International J. of Math. Sci. & Engg. Appls. (IJMSEA)
ISSN 0973-9424, Vol. 11 No. II (August, 2017), pp. 99-110

CONVOLUTION IDENTITIES INVOLVING FIXED POWER OF EXPANDING VARIABLE, HYBRID FIBONACCI AND LUCAS POLYNOMIALS

R. RANGARAJAN[1], C. K. HONNEGOWDA[2] AND RANGASWAMY[3]
[1,2] Department of Studies in Mathematics,
University of Mysore, Manasagangotri, Mysuru - 570 006, India
[3] Department of Mathematics, B.M.S. College of Engineering,
Bull Temple Road, Basavanagudi, Bengaluru - 560 019, India

Abstract

Recently, hybrid Fibonacci polynomials in two variables are defined in [8] which contains Fibonacci numbers and two types of Fibonacci polynomials in one variable, namely, Catalan polynomials and Jacobsthal polynomials as special cases and which exhibits many interesting combinatorial properties useful for research workers in combinatorics. In a similar way, hybrid Lucas polynomials in two variables are defined in [9] corresponding to the hybrid Fibonacci polynomials in two variables. In the present paper, Convolution Identities of hybrid Fibonacci and Lucas polynomials in two variables with a fixed power of expanding variable is stated and proved up to third degree.

1. Introduction

The simple and natural recurrence relation with three terms is $F_{n+1} = F_n + F_{n-1}$,

---

Key Words : *Combinatorial identities, Fibonacci and Lucas numbers and Convolution identities.*

AMS Subject Classification : 05A19, 11B33 and 30B70.

$F_0 = 0,\ F_1 = 1, n = 1, 2, 3, \dots$ given by Fibonacci numbers.

Closely connected to this is one more such relation $L_{n+1} = L_n + L_{n-1},\ L_0 = 2,\ L_1 = 1,$ $n = 1, 2, 3, \dots$ given by Lucas numbers [1, 2, 3, 12].

Motivated by the above relations, a twin pair of sequences $\{l_n^{(C)}(x), f_n^{(C)}(x)\}$ called Catalan Lucas and Fibonacci polynomials and $\{l_n^{(J)}(x), f_n^{(J)}(x)\}$ called Jacobsthal Lucas and Fibonacci polynomials are defined as follows in the literature [3]:

$$l_{n+1}^{(C)}(x) = x l_n^{(C)}(x) + l_{n-1}^{(C)}(x), \text{ with } l_0^{(C)}(x) = 2,\ l_1^{(C)}(x) = x, n = 1, 2, 3, \dots\ ;$$

$$f_{n+1}^{(C)}(x) = x f_n^{(C)}(x) + f_{n-1}^{(C)}(x), \text{ with } f_0^{(C)}(x) = 0,\ f_1^{(C)}(x) = 1, n = 1, 2, 3, \dots\ ;$$

$$l_{n+1}^{(J)}(y) = l_n^{(J)}(x) + y l_{n-1}^{(J)}(x), \text{ with } l_0^{(J)}(y) = 2,\ l_1^{(J)}(y) = 1, n = 1, 2, 3, \dots\ ;$$

and $$f_{n+1}^{(J)}(y) = f_n^{(J)}(x) + y f_{n-1}^{(J)}(y), \text{ with } f_0^{(J)}(Y) = 0,\ f_1^{(J)}(y) = 1, n = 1, 2, 3, \dots\ .$$

The following three term recurrence relations of hybrid Fibonacci and Lucas polynomials [8, 9] are useful for working out convolution identities.

$$f_{n+1}^{(H)}(x, y) = x f_n^{(H)}(x, y) + y f_{n-1}^{(H)}(x, y), \tag{1.1}$$

$$l_{n+1}^{(H)}(x, y) = x\ l_n^{(H)}(x, y) + y\ l_{n-1}^{(H)}(x, y) \tag{1.2}$$

and

$$l_n^{(H)}(x, y) = x\ f_n^{(H)}(x, y) + 2y\ f_{n-1}^{(H)}(x, y). \tag{1.3}$$

The generalized hybrid Fibonacci and Lucas polynomials in two variables x and y of degree n, are given by the following binet forms:

$$f_n^{(H)}(x, y) = \frac{1}{\sqrt{x^2 + 4y}} \left[\Big(\frac{x + \sqrt{x^2 + 4y}}{2} \Big)^n - \Big(\frac{x - \sqrt{x^2 + 4y}}{2} \Big)^n \right].$$

$$l_n^{(H)}(x, y) = \left[\Big(\frac{x + \sqrt{x^2 + 4y}}{2} \Big)^n + \Big(\frac{x - \sqrt{x^2 + 4y}}{2} \Big)^n \right].$$

Put $\alpha = \Big(\frac{x + \sqrt{x^2 + 4y}}{2} \Big)$ and $\beta = \Big(\frac{x - \sqrt{x^2 + 4y}}{2} \Big)$. Then

$$f_n^{(H)}(x, y) = \frac{\alpha^n - \beta^n}{\alpha - \beta}\ , \qquad l_n^{(H)}(x, y) = \alpha^n + \beta^n. \tag{1.4}$$

