International J. of Math. Sci. & Engg. Appls. (IJMSEA)
ISSN 0973-9424, Vol. 11 No. II (August, 2017), pp. 111-122

A TWO DIMENSIKONAL OPEN TYPE MIXED CUBATURE BASED ON ANTI-GAUSS CUBATURE IN ADAPTIVE ENVIRONMENT

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Abstract

In this paper in a systematic fashion a mixed cubature rule in double variable has been brought into focus which is of precision five. As a matter of fact here anti-Gauss 3-point cubature rule and Fejer's second 3-point cubature rule have been composed of convexly in two dimensions where each rule possesses a precision 3. With the aid of an adaptive cubature algorithm this mixed cubature rule has been strengthen up by fixing up a termination criterion which was shown by evaluating some test integrals.

1. Introduction

The centre of interest of this current proposed work is primarily laid upon the following

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Key Words : *Anti-Gauss 3-point rule* $\{R^2_{aG_3}(f)\}$, *Fejers second 3-point rule* $\{R^2_{2_{F_3}}(f)\}$, *Mixed cubature rule* $\{R^2_{aG_3 2_{F_3}}(f)\}$, *open type cubature rule, Adaptive cubature routine.*

AMS Subject Classification : 65D30, 65D32.

discussion. Suppose we wish to approximate the area of the region under the surface $f(x,y)$, where $f(x,y)$ does not possess large functional variations and it is sufficiently well behaved and defined in a prescribed domain $[a,b]\times[c,d]$, then we move forward as follows.

Let $I(f)$ be the approximation to the integral

$$\int_a^b\int_c^d f(x,y)dxdy \approx \sum_{i=0}^{n}\sum_{j=0}^{m} w_i w_j f(x_i,y_j). \tag{1.1}$$

Now our goal is to evaluate the above integral in such a way such that a better result will be achieved minimizing the error and with a less computation with comfort and ease. Some of the proposed works are cited in [8, 15, 2, 1, 3, 9].

R. N. Das and G. Pradhan [6, 7] in 1996 played the big role and lead role to establish a new quadrature rule which is called the mixed quadrature rule. Initially they took Simpson's $\frac{1}{3}$ rule and Gauss-Legendre 2-point rule each having precision 3 to produce a mixed quadrature rule. They were quite brilliant in showing that mixed rule is dominant over the corresponding constituent rules. In the same tune many other mixed quadrature rules were developed by several authors [4, 5, 12, 13, 11].

Here, in this present development, P. Patra and R. B. Dash are first to use anti- Gaussian rule to develop an open mixed cubature rule in two dimensions blending two other open type rules. Previously many mixed quadrature rules [16, 17, 14] in one dimension were formed with special reference to anti-Gauss rule [10]. Also an adaptive cubature algorithm is devised to boost up the mixed rule with the numerical evaluation which was reflected in Table-2.

2. Framing of Open type Quadrature Rules in Two Dimensions

Let us consider the integral

$$I(f)=\int_a^b\int_c^d f(x,y)dxdy=\int_{-1}^{1}\int_{-1}^{1} f(x,y)dxdy \tag{2.1}$$

in two dimensions where $f(x,y)$ is defined over the domain $[-1,1]\times[-1,1]$.

Now we can bring back anti-Gauss 3-point rule for approximating (2.1) as

$$I(f)=\int_a^b\int_c^d f(x,y)dxdy=\int_{-1}^{1}\int_{-1}^{1} f(x,y)dxdy \approx R^2_{aG_3}(f)$$

where,

$$\begin{aligned} R^2_{aG_3}(f) &= \frac{1}{169}\left[256f(0,0)+25\left\{f\left(-\sqrt{\frac{13}{15}},-\sqrt{\frac{13}{15}}\right)\right.\right. \\ &+f\left(-\sqrt{\frac{13}{15}},\sqrt{\frac{13}{15}}\right)+f\left(\sqrt{\frac{13}{15}},-\sqrt{\frac{13}{15}}\right)+f\left.\left(\sqrt{\frac{13}{15}},\sqrt{\frac{13}{15}}\right)\right\} \\ &\left.+80\left\{f\left(-\sqrt{\frac{13}{15}},0\right)+f\left(\sqrt{\frac{13}{15}},0\right)+f\left(0,-\sqrt{\frac{13}{15}}\right)+f\left(0,\sqrt{\frac{13}{15}}\right)\right\}\right] \end{aligned} \tag{2.2}$$

