call the limit $x^* \in X$. Then, $x_n \to x^*$ as $n \to +\infty$. By continuity of T we have

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$$\rho(Tx^*, Ty^*, t) \geq \beta \frac{\rho(x^*, Tx^*, t)\rho(y^*, Ty^*, t)}{\rho(x^*, y^*, t)} + \gamma\rho(x^*, y^*t) - \alpha\rho(y^*, Tx^*, t)$$

This implies that

$$\begin{aligned} \rho(x^*, y^*, t) &\geq \gamma\rho(x^*, y^*, t) - \alpha\rho(x^*, y^*, t) \\ \rho(x^*, y^*, t) &\geq (\gamma - \alpha)\rho(x^*, y^*, t) \\ \Rightarrow \rho(x^*, y^*, t) &\leq \frac{1}{\gamma - \alpha}\rho(x^*, y^*, t) \end{aligned}$$

This is true only when $\rho(x^*, y^*, t) = 0$. So $x^* = y^*$. Hence T has a unique fixed point in X. □

Theorem 3.4 : Let (X, ρ, s) be a complete parametric b-metric space and T a continuous mapping satisfying the following condition

$$\rho(Tx, Ty, t) \geq \beta \frac{\rho(x, Tx, t)[\delta + \rho(y, Ty, t)]}{\delta + \rho(x, y, t)} + \gamma\rho(x, y, t) - \alpha \min\{\rho(x, Ty, t), \rho(y, Tx, t)\}$$

for all $x, y \in X$, $x \neq y$, and for all $\delta, t > 0$, where $\alpha, \beta, \gamma \geq 0$ are real constants and $s\beta + \gamma > (1 + \alpha)s + s^2\alpha, \gamma > 1 + \alpha$. Then T has a unique fixed point in X.

Proof : Choose $x_0 \in X$ be arbitrary, to define the iterative sequence $\{x_n\}_{n\in N}$ as follows, $Tx_n = x_{n-1}$ for $n = 1, 2, 3, ...$ Taking $x = x_{n+1}$ and $y = x_{n+2}$ we obtain

$$\begin{aligned} \rho(Tx_{n+1}, Tx_{n+2}, t) &\geq \beta \frac{\rho(x_{n+1}, Tx_{n+1}, t)[\delta + \rho(x_{n+2}, Tx_{n+2}, t)]}{\delta + \rho(x_{n+1}, x_{n+2}, t)} + \gamma\rho(x_{n+1}, x_{n+2}, t) \\ &- \alpha \min\{\rho(x_{n+1}, Tx_{n+2}, t), \rho(x_{n+2}, Tx_{n+1}t)\} \\ \rho(x_n, x_{n+1}, t) &\geq \beta \frac{\rho(x_{n+1}, x_n, t)[\delta + \rho(x_{n+2}, x_{n+1}, t)]}{\delta + \rho(x_{n+1}, x_{n+2}, t)} + \gamma\rho(x_{n+1}, x_{n+2}, t) \\ &- \alpha \min\{\rho(x_{n+1}, x_{n+1}, t), \rho(x_{n+2}, x_n, t)\} \\ &\geq \beta \frac{\rho(x_{n+1}, x_n, t)[\delta + \rho(x_{n+1}, x_{n+2}, t)]}{\delta + \rho(x_{n+1}, x_{n+2}, t)} + \gamma\rho(x_{n+1}, x_{n+2}, t) \\ &- \alpha \min\{\rho(x_{n+1}, x_{n+1}, t), \rho(x_{n+2}, x_n, t)\} \\ &\geq \beta\rho(x_n, x_{n+1}, t) + \gamma\rho(x_{n+1}, x_{n+2}, t) - \alpha\rho(x_n, x_{n+2}, t) \\ &\geq \beta\rho(x_n, x_{n+1}, t) + \gamma\rho(x_{n+1}, x_{n+2}, t) - \alpha s[\rho(x_n, x_{n+1}, t) + \rho(x_{n+1}, x_{n+2}, t)] \\ &\geq \frac{(\gamma - s\alpha)}{(1 + s\alpha - \beta)}\rho(x_{n+1}, x_{n+2}, t) \end{aligned}$$

for all $t > 0$. The last inequality gives

$$\rho(x_{n+1}, x_{n+2}, t) \leq \frac{1 + s\alpha - \beta}{\gamma - s\alpha}\rho(x_n, x_{n+1}, t) = k\rho(x_n, x_{n+1}, t)$$

for all $t > 0$, where $k = \frac{1+s\alpha-\beta}{\gamma-s\alpha} < \frac{1}{s}$. Hence by induction, we obtain

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$$\begin{aligned}\rho(Tx^*, Ty^*, t) \geq \beta\frac{\rho(x^*, Tx^*, t)[\delta + \rho(y^*, Ty^*, t)]}{\delta + \rho(x^*, y^*, t)} + \gamma\rho(x^*, y^*t)\\ - \alpha\min\{\rho(x^*, Ty^*, t), \rho(y^*, Tx^*, t)\}\end{aligned}$$

This implies that

$$\begin{aligned}\rho(x^*, y^*, t) &\geq \gamma\rho(x^*, y^*, t) - \alpha\rho(x^*, y^*, t)\\ \rho(x^*, y^*, t) &\geq (\gamma - \alpha)\rho(x^*, y^*, t)\\ \Rightarrow \rho(x^*, y^*, t) &\leq \frac{1}{\gamma - \alpha}\rho(x^*, y^*, t)\end{aligned}$$

This is true only when $\rho(x^*, y^*, t) = 0$. So $x^* = y^*$. Hence T has a unique fixed point in X. □

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