Also,

$$\alpha+\beta = x, \quad \alpha-\beta = \sqrt{x^2+4y} \quad \text{and} \quad \alpha\beta = -y. \tag{1.5}$$

Hybrid Fibonacci and Lucas polynomials in two variables exhibit many interesting combinatorial properties useful for research workers in combinatorics [1, 2, 3, 10, 12]. They are also related to Tchebyshev polynomials [6, 7, 8, 9, 11].
In the next section, Convolution Identities of hybrid Fibonacci and Lucas polynomials in two variables with a fixed power of expanding variable, $k^m, m = 0\ ,\ 1$ are stated and proved. In the last section, Convolution Identities of hybrid Fibonacci and Lucas polynomials in two variables with a fixed power of expanding variable, $k^m, m = 2\ ,\ 3$ are stated and proved.

2. Convolution Identities with a Fixed Power of Expanding Variable

One of the remarkable identities is the following well known Bernoulli's identity [1] : If $S_n(m) = \sum_{k=1}^{n} k^m$, then

$$\binom{m}{1}S_n(m-1)+\binom{m}{2}S_n(m-2)+\cdots+\binom{m}{1}S_n(1)+S_n(0)=\Big[(n+1)^m-1^m\Big].$$

Put $\sigma_n(m,x) = \sum_{k=1}^{n} k^m x^k$. By a simple manipulation $k^m = \Big((k-1)+1\Big)^m$ and a binomial expansion readily gives

$$\begin{aligned}\sigma_n(m,x) &= x\Big[\sigma_n(m,x)+\binom{m}{1}\sigma_n(m-1,x)+\binom{m}{2}\sigma_n(m-2,x)+\cdots+\binom{m}{1}\sigma_n(1,x)\\ &+ \sigma_n(0,x)\Big]-\Big[(n+1)^m-1^m\Big].\end{aligned}$$

Hence we arrive at a beautiful generalized Bernoulli's identity:

$$\begin{aligned}(x-1)\sigma_n(m,x) &+ \binom{m}{1}x\sigma_n(m-1,x)+\binom{m}{2}x\sigma_n(m-2,x)\\ &+\cdots+\binom{m}{1}x\sigma_n(1,x)+\sigma_n(0,x)\\ &= x^{n+1}\Big[(n+1)^m-1^m\Big].\end{aligned}$$

For $x = 1$, we get back Bernoulli's identity [1].

Table 1

m	$\sigma_n(m,x)$
0	$\dfrac{x^{n+1}-x}{x-1}$
1	$\dfrac{nx^{n+2}-(n+1)x^{n+1}+x}{(x-1)^2}$
2	$\dfrac{1}{(x-1)^3}[n^2x^{n+3}-(2n^2+2n-1)x^{n+2}+(n+1)^2x^{n+1}-x^2-x]$
3	$\dfrac{1}{(x-1)^4}[n^3x^{n+4}+a_{31}x^{n+3}+a_{32}x^{n+2}+a_{33}x^{n+1}+x^3+4x^2+x]$

where $a_{31}=(-3n^3-3n^2+3n-1)$, $a_{32}=(3n^3+6n^2-4)$ and $a_{33}=(-n^3-3n^2-3n-1)$. One can apply generalized Bernoulli's identity to compute for any value of m. But in the present paper we work out only up to 3. The Convolution identities are stated and proved in this section for $m=0,1$ and continued in the next section for $m=2,3$. In a different approach, Convolution identities at levels $m=1,2$ are available in the literature [4, 5].

Theorem 1 : The convolution identities at the level $m = 0$ are

$$(1.1)\quad \sum_{k=1}^{n} f_k^{(H)}(x,y)f_{n-k}^{(H)}(x,y) = \frac{n\ l_n^{(H)}(x,y)-xf_n^{(H)}(x,y)}{(x^2+4y)}$$

$$(1.2)\quad \sum_{k=1}^{n} l_k^{(H)}(x,y)l_{n-k}^{(H)}(x,y) = n\ l_n^{(H)}(x,y)+xf_n^{(H)}(x,y)$$

$$(1.3)\quad \sum_{k=1}^{n} l_k^{(H)}(x,y)f_{n-k}^{(H)}(x,y) = (n-1)f_n^{(H)}(x,y)$$

$$(1.4)\quad \sum_{k=1}^{n} f_k^{(H)}(x,y)l_{n-k}^{(H)}(x,y) = (n+1)f_n^{(H)}(x,y)$$