In the same approach the Fejer's second 3-point rule for approximating (2.1) is derived as

$$I(f)=\int_a^b\int_c^d f(x,y)dxdy=\int_{-1}^1\int_{-1}^1 f(x,y)dxdy\approx R^2_{2_{F_3}}(f)$$

where,

$$\begin{aligned} R^2_{2_{F_3}}(f) &= \frac{4}{9}\left[f(0,0)+f\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)+f\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)\right. \\ &+f\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)+f\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)+f\left(-\frac{1}{\sqrt{2}},0\right) \\ &\left.+f\left(\frac{1}{\sqrt{2}},0\right)+f\left(0,-\frac{1}{\sqrt{2}}\right)+f\left(0,\frac{1}{\sqrt{2}}\right)\right]. \end{aligned} \tag{2.3}$$

Let us expand the exact integral[1] as given in (2.1) using Maclaurin's expansion of functions in two dimensions.

$$\begin{aligned} I(f) &= \int_{-1}^1\int_{-1}^1 f(x,y)dxdy \\ &= \int_{-1}^1\int_{-1}^1\left[f(0,0)+\left\{x\frac{\partial f(0,0)}{\partial x}+y\frac{\partial f(0,0)}{\partial y}\right\}\right. \\ &+\frac{1}{2!}\left\{x^2\frac{\partial^2 f(0,0)}{\partial x^2}+2xy\frac{\partial^2 f(0,0)}{\partial x\partial y}+y^2\frac{\partial^2 f(0,0)}{\partial y^2}\right\} \end{aligned}$$

[1]The function $f(x,y)$ is continuously differentiable in the domain $[-1,1]\times[-1,1]$.

$$
\begin{aligned}
&+\frac{1}{3!}\left\{x^3\frac{\partial^3 f(0,0)}{\partial x^3}+3x^2y\frac{\partial^3 f(0,0)}{\partial x^2\partial y}+3xy^2\frac{\partial^3 f(0,0)}{\partial x\partial y^2}+y^3\frac{\partial^2 f(0,0)}{\partial y^3}\right\}\\
&+\frac{1}{4!}\left\{x^4\frac{\partial^4 f(0,0)}{\partial x^4}+4x^3y\frac{\partial^4 f(0,0)}{\partial x^3\partial y}+6x^2y^2\frac{\partial^4 f(0,0)}{\partial x^2\partial y^2}+4xy^3\frac{\partial^4 f(0,0)}{\partial x\partial y^3}\right.\\
&\left.+y^4\frac{\partial^4 f(0,0)}{\partial y^4}\right\}\\
&+\frac{1}{5!}\left\{x^5\frac{\partial^5 f(0,0)}{\partial x^5}+5x^4y\frac{\partial^5 f(0,0)}{\partial x^4\partial y}+10x^3y^2\frac{\partial^5 f(0,0)}{\partial x^3\partial y^2}+10x^2y^3\frac{\partial^5 f(0,0)}{\partial x^2\partial y^3}\right.\\
&\left.+5xy^4\frac{\partial^5 f(0,0)}{\partial x\partial y^4}+y^5\frac{\partial^5 f(0,0)}{\partial y^5}\right\}\\
&+\frac{1}{6!}\left\{x^6\frac{\partial^5 f(0,0)}{\partial x^6}+6x^5y\frac{\partial^6 f(0,0)}{\partial x^5\partial y}+15x^4y^2\frac{\partial^6 f(0,0)}{\partial x^4\partial y^2}+20x^3y^3\frac{\partial^6 f(0,0)}{\partial x^3\partial y^3}\right.\\
&\left.\left.+15x^2y^4\frac{\partial^6 f(0,0)}{\partial x^2\partial y^4}+6xy^5\frac{\partial^6 f(0,0)}{\partial x\partial y^5}+y^6\frac{\partial^6 (0,0)}{\partial y^6}\right\}+\cdots\right]dxdy.
\end{aligned}
$$

