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TOPOLOGICAL ENTROPY FOR MODIFIED DUFFING MAP

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Abstract

In this work, we study the general properties of modified Duffing map in the form $F^1_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -bx + ay - \tan y \end{pmatrix}$. We prove it has positive Lyapunov exponent if $|b| = 1$, but it has positive Lyapunov exponent if $b > 0$ with some conditions on x. So we give estimate of topological entropy of modified Duffing map. Finally, we prove it is disital if $|b| > 1$ and proximal if $|b| < 1$.

1. Introduction

Duffing map has many forms, in [4] they study Duffing map with form $F_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -bx + ay - y^3 \end{pmatrix}$. In this work, we changed this form of Duffing map to modified Duffing map by replacing y^3 by tan y and we denoted this modified Duffing map by $F^1_{a,b}$. We study two important chaotic properties of it, one of them is the topological entropy, and the other one is Lyapunov exponents.

During the last two decades dynamical systems focused its study on important affair which is chaotic behavior. Virtually, some types of random behavior has been indicated by different systems that idiom chaos was stratified in their [6].

Key Words : *Lyapunov exponent, Topological entropy, Duffing map.*

In 2008, Newhous, Berz, Grot and Makino found that a quantitative measure of orbit complexity is gave by an important invariant which is topological entropy of a discrete dynamical system. However this has several definitions but, for example, in symbolic system it is not easy to compute or estimate , and then algorithms were developed for the estimate of one dimensional [7].

Let $F : X \to X$ be a continuous differential map of a compact metric space X, for $\epsilon > 0$ and $n \in \mathbb{Z}_+$, we say $E \subset X$ is an (n, ϵ) - separated set if for every $x, y \in E$, then exists i.e. $0 \leq i < n$, such that $d(F^i(x), F^i(y)) > \epsilon$ then the topological entropy of F, denoted by $h_{top}(F)$ is defined to be $h_{top}(F) = \lim_{\epsilon \to 0} \lim_{n \to \infty} \sup \frac{1}{n} \log r(n, \epsilon)$ where $r(n, \epsilon)$ is the maximum cardinally of all $r(n, \epsilon)$-separated sets [6]. Purely probabilistic concept measure the growth in randomness or number of typical orbits it is said metric entropy, while the rates, at which nearly orbits diverge, are measured by Lyapunov exponents which is primarily geometric. [7]

The Lyapunov exponent was defined of a map F at x (all x in X in direction v) by $L^{\pm}(x, v) = \lim_{n \to \infty} \frac{1}{n} ln \|DF_x^n v\|$ whenever the limit exists. In higher dimensions, for example in $\mathbb{R}^n$ the map F will have n Lyapunov exponents, say $L_1^{\pm}(x, v_1), L_2^{\pm}(x, v_2), \cdots, L_n^{\pm}(x, v_n)$ for a minimum n Lyapunov exponent that is $L^{\pm}(x, v) = \max\{L_1^{\pm}(x, v_1), L_2^{\pm}(x, v_2), \cdots,$ $L_n^{\pm}(x, v_n)\}$, where $v = (v_1, v_2, \cdots, v_n)$ [6].

Let $F^1 : \mathbb{R}^2 \to \mathbb{R}^2$ be a modified Duffing map. We say F^1 is C^∞, if its mixed k-th partial derivatives exist and are continuous for all $k \in \mathbb{Z}_+$ and it is called a diffeomorphism if it is one-to-one, onto, C^∞ and its inverse is C^∞.