Proof :

$$(1.1):\quad \sum_{k=1}^{n} f_k^{(H)}(x,y)f_{n-k}^{(H)}(x,y) = \sum_{k=1}^{n}\left(\frac{\alpha^k-\beta^k}{\alpha-\beta}\right)\left(\frac{\alpha^{n-k}-\beta^{n-k}}{\alpha-\beta}\right)$$

$$= \frac{1}{(\alpha-\beta)^2}\Big[\sum_{k=1}^{n}(\alpha^n+\beta^n)-\alpha^n\sum_{k=1}^{n}\left(\frac{\beta}{\alpha}\right)^k-\beta^n\sum_{k=1}^{n}\left(\frac{\alpha}{\beta}\right)^k\Big]$$

$$= \frac{1}{(x^2+4y)}\Big[n(\alpha^n+\beta^n)) - \alpha^n\frac{\beta}{\alpha}\left(\frac{1-(\frac{\beta}{\alpha})^n}{1-(\frac{\beta}{\alpha})}\right) - \beta^n\frac{\alpha}{\beta}\left(\frac{1-(\frac{\alpha}{\beta})^n}{1-(\frac{\alpha}{\beta})}\right)\Big]$$

(by using table1, $m=0$)

$$= \frac{1}{(x^2+4y)}\Big[n(\alpha^n+\beta^n)) - \left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)(\alpha+\beta)\Big]$$

$$= \frac{nl_n^{(H)}(x,y) - xf_n^{(H)}(x,y)}{(x^2+4y)}$$

(by using (1.4) and (1.5)).

The proof of (1.2) is similar to that of (1.1).

$$(1.3): \quad \sum_{k=1}^{n} l_k^{(H)}(x,y) f_{n-k}^{(H)}(x,y) = \sum_{k=1}^{n}(\alpha^k+\beta^k)\left(\frac{\alpha^{n-k}-\beta^{n-k}}{\alpha-\beta}\right)$$

$$= \frac{1}{(\alpha-\beta)}\Big[\sum_{k=1}^{n}(\alpha^n-\beta^n) - \beta^n\sum_{k=1}^{n}\left(\frac{\alpha}{\beta}\right)^k + \alpha^n\sum_{k=1}^{n}\left(\frac{\beta}{\alpha}\right)^k\Big]$$

$$= \sum_{k=1}^{n}\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right) - \alpha\frac{1}{(\alpha-\beta)}\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right) + \beta\frac{1}{(\alpha-\beta)}\left(\frac{\alpha^n-\beta^n}{\alpha-\beta}\right)$$

(by using table1, $m=0$ and by direct simplification)

$$= (n-1)f_n^{(H)}(x,y)$$

(by using (1.4)).

The proof of (1.4) is similar to that of (1.3).

Theorem 2 : The convolution identities at the level $m = 1$ are

$$(2.1)\quad \sum_{k=1}^{n} kf_k^{(H)}(x,y)f_{n-k}^{(H)}(x,y) = \frac{n(n+1)l_n^{(H)}(x,y)}{2(x^2+4y)} - \frac{nf_{n+1}^{(H)}(x,y)}{(x^2+4y)}$$

$$(2.2)\quad \sum_{k=1}^{n} k\ l_k^{(H)}(x,y)l_{n-k}^{(H)}(x,y) = \frac{n(n+1)l_n^{(H)}(x,y)}{2} + nf_{n+1}^{(H)}(x,y)$$

$$(2.3)\quad \sum_{k=1}^{n} kf_k^{(H)}(x,y)l_{n-k}^{(H)}(x,y) = \frac{n(n+1)f_n^{(H)}(x,y)}{2} - \frac{[nxf_{n+1}^{H} + 2(n+1)yf_n^{H}(x,y]}{(x^2+4y)}$$

$$(2.4)\quad \sum_{k=1}^{n} kl_k^{(H)}(x,y)f_{n-k}^{(H)}(x,y) = \frac{n(n+1)f_n^{(H)}(x,y)}{2} + \frac{[nxf_{n+1}^{H} + 2(n+1)yf_n^{H}(x,y]}{(x^2+4y)}$$

Proof:

$$(2.1): \quad \sum_{k=1}^{n} k\ f_k^{(H)}(x,y) f_{n-k}^{(H)}(x,y) = \sum_{k=1}^{n} k\left(\frac{\alpha^k-\beta^k}{\alpha-\beta}\right)\left(\frac{\alpha^{n-k}-\beta^{n-k}}{\alpha-\beta}\right)$$

$$= \frac{1}{(\alpha-\beta)^2}\Big[\sum_{k=1}^{n} k(\alpha^n+\beta^n) - \alpha^n \sum_{k=1}^{n} k\Big(\frac{\beta}{\alpha}\Big)^k - \beta^n \sum_{k=1}^{n} k\Big(\frac{\alpha}{\beta}\Big)^k\Big]$$

$$= \frac{1}{(x^2+4y)}\Big[\frac{n(n+1)}{2} l_n^{(H)}(x,y) - \frac{\alpha^{n+2}}{(\alpha-\beta)^2}\Big(n\frac{\beta^{n+2}}{\alpha^{n+2}} - (n+1)\frac{\beta^{n+1}}{\alpha^{n+1}} + \frac{\beta}{\alpha}\Big)$$

$$- \frac{\beta^{n+2}}{(\alpha-\beta)^2}\Big(n\frac{\alpha^{n+2}}{\beta^{n+2}} - (n+1)\frac{\alpha^{n+1}}{\beta^{n+1}} + \frac{\alpha}{\beta}\Big)\Big]$$