Integration yields

$$
\begin{aligned}
I(f) &= \int_{-1}^{1}\int_{-1}^{1} f(x,y)dxdy\\
&= 4f(0,0)+\frac{4}{3\times 2!}\left[\frac{\partial^2 f(0,0)}{\partial x^2}+\frac{\partial^2 f(0,0)}{\partial y^2}\right]\\
&\quad+\frac{1}{4!}\left[\frac{4}{5}\left\{\frac{\partial^4 f(0,0)}{\partial x^4}+\frac{\partial^4 f(0,0)}{\partial y^4}\right\}+\frac{8}{3}\left.\right\}\frac{\partial^4 f(0,0)}{\partial x^2\partial y^2}\right\}\Bigg]\\
&\quad+\frac{4}{6!}\left[\frac{1}{7}\left\{\frac{\partial^6 f(0,0)}{\partial x^6}+\frac{\partial^6 f(0,0)}{\partial y^6}\right\}+\left\{\frac{\partial^6 f(0,0)}{\partial x^4\partial y^2}+\frac{\partial^6 f(0,0)}{\partial x^2\partial y^4}\right\}\right]+\cdots
\end{aligned}
\tag{2.4}
$$

Equation (2.4) may be re-phrased in a bit simplified notation as

$$
\begin{aligned}
I(f) &= \int_{-1}^{1}\int_{-1}^{1} f(x,y)dxdy\\
&= 4f(0,0)+\frac{2}{3}[f_{2,0}(0,0)+f_{0,2}(0,0)]+\frac{1}{30}[f_{4,0}(0,0)+f_{0,4}(0,0)]\\
&\quad+\frac{1}{9}f_{2,2}(0,0)+\frac{4}{7!}[f_{6,0}(0,0)+f_{0,6}(0,0)]+\frac{1}{180}[f_{4,2}(0,0)+f_{2,4}(0,0)]+\cdots
\end{aligned}
\tag{2.5}
$$

3. Casting of the Mixed Cubature Rule of Precision Five

Let $E^2_{aG_3}(f)$ and $E^2_{2_{F_3}}(f)$ stood for the error terms in approximating the integral $I(f)$ with the aid of the rules (2.2) and (2.3) respectively. Then

$$I(f) = R^2_{aG_3}(f) + E^2_{aG_3}(f) \tag{3.1}$$

$$I(f) = R^2_{2_{F_3}}(f) + R^2_{2_{F_3}}(f) \tag{3.2}$$

Assuming $f(x, y)$ to be continuously differentiable in $[-1, 1] \times [-1, 1]$ and evaluating $R^3_{aG_3}(f), R^2_{2_{F_3}}(f)$ employing Maclaurin's series expansion as we did in (2.4) we get

$$\begin{aligned} E^2_{aG_3}(f) &= I(f) - R^2_{aG_3}(f) \\ &= -\frac{2}{135}[f_{4,0}(0,0) + f_{0,4}(0,0)] - \frac{127}{212625}[f_{6,0}(0,0) + f_{0,6}(0,0)] \\ &\quad -\frac{1}{405}[f_{4,2}(0,0) + f_{2,4}(0,0)] - \cdots \end{aligned}$$

or

$$\begin{aligned} E^2_{aG_3}(f) &= -\left[\frac{2}{135}[f_{4,0}(0,0) + f_{0,4}(0,0)] + \frac{127}{212625}[f_{6,0}(0,0) + f_{0,6}(0,0)] \right. \\ &\quad \left. +\frac{1}{405}[f_{4,2}(0,0) + f_{2,4}(0,0)] + \cdots \right] \end{aligned} \tag{3.3}$$

which is identical in measure but of opposite in sign to the Gauss-Legendre 2-point rule in two dimension as derived in [9].