Let V be a subset of $\mathbb{R}^2$ and v_0 be any element in $\mathbb{R}^2$. Consider $F : V \to \mathbb{R}^2$ be a map. Furthermore assume that the first partials of the coordinate maps f and g of F exist at v_0. The differential of F at v_0 is the linear map, then $DF(v_0)$ defined on $\mathbb{R}^2$ by $DF(v_0) = \begin{pmatrix} \frac{\partial f}{\partial x}(v_0) & \frac{\partial f}{\partial y}(v_0) \\ \\ \frac{\partial g}{\partial x}(v_0) & \frac{\partial g}{\partial y}(v_0) \end{pmatrix}$, for all v_0 in $\mathbb{R}^2$. The determinant of $DF(v_0)$ is called the Jacobian of F at v_0 and it is denoted by $JF(v_0) = det\ DF(v_0)$, so F is said to be area-contracting at v_0 if $|det\ DF(v_0)| < 1$ and F is said to be area-expanding at v_0 if $|det\ DF(v_0)| > 1$. Let A be an $n \times n$ matrix, the real number λ is called eigenvalue of A if there exists a non zero vector X in $\mathbb{R}^n$ such that $AX = \lambda X$, every non zero vector X satisfying this equation is called an eigen vector of A associated with the eigenvalue λ. The fixed point, which defined as : $\begin{pmatrix} x \\ y \end{pmatrix}$ is called fixed point if $F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -bx + ay - \tan y \end{pmatrix}$. A matrix $A \in GL(2, \mathbb{Z})$ with $Det(A) = \pm 1$ is called hyperbolic matrix if $|\lambda_1| > 1 > |\lambda_2|$ where λ_1 and λ_2 are the eigenvalues of A [1].

2. General Properties

In this work, we changed this form $F_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -bx + ay - y^3 \end{pmatrix}$ of Duffing map to modified Duffing map by replacing y^3 by $\tan y$ and we denoted this modified Duffing map by $F^1_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -bx + ay - \tan y \end{pmatrix}$.

In other words, the Duffing map to modified Duffing map have different properties of dynamic qualities but they are similar in their characteristics chaotic properties. In this section,we study general properties of modified Duffing map.

Proposition 2.1 : Let $F^1_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -bx + ay - \tan y \end{pmatrix}$ be a modified Duffing map, then the Jacobian of $F^1_{a,b}$ is b.

Proof : $DF^1_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -b & a - sec^2\ y \end{pmatrix}$ so

$$JF^1_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} = det\begin{pmatrix} 0 & 1 \\ -b & a - sec^2\ y \end{pmatrix} = b$$ □

Remark 2.2 : If $|b| < 1$ then $F^1_{a,b}$ is area-contracting and it is area-expanding if $|b| > 1$.

Proof : If $|b| < 1$ then by Proposition 2.1, $\left|det\left(DF^1_{ab}\begin{pmatrix} x \\ y \end{pmatrix}\right)\right| < 1$ that is $F^1_{a,b}$ is area-contracting and $|b| > 1$ then $\left|det\left(DF^1_{ab}\begin{pmatrix} x \\ y \end{pmatrix}\right)\right| > 1$ this implies that $F^1_{a,b}$ is area-expanding. □

Proposition 2.3 : Let $F^1_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -bx + ay - \tan y \end{pmatrix}$ be a modified Duffing map, for all $a, b \in \mathbb{R}$ and $b = a - 2$, it has one fixed point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ only.

Proof : By definition of fixed point $F^1_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -bx + ay - \tan y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$ then $y = x$ and $-bx + ay - \tan y = y$, so $\tan x = (-b + a - 1)x$, since $b = a - 2$ then $x = 0$, so $y = 0$ therefore $P = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ the only fixed point. □

Proposition 2.4 : If $b \leq \left(\frac{a+\sec^2 y}{2}\right)^2$ then the eigenvalues of $DF^1_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}$ are $\lambda_\pm = \frac{(a+\sec^2 y)\pm\sqrt{(a+\sec^2 y)^2-4b}}{2}$.

Proof : If λ is the eigenvalue of $DF^1_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}$ then it must be satisfied the characteristic equation $det\begin{pmatrix} \lambda & -1 \\ b & \lambda - a - \sec^2\ y \end{pmatrix} = 0$ then $\lambda^2 - a\lambda - \lambda\ \sec^2 y + b = 0$ and the solution of this equation is $\lambda_\pm = \frac{(a+\sec^2\ y)\pm\sqrt{(a+\sec^2\ y)^2-4b}}{2}$. □

Remark 2.5 :

1. It is clear that if $|a+\sec^2\ y| > 2\sqrt{b}, b > 0$ then the eigenvalue of $DF^1_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}$ are real.
2. The eigenvalues of $DF^1_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}$ at fixed point $P = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ are $\lambda_0 = \frac{a\pm\sqrt{a^2-4b}}{2}$.