(by using table1, $m = 1$)

$$= \frac{1}{(x^2+4y)}\Big[\frac{n(n+1)}{2} l_n^{(H)}(x,y)) - \frac{n}{(\alpha-\beta)^2}\Big(l_{n+2}^{(H)}(x,y) + y l_n^{(H)}(x,y)\Big)\Big]$$

(by using (1.4) and (1.5))

$$= \frac{n(n+1)}{2(x^2+4y)} l_n^{(H)}(x,y) - \frac{n}{(x^2+4y)} f_{n+1}^{(H)}(x,y)$$

(by repeated deductions using (1.1) , (1.2) and (1.3))

The proof of (2.2) is similar to that of (2.1).

$$(2.3): \quad \sum_{k=1}^{n} k\ l_k^{(H)}(x,y) f_{n-k}^{(H)}(x,y) = \sum_{k=1}^{n} k(\alpha^k+\beta^k)\Big(\frac{\alpha^{n-k}-\beta^{n-k}}{\alpha-\beta}\Big)$$

$$= \Big[\sum_{k=1}^{n} k\Big(\frac{\alpha^n-\beta^n}{\alpha-\beta}\Big) + \frac{\alpha^n}{\alpha-\beta}\sum_{k=1}^{n} k\Big(\frac{\beta}{\alpha}\Big)^k - \frac{\beta^n}{\alpha-\beta}\sum_{k=1}^{n} k\Big(\frac{\alpha}{\beta}\Big)^k\Big]$$

$$= \Big[\frac{n(n+1)}{2} f_n^{(H)}(x,y) + \frac{\alpha^{n+2}}{(\alpha-\beta)^3}\Big(n\frac{\beta^{n+2}}{\alpha^{n+2}} - (n+1)\frac{\beta^{n+1}}{\alpha^{n+1}} + \frac{\beta}{\alpha}\Big)$$

$$- \frac{\beta^{n+2}}{(\alpha-\beta)^3}\Big(n\frac{\alpha^{n+2}}{\beta^{n+2}} - (n+1)\frac{\alpha^{n+1}}{\beta^{n+1}} + \frac{\alpha}{\beta}\Big)\Big]$$

(by using table1, $m = 1$)

$$= \Big[\frac{n(n+1)}{2} f_n^{(H)}(x,y)) - \frac{1}{(x^2+4y)}\Big(n f_{n+2}^{(H)}(x,y) + (n+2) y f_n^{(H)}(x,y)\Big)\Big]$$

(by using (1.4) and (1.5))

$$= \frac{n(n+1)}{2} f_n^{(H)}(x,y)) - \frac{1}{(x^2+4y)}\Big[n x f_{n+1}^{(H)}(x,y) + 2(n+1) y f_n^{(H)}(x,y)\Big]$$

(by using (1.1))

The proof of (2.4) is similar to that of (2.3).

3. The Convolution Identities at the Levels $m = 2$ and $m = 3$

In this section, we continue the computation of the Convolution identities at higher levels.

Theorem 3 : The convolution identities at the level $m = 2$ are

$$(3.1)\qquad \sum_{k=1}^{n} k^2 f_k^{(H)}(x,y) f_{n-k}^{(H)}(x,y)$$
$$= \frac{n(n+1)(2n+1)}{6(x^2+4y)} l_n^{(H)}(x,y) - \frac{[n^2(x^2+4y)+4ny]}{(x^2+4y)^2} f_{n+1}^{(H)}(x,y) + \frac{2xy(n+1)}{(x^2+4y)^2} f_n^H(x,y)$$

$$(3.2)\qquad \sum_{k=1}^{n} k^2 l_k^{(H)}(x,y) l_{n-k}^{(H)}(x,y)$$
$$= \frac{n(n+1)(2n+1)}{6} l_n^{(H)}(x,y) + \frac{[n^2(x^2+4y)+4ny]}{(x^2+4y)} f_{n+1}^{(H)}(x,y) - \frac{2xy(n+1)}{(x^2+4y)} f_n^H(x,y)$$

$$(3.3)\qquad \sum_{k=1}^{n} k^2 f_k^{(H)}(x,y) l_{n-k}^{(H)}(x,y)$$
$$= \frac{n(n+1)(2n+1)}{6} f_n^{(H)}(x,y) - \frac{n[nxf_{n+1}^H(x,y)+2(n+1)yf_n^H(x,y)]}{(x^2+4y)}$$

$$(3.4)\qquad \sum_{k=1}^{n} k^2 l_k^{(H)}(x,y) f_{n-k}^{(H)}(x,y)$$
$$= \frac{n(n+1)(2n+1)}{6} f_n^{(H)}(x,y) + \frac{n[nxf_{n+1}^H(x,y)+2(n+1)yf_n^H(x,y)]}{(x^2+4y)}$$

Proof :

$$(3.1):\qquad \sum_{k=1}^{n} k^2\ f_k^{(H)}(x,y) f_{n-k}^{(H)}(x,y) = \sum_{k=1}^{n} k^2 \left(\frac{\alpha^k-\beta^k}{\alpha-\beta}\right)\left(\frac{\alpha^{n-k}-\beta^{n-k}}{\alpha-\beta}\right)$$
$$= \frac{1}{(\alpha-\beta)^2}\Big[\sum_{k=1}^{n} k^2(\alpha^n+\beta^n) - \alpha^n \sum_{k=1}^{n} k^2\Big(\frac{\beta}{\alpha}\Big)^k - \beta^n \sum_{k=1}^{n} k^2\Big(\frac{\alpha}{\beta}\Big)^k\Big]$$
$$= \frac{1}{(x^2+4y)}\Big[\frac{n(n+1)(2n+1)}{6} l_n^{(H)}(x,y)$$
$$+ \frac{\alpha^{n+3}}{(\alpha-\beta)^3}\Big(n^2\frac{\beta^{n+3}}{\alpha^{n+3}} - (2n^2+2n-1)\frac{\beta^{n+2}}{\alpha^{n+2}} + (n+1)^2\frac{\beta^{n+1}}{\alpha^{n+1}} - \frac{\beta^2}{\alpha^2} - \frac{\beta}{\alpha}\Big)$$

$$- \frac{\beta^{n+3}}{(\alpha-\beta)^2}\Big(n^2\frac{\alpha^{n+3}}{\beta^{n+3}} - (2n^2+2n-1)\frac{\alpha^{n+2}}{\beta^{n+2}} + (n+1)^2\frac{\alpha^{n+1}}{\beta^{n+1}} - \frac{\alpha^2}{\beta^2} - \frac{\alpha}{\beta}\Big)\Big]$$

(by using table1, $m = 2$)

$$= \frac{1}{(x^2+4y)}\Big[\frac{n(n+1)(2n+1}{6}l_n^{(H)}(x,y)\Big] - \frac{1}{(\alpha-\beta)^2}\Big[n^2\Big(\frac{\alpha^{n+3}-\beta^{n+3}}{\alpha-\beta}\Big)$$

$$- (2n^2+2n-1)\alpha\beta\Big(\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\Big) + (n+1)^2 f_{n-1}^{(H)}(x,y)(\alpha\beta)^2\Big(\frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta}\Big)$$

$$+ (\alpha\beta)^2\Big(\frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta}\Big) + (\alpha\beta)\Big(\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\Big)\Big]$$

$$= \frac{1}{(x^2+4y)}\Big[\frac{n(n+1)(2n+1}{6}l_n^{(H)}(x,y)\Big]$$

$$- \frac{1}{(\alpha-\beta)^2}\Big[n^2 f_{n+3}^{(H)} + (2n^2+2n-1)y f_{n+1}^{(H)}(x,y) + (n+1)^2 y^2 f_{n-1}^{(H)}(x,y)$$

$$+ y^2 f_{n-1}^{(H)}(x,y) - y f_{n+1}^{(H)}(x,y)\Big]$$

(by using (1.4) and (1.5))

$$= \frac{n(n+1)(2n+1)}{6(x^2+4y)}l_n^{(H)}(x,y) - \frac{1}{(x^2+4y)^2}\Big[(n^2(x^2+4y)+4ny)f_{n+1}^{(H)}(x,y)$$

$$- 2xy(n+1)f_n^{(H)}(x,y)\Big]$$

(by repeated deductions using (1.1) , (1.2) and (1.3))

The proof of (3.2) is similar to that of (3.1).

(3.3) :
$$\sum_{k=1}^{n} k^2\, l_k^{(H)}(x,y) f_{n-k}^{(H)}(x,y) = \sum_{k=1}^{n} k^2(\alpha^k+\beta^k)\Big(\frac{\alpha^{n-k}-\beta^{n-k}}{\alpha-\beta}\Big)$$

$$= \frac{1}{\alpha-\beta}\Big[\sum_{k=1}^{n} k^2(\alpha^n-\beta^n) + \alpha^n\sum_{k=1}^{n} k^2\Big(\frac{\beta}{\alpha}\Big)^k - \beta^n\sum_{k=1}^{n} k^2\Big(\frac{\alpha}{\beta}\Big)^k\Big]$$

$$= \Big[\frac{n(n+1)(2n+1)}{6}f_n^{(H)}(x,y)$$

$$- \frac{\alpha^{n+3}}{(\alpha-\beta)^4}\Big(n^2\frac{\beta^{n+3}}{\alpha^{n+3}} - (2n^2+2n-1)\frac{\beta^{n+2}}{\alpha^{n+2}} + (n+1)^2\frac{\beta^{n+1}}{\alpha^{n+1}} - \frac{\beta^2}{\alpha^2} - \frac{\beta}{\alpha}\Big)$$

$$- \frac{\beta^{n+3}}{(\alpha-\beta)^4}\Big(n^2\frac{\alpha^{n+3}}{\beta^{n+3}} - (2n^2+2n-1)\frac{\alpha^{n+2}}{\beta^{n+2}} + (n+1)^2\frac{\alpha^{n+1}}{\beta^{n+1}} - \frac{\alpha^2}{\beta^2} - \frac{\alpha}{\beta}\Big)\Big]$$