Proposition 2.6 : $DF^1_{a,b}$ is non hyperbolic matrix if $\sec y = \sqrt{2-a}$ and $b = 1$. Also it is non hyperbolic if $\sec\ y = \pm\sqrt{-a}$ and $b = -1$.

Proof : if $\sec\ y = \sqrt{2-a}$ then $1 - a - \sec^2\ y = -1$so $4 - 4(a+\sec^2\ y) + (a+\sec^2\ y)^2 = (a+\sec^2\ y)^2 - 4$ that means $[-2+(a+\sec^2\ y)]^2 = (a+\sec^2 y)^2 - 4$ then $\frac{(a+\sec^2\ y)\pm\sqrt{(a+\sec^2 y)^2-4b}}{2} = 1$ hence $\lambda = 1$. By following the same steps we get $\lambda = 1$ if $\sec\ y = \pm\sqrt{-a}$ and $b = -1$. □

Proposition 2.7 : If $b \neq 0$ then $F^1_{a,b}$ is one-to-one.

Proof : Let $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in \mathbb{R}^2$ such that $F^1_{a,b}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = F^1_{a,b}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ then $\begin{pmatrix} y_1 \\ -bx_1 + ay_1 - \tan y_1 \end{pmatrix} = \begin{pmatrix} y_2 \\ -bx_2 + ay_2 - \tan y_2 \end{pmatrix}$ so $y_1 = y_2$ and $-bx_1 + ay_1 - \tan y_1 = -bx_2 + ay_2 - \tan y_2$ hence $-bx_1 = -bx_2$ and $b \neq 0$ so $x_1 = x_2$, that is $F^1_{a,b}$ is one-to-one. □

Remark 2.8 : If $b = 0$ then $F^1_{a,0}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ ay - \tan y \end{pmatrix}$, so $ker(F^1_{a,0}) = \left\{\begin{pmatrix} x \\ y \end{pmatrix} : x \in \mathbb{R}\right\}$ then $F^1_{a,0}$ is not one-to-one, that is $F^1_{a,0}$ is not diffeomorphism.

Proposition 2.9 : If $\cos x \neq 0$ then $F^1_{a,b}$ is C^∞.

Proof : All first partial derivatives exist and continuous. Note that $\frac{\partial^n f}{\partial x^n} = 0 \; \forall \;\; n \in \mathbb{N}$ and $\frac{\partial^n f}{\partial y^n} = 0, \frac{\partial^n g}{\partial x^n} = 0 \;\; \forall \;\; n \geq 2$, since $\cos x \neq 0$ then $\frac{\partial^n g}{\partial y^n}$ exists and continuous. □

Proposition 2.10 : If $b \neq 0$ then $F^1_{a,b}$ is onto.

Proof : let $\begin{pmatrix} v \\ w \end{pmatrix}$ any element in $\mathbb{R}^2$ such that $y = v$ and $x = \frac{w+\tan y - av}{-b}$ then there exists $\begin{pmatrix} \frac{w+\tan y-av}{-b} \\ v \end{pmatrix} \in \mathbb{R}^2 \ni F^1_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} v \\ w \end{pmatrix}$ then $F^1_{a,b}$ is onto. □

Remark 2.11 : Since $F^1_{a,b}$ is one-to-one and onto from Propositions 2.7 and 2.10, then $F^1_{a,b}$ is invertible.

Proposition 2.12 : For all $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$, $\sec y \neq \pm\sqrt{2-a}$, $\sec y \neq \pm\sqrt{-a}$ and $|a| > 2\sqrt{b}$, $DF^1_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}$ is hyperbolic matrix if and only if $|b| = 1$.

Proof : $\Rightarrow$ Since $DF^1_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} \in GL(2, \mathbb{Z})$ and $det\left(DF^1_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}\right) = |b|$, then $b = 1$ or $b = -1$.