(by using table1, $m = 2$)

$$= \Big[\frac{n(n+1)(2n+1)}{6} f_n^{(H)}(x,y)\Big]$$
$$- \frac{1}{(\alpha-\beta)^4}\Big[n^2(\alpha^{n+3}+\beta^{n+3}) - (2n^2+2n-1)\alpha\beta(\alpha^{n+1}+\beta^{n+1})$$
$$+ (n+1)^2(\alpha\beta)^2(\alpha^{n-1}+\beta^{n-1}) - (\alpha\beta)^2(\alpha^{n-1}+\beta^{n-1}) - (\alpha\beta)(\alpha^{n+1}+\beta^{n+1})\Big]$$
$$= \Big[\frac{n(n+1)(2n+1)}{6} f_n^{(H)}(x,y)\Big]$$
$$- \frac{1}{(x^2+4y)^2}\Big[n^2 l_{n+3}^{(H)} + (2n^2+2n)y l_{n+1}^{(H)}(x,y) + (n^2+2n)y^2 l_{n-1}^{(H)}(x,y)\Big]$$

(by using (1.4) and (1.5))

$$= \frac{n(n+1)(2n+1)}{6} f_n^{(H)}(x,y) - \frac{n}{(x^2+4y)}\Big[nx f_{n+1}^{(H)}(x,y) + 2(n+1)y f_n^{(H)}(x,y)\Big]$$

(by repeated deductions using (1.1) , (1.2) and (1.3))

The proof of (3.4) is similar to that of (3.3).

Theorem 4 : The convolution identities at the level $m = 3$ are

(4.1) $$\sum_{k=1}^{n} k^3 f_k^{(H)}(x,y) f_{n-k}^{(H)}(x,y)$$
$$= \frac{n^2(n+1)^2}{4(x^2+4y)} l_n^{(H)}(x,y) - \frac{[n^3(x^2+4y)+6n^2y]}{(x^2+4y)^2} f_{n+1}^{(H)}(x,y) - \frac{(n^2+n)3xy}{(x^2+4y)^2} f_n^{H}(x,y)$$

(4.2) $$\sum_{k=1}^{n} k^3 l_k^{(H)}(x,y) l_{n-k}^{(H)}(x,y)$$
$$= \frac{n^2(n+1)^2}{4} l_n^{(H)}(x,y) + \frac{[n^3(x^2+4y)+6n^2y]}{(x^2+4y)} f_{n+1}^{(H)}(x,y) - \frac{(n^2+n)3xy}{(x^2+4y)} f_n^{H}(x,y)$$

(4.3) $$\sum_{k=1}^{n} k^3 l_k^{(H)}(x,y) f_{n-k}^{(H)}(x,y)$$
$$= \frac{n^2(n+1)^2}{4} f_n^{(H)}(x,y) - \frac{1}{(x^2+4y)^2}\Big[[n^3x(x^2+4y) - 6nxy] f_{n+1}^{(H)}(x,y)$$
$$+[2n^3y(x^2+4y) + 3n^2y(x^2+4y) + 2y(x^2+4y) + 3y(nx^2-4y)] f_n^{(H)}(x,y)\Big]$$

(4.4) $$\sum_{k=1}^{n} k^3 f_k^{(H)}(x,y) l_{n-k}^{(H)}(x,y)$$
$$= \frac{n^2(n+1)^2}{4} f_n^{(H)}(x,y) + \frac{1}{(x^2+4y)^2}\Big[[n^3x(x^2+4y) - 6nxy] f_{n+1}^{(H)}(x,y)$$
$$+ [2n^3y(x^2+4y) + 3n^2y(x^2+4y) + 2y(x^2+4y) + 3y(nx^2-4y)] f_n^{(H)}(x,y)\Big]$$

Proof:

(4.1) :
$$\sum_{k=1}^{n} k^3 \ f_k^{(H)}(x,y) f_{n-k}^{(H)}(x,y) = \sum_{k=1}^{n} k^3 \left(\frac{\alpha^k - \beta^k}{\alpha - \beta}\right)\left(\frac{\alpha^{n-k} - \beta^{n-k}}{\alpha - \beta}\right)$$

$$= \frac{1}{(\alpha-\beta)^2}\Big[\sum_{k=1}^{n} k^3(\alpha^n+\beta^n) - \alpha^n \sum_{k=1}^{n} k^3\Big(\frac{\beta}{\alpha}\Big)^k - \beta^n \sum_{k=1}^{n} k^3\Big(\frac{\alpha}{\beta}\Big)^k\Big]$$