$\Leftarrow$ Let $b = 1$, then since $det\left(DF^1_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}\right) = b$ we have $\lambda_1\lambda_2 = b$ that is $\lambda_1\lambda_2 = 1$ then $\lambda_1 = \frac{1}{\lambda_2}$. If $\lambda_1 > 1$ therefore $\lambda_2 < 1$ or $\lambda_1 < 1$ so $\lambda_2 > 1$. So $DF^1_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}$ is hyperbolic matrix. □

3. Topological Entropy

Topological entropy is the exponential growth rate of the number of essentially different orbit segments of the length n. The topological entropy is a property of F alone and is not associated with any metric properties of the dynamics [6].

In this section we recall two important theorems of topological entropy.

We recall the theorem (3.25) on [5] by

Theorem 3.1 : Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous map then $h_{top}(F) \geq \log|\lambda|$ where λ is the largest eigenvalue of $DF(v)$ where $v \in \mathbb{R}^2$.

Proposition 3.2 : If $a > 0$ and $a^2 > 4b$ then $h_{top}(F^1_{a,b}) > \frac{1}{2}\log(a^2 - 4b) - \log 2$.

Proof : since $a^2 > 4b$ then $\sqrt{a^2 - 4b} > 0$. Since $a > 0$ then $a + \sqrt{a^2 - 4b} > \sqrt{a^2 - 4b}$ so $a + \sec^2 y\sqrt{a^2 + 2a\sec^2 y + (\sec^2 y)^2 - 4b} > \sqrt{a^2 - 4b}$. Therefore $\log|\lambda_1| > \log\left(\frac{\sqrt{a^2-4b}}{2}\right)$, then $h_{top}(F^1_{a,b}) > \frac{1}{2}\log(a^2 - 4b) - \log 2$. □

We recall the theorem (3.35) in (2) by

Theorem 3.3: Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be $h_{top}(F) \leq \log \max\limits_{x\in\mathbb{R}^2} \max\limits_{L\subset T_X\mathbb{R}^2} |det(DF(x))|L|$.

Proposition 3.4 : The upper estimate of topological entropy of $F^1_{a,b}$ is $h_{top}(F^1_{a,b}) \leq \log |b|$.

Proof : by Theorem 3.3 we get $h_{top}(F^1_{a,b} \leq \log \max\limits_{x\in\mathbb{R}^2} \max\limits_{L\subset T_X\mathbb{R}^2} |det(DF^1_{a,b}(x))|L|$
$\leq \log \max\limits_{x\in\mathbb{R}^n} \max\limits_{L\in\mathbb{R}^n} |b| \log |b|$. □

4. Lyapunov Exponent

The Lyapunov exponents give the average exponential rate of divergence or convergence of nearly orbits in the phase-space. In systems exhibiting exponential orbital divergence, small initial differences which we may not be able to resolve get magnified rapidly leading to less of predictability any system containing at least one positive Lyapunov exponent is defined to be chaotic, with the magnitude of the exponent reflecting the time scale on which system dynamics become unpredictable [3].

In this section we prove that F has positive Lyapunov exponent.

Proposition 4.1 : For all $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$, if $|b| = 1$ and either $\sec y \neq \pm\sqrt{2-a}$ or $\sec y \neq \pm\sqrt{-a}$ then $F^1_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}$ has positive Lypunov exponent.