$$= \frac{1}{(x^2+4y)}\Big[\frac{n^2(n+1)^2}{4} l_n^{(H)}(x,y)$$
$$- \frac{\alpha^{n+4}}{(\alpha-\beta)^6}\Big(n^3\frac{\beta^{n+4}}{\alpha^{n+4}} + (-3n^3-3n^2+3n-1)\frac{\beta^{n+3}}{\alpha^{n+3}}$$
$$+ (3n^3+6n^2-4)\frac{\beta^{n+2}}{\alpha^{n+2}} + (-n^3-3n^2-3n-1)\frac{\beta^{n+1}}{\alpha^{n+1}} + \frac{\beta^3}{\alpha^3} + 4\frac{\beta^2}{\alpha^2} + \frac{\beta}{\alpha}\Big)$$
$$- \frac{\beta^{n+4}}{(\alpha-\beta)^6}\Big(n^3\frac{\alpha^{n+4}}{\beta^{n+4}} + (-3n^3-3n^2+3n-1)\frac{\alpha^{n+3}}{\beta^{n+3}}$$
$$+ (3n^3+6n^2-4)\frac{\alpha^{n+2}}{\beta^{n+2}} + (-n^3-3n^2-3n-1)\frac{\alpha^{n+1}}{\beta^{n+1}} + \frac{\alpha^3}{\beta^3} + 4\frac{\alpha^2}{\beta^2} + \frac{\alpha}{\beta}\Big)\Big]$$

(by using table1, $m = 3$)

$$= \frac{n^2(n+1)^2}{4(x^2+4y)} l_n^{(H)}(x,y) - \frac{1}{(\alpha-\beta)^6}\Big[n^3(\alpha^{n+4}+\beta^{n+4})$$
$$+ (-3n^3-3n^2+3n-1)\alpha\beta(\alpha^{n+2}+\beta^{n+2}) + (3n^3+6n^2-4)(\alpha\beta)^2(\alpha^n+\beta^n)$$
$$+ (-n^3-3n^2-3n-1)(\alpha\beta)^3(\alpha^{n-2}+\beta^{n-2}) + (\alpha\beta)^3(\alpha^{n-2}+\beta^{n-2})$$
$$+ 4(\alpha\beta)^2(\alpha^n+\beta^n) + \alpha\beta(\alpha^{n+2}+\beta^{n+2})\Big]$$

(by using (1.4) and (1.5))

$$= \frac{n^2(n+1)^2}{4(x^2+4y)} l_n^{(H)}(x,y) - \frac{1}{(\alpha-\beta)^6}\Big[n^3 l_{n+4}^{(H)} + (3n^3+3n^2-3n)y l_{n+2}^{(H)}(x,y)$$
$$+ (3n^3+6n^2)y^2 l_n^{(H)}(x,y) + (n^3+3n^2+3n)y^3 l_{n-2}^{(H)}(x,y)\Big]$$

$$= \frac{n^2(n+1)^2}{4(x^2+4y)} l_n^{(H)}(x,y) - \frac{1}{(x^2+4y)^2}\Big[(n^3(x^2+4y)+6n^2y) f_{n+1}^{(H)}(x,y)$$
$$-((n^2+n)3xy) f_n^{(H)}(x,y))\Big]$$

(by repeated deductions using (1.1) ,(1.2) and (1.3))

The proof of (4.2) is similar to that of (4.1).

(4.3) :
$$\sum_{k=1}^{n} k^3 \ l_k^{(H)}(x,y) f_{n-k}^{(H)}(x,y) = \sum_{k=1}^{n} k^3(\alpha^k+\beta^k)\Big(\frac{\alpha^{n-k}-\beta^{n-k}}{\alpha-\beta}\Big)$$