Proof : Let $X = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$, the Lyapunov exponent of $F^1_{a,b}$ is given by the formula $L_1\left(\begin{pmatrix} x \\ y \end{pmatrix}, v_1\right) = \lim\limits_{n\to\infty} \frac{1}{n} ln \left\| DF^n_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} v_1 \right\|$. From Proposition 2.12 $|\lambda_1| = \frac{1}{|\lambda_2|}$ where $\sec y \neq \pm\sqrt{2-a}$, if $|\lambda_1| < 1$ then
$L_1\left(\begin{pmatrix} x \\ y \end{pmatrix}, v_1\right) = \lim\limits_{n\to\infty} \frac{1}{n} ln \left\| \left(DF_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} v_1 \right)^n \right\| > ln \left\| \frac{(a+\sec^2 y)\pm\sqrt{(a+\sec^2 y)^2-4b}}{2} \right\},$
by hypothesis $L_1 > 0$ so if $|\lambda_1| > 1$
then $L_2\left(\begin{pmatrix} x \\ y \end{pmatrix}, v_2\right) = \lim\limits_{n\to\infty} \frac{1}{n} ln \left\| \left(DF_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} v_2 \right)^n \right\| < ln \left\| \frac{(a+\sec^2 y)\pm\sqrt{(a+\sec^2 y)^2-4b}}{2} \right\},$
thus the Lyapunov exponent $L(x, v) = \max\{L_1(x, v_1), L_2(x, v_2)\}$ hence the Lyapunov exponent of $F^1_{a,b}$ is positive. Also where $\sec y \neq \pm\sqrt{-a}$ we get positive Lyapunov exponent by follow the same steps. □

Proposition 4.2 : If $a > |1-\sec^2 y+b|$ and $b > 0$ then $F^1_{a,b}\begin{pmatrix} x \\ y \end{pmatrix}$ has positive Lypunov exponent.

Proof : If $a > |1-\sec^2 y+b|$, since $\sec^2 y > 0$ then $-a-2\sec^2 y < 1-\sec^2 y+b < a$ so $4+4(a+\sec^2 y)+(a+\sec^2 y)^2 > (a+\sec^2 y)^2-4b > 4-4(a+\sec^2 y)+(a+\sec^2 y)^2$ that means

$-2 > a + \sec^2 y \mp \sqrt{(a+\sec^2 y)^2 - 4b} > 2$, then $|\lambda_{1,2}| > 1$. Let $X = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$, the Lyapunov exponent of $F^1_{a,b}$ is given by $L_1\left(\begin{pmatrix} x \\ y \end{pmatrix}, v_1\right) = \lim_{n\to\infty} \frac{1}{n} ln \left\| DF^1_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} v_1 \right\|$, sicne $|\lambda_{1,2}| > 1$. Then $L_1\left(\begin{pmatrix} x \\ y \end{pmatrix}, v_1\right) = \lim_{n\to\infty} \frac{1}{n} ln \left\| DF^1_{a,b}\begin{pmatrix} x \\ y \end{pmatrix} v_1 \right\| > ln\ 1 = 0$ that is the Lyapunov exponent $L(x,v) = \max\{L_1(x,v_1), L_2(x,v_2)\}$ is positive. □

5. Disital and Proximal

In this section, we describe two properties related to the asymptotic behavior of the distance between corresponding points on pairs of orbits.

A homeomorphism $F : X \to X$ is distal if every pair of distinct points $x, y \in X$ are distal. If (X, d) is a compact metric space, then $x, y \in X$ are proximal if there is a sequence $n_k \in \mathbb{Z}$ such that $d(F^{n_K}(x), F^{n_k}(y)) \to 0$ as $k \to \infty$. If two points x and y are not proximal, they are called disital [4].

Proposition 5.1 : $F^1_{a,b}$ is distal if $|b| > 1$.

Proof : Since $|b| > 1$ then by Remark 2.3 $F^1_{a,b}$ has area-expanding, that is, for all $X, Y \in \mathbb{R}^2, d(F^n_{a,b}(X), F^n_{a,b}(Y)) > \epsilon$ for some $\epsilon > 0, n \in \mathbb{Z}$. By definition of distal points, all points X, Y are distal. Thus $F^1_{a,b}$ is distal. □

Proposition 5.2 : $F^1_{a,b}$ is proximal if $|b| < 1$.

Proof : Since $|b| < 1$ then by Remark 2.3 $F^1_{a,b}$ has area-contracting, that is, for sequence $n_k \in \mathbb{Z}$ and for all $X, Y \in \mathbb{R}^2, d(F^n_{a,b}(X), F^n_{a,b}(Y)) \to 0$ as $k \to \infty$. By definition of proximal points, all points X, Y are proximal. Thus $F^1_{a,b}$ is proximal. □

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