$$
\begin{aligned}
&= \frac{1}{\alpha-\beta}\Big[\sum_{k=1}^{n} k^3(\alpha^n-\beta^n) + \alpha^n\sum_{k=1}^{n} k^3\Big(\frac{\beta}{\alpha}\Big)^k - \beta^n\sum_{k=1}^{n} k^3\Big(\frac{\alpha}{\beta}\Big)^k\Big] \\
&= \Big[\frac{n^2(n+1)^2}{4} f_n^{(H)}(x,y) \\
&+ \frac{\alpha^{n+4}}{(\alpha-\beta)^5}\Big(n^3\frac{\beta^{n+4}}{\alpha^{n+4}} + (-3n^3-3n^2+3n-1)\frac{\beta^{n+3}}{\alpha^{n+3}} + (3n^3+6n^2-4)\frac{\beta^{n+2}}{\alpha^{n+2}} \\
&+ (-n^3-3n^2-3n-1)\frac{\beta^{n+1}}{\alpha^{n+1}} + \frac{\beta^3}{\alpha^3} + 4\frac{\beta^2}{\alpha^2} + \frac{\beta}{\alpha}\Big) \\
&- \frac{\beta^{n+4}}{(\alpha-\beta)^5}\Big(n^3\frac{\alpha^{n+4}}{\beta^{n+4}} + (-3n^3-3n^2+3n-1)\frac{\alpha^{n+3}}{\beta^{n+3}} + (3n^3+6n^2-4)\frac{\alpha^{n+2}}{\beta^{n+2}} \\
&+ (-n^3-3n^2-3n-1)\frac{\alpha^{n+1}}{\beta^{n+1}} + \frac{\alpha^3}{\beta^3} + 4\frac{\alpha^2}{\beta^2} + \frac{\alpha}{\beta}\Big)\Big] \\
&\quad \text{(by using table1, } m=3) \\
&= \Big[\frac{n^2(n+1)^2}{4} f_n^{(H)}(x,y)\Big] \\
&- \frac{1}{(\alpha-\beta)^4}\Big[n^3\Big(\frac{\alpha^{n+4}-\beta^{n+4}}{\alpha-\beta}\Big) + (3n^3+3n^2-3n+1)\alpha\beta\Big(\frac{\alpha^{n+2}-\beta^{n+2}}{\alpha-\beta}\Big) \\
&+ (3n^3+6n^2-4)(\alpha\beta)^2\Big(\frac{\alpha^n-\beta^n}{\alpha-\beta}\Big) + (n^3+3n^2+3n-1)(\alpha\beta)^3\Big(\frac{\alpha^{n-2}-\beta^{n-2}}{\alpha-\beta}\Big) \\
&- (\alpha\beta)^3\Big(\frac{\alpha^{n-2}-\beta^{n-2}}{\alpha-\beta}\Big) - 4(\alpha\beta)^2\Big(\frac{\alpha^n-\beta^n}{\alpha-\beta}\Big) - (\alpha\beta)\Big(\frac{\alpha^{n+2}-\beta^{n+2}}{\alpha-\beta}\Big)\Big] \\
&\quad \text{(by using (1.4) and (1.5))} \\
&= \Big[\frac{n^2(n+1)^2}{4} f_n^{(H)}(x,y)\Big] \\
&- \frac{1}{(x^2+4y)^2}\Big[n^3 f_{n+4}^{(H)} + (3n^3+3n^2-3n+2)y f_{n+2}^{(H)}(x,y) \\
&+ (3n^3+6n^2-8)y^2 f_n^{(H)}(x,y) + (n^3+3n^2+3n+2)y^3 f_{n-2}^{(H)}(x,y)\Big] \\
&= \frac{n^2(n+1)^2}{4} f_n^{(H)}(x,y) - \frac{n}{(x^2+4y)^2}\Big[[n^3x(x^2+4y) - 6nxy] f_{n+1}^{(H)}(x,y) \\
&+ 2n^3y(x^2+4y) + 3n^2y(x^2+4y) + 2y(x^2+4y) + 3y(nx^2-4y) f_n^{(H)}(x,y)\Big] \\
&\quad \text{(by repeated deductions using (1.1) ,(1.2) and (1.3))}
\end{aligned}
$$

The proof of (4.4) is similar to that of (4.3). The same procedure of employing generalized Bernouli identity can be applied to compute convolution identities at any level.

Acknowledgement

The second author would like to thank both UGC -SWRO,F. No. FIP/12th Plan/KADA018 TF02 and Govt. of Karnataka (DCE). All the authors thank Prof. B. S. Kirangi and Dr. Shashikala P. for valuable discussions.

References

[1] Anglin W. S., The Queen of Mathematics, An Introduction to Number Theory, Kluwer Academy Publishers, (1995).

[2] Koshy T., Fibonacci and Lucas Numbers with Applications, A Wiley Interscience Publication, New York, (2001).

[3] Koshy T., Elematary Number Theory with Applications, Academic press, Second edition, (2007).

[3] Kim A., Convolution sums related to fibonacci numbers and lucas numbers, Asian Research journal of mathematics, 1(1) (2016), 1-17.

[4] Kim A., Generalization of convolution sums with fibonacci numbers and lucas numbers, Asian Research journal of mathematics, 1(1) (2016), 1-10.

[5] Rangarajan R., Shashikala P. and Honne gowda C. K., Certain combinatorial identities of twin pairs related to Tchebychev polynomials, International Journal of Computer Applications, 154(05) (2016), 01-05.

[6] Rangarajan R., Shashikala P. and Honne gowda C. K., Certain combinatorial properties of twin triplets related to Tchebychev polynomials, International Journal of Mathematics and its Applications, 4(4)(2016), 01-10.

[7] Rangarajan R., Honne gowda C. K. and Shashikala P., Certain combinatorial results on two variable hybrid fibonacci polynomials, (Accepted).

[8] Rangarajan R., Honne gowda C. K. and Shashikala P., A two variable lucas polynomials correspondibg to hybrid fibonacci polynomials, Global Journal of pure and Applied mathematics, 12(6) (2017), 1125-1134.

[9] Riordan J., Combinatorial Identities, Robert E. Krieger Publishing Company, New York, (1979).

[10] Shashikala P. and Rangarajan R., Tchebychev and Brahmagupta polynomials and golden ratio : two new interconections, International J. Math. Combin., 3 (2016), 57-67.

[11] Vajda S., Fibonacci and Lucas Numbers and Golden Section, Theory and Applications, Ellis-Horwood, London.