Next, in a similar manner we estimate the error of Fejer's second 3-point rule in two dimensions as follows.

$$\begin{aligned} E^2_{2_{F_3}}(f) &= I(f) - R^2_{2_{F_3}}(f) \\ &= -\frac{1}{180}[f_{4,0}(0,0) + f_{0,4}(0,0)] + \frac{1}{3024}[f_{6,0}(0,0) + f_{0,6}(0,0)] \\ &\quad +\frac{1}{1080}[f_{4,2}(0,0) + f_{2,4}(0,0)] + \cdots \end{aligned} \tag{3.4}$$

From (3.3) and (3.4) it is revealed that each of the rules (2.2) and (2.3) is of **precision 3**.

Now multiplying (3.1) by 3 and (3.2) by 8 respectively and then summing the outcomes we attain

$$I_{mix}(f) = \frac{1}{11}[3R^2_{aG_3}(f) + 8R^2_{2_{F_3}}(f)] + \frac{1}{11}[3E^2_{aG_3}(f) + 8E^2_{2_{F_3}}(f)]$$

or

$$I_{mix}(f) = R^2_{aG_3 2_{F_3}}(f) + E^2_{aG_3 2_{F_3}}(f) \tag{3.5}$$

where

$$R^2_{aG_3 2_{F_3}}(f) = \frac{1}{11}[3R^2_{aG_3}(f) + 8R^2_{2_{F_3}}(f)] \tag{3.6}$$

and

$$E^2_{aG_3 2_{F_3}}(f) = \frac{1}{11}[3E^2_{aG_3}(f) + 8E^2_{2_{F_3}}(f)] \tag{3.7}$$

Equation (3.6) expresses the desired mixed cubature rule for approximating $I(f)$ and equation (3.7) is the error generated by the rule (3.6). Hence

$$E^2_{aG_3 2_{F_3}}(f) = \frac{11}{141750}[f_{6,0}(0,0) + f_{0,6}(0,0)] + \cdots \tag{3.8}$$

which shows that the mixed cubature rule $R^2_{aG_3 2_{F_3}}(f)$ has degree of exactness equal to 5, i. e.; it integrates all polynomials of degree ≤ 5 in x and y, as the first term of $E^2_{aG_3 2_{F_3}}(f)$ starts from 6th order partial derivative. The rule (3.6) can be entitled as an open type mixed rule in two dimensions as it is composed of on the platform of two different types of open cubature rules carrying out identical precision (i. e.; precision 3).

4. Error Analysis

Theorem 4.1 : When $f(x, y)$ is given as a continuously differentiable function in $[-1, 1] \times [-1, 1]$ then the error $E^2_{aG_3 2_{F_3}}(f)$ executed with the rule $R^2_{aG_3 2_{F_3}}(f)$ is given

$$|E^2_{aG_3 2_{F_3}}(f)| \approx \frac{11}{141750}|[f_{6,0}(0,0) + f_{0,6}(0,0)]|.$$

Proof : Directly follows from equation (3.8).

Theorem 4.2 : The bounds for the truncation error $E^2_{aG_3 2_{F_3}}(f) = I(f) - R^2_{aG_3 2_{F_3}}(f)$ is presented as

$$|E^2_{aG_3 2_{F_3}}(f)| \leq \frac{2M}{495}|\xi_2 - \xi_1| \times |\eta_2 - \eta_1|$$

where

$$M = \max_{\substack{-1 \le x < 1 \\ -1 \le y \le 1}} |[f_{5,0}(x,0) + f_{0,5}(0,y)]|.$$

Proof : In the light of (3.3) and (3.4) we hold

$$E^2_{aG_3}(f) \approx -\frac{2}{135}[f_{4,0}(\xi_1,\eta_1) + f_{0,4}(\xi_1,\eta_1)], \quad (\xi_1,\eta_1) \in [-1,1] \times [-1,1]$$

$$E^2_{2_{f_3}}(f) \approx -\frac{1}{180}[f_{4,0}(\xi_2,\eta_2) + f_{0,4}(\xi_2,\eta_2)], \quad (\xi_2,\eta_2) \in [-1,1] \times [-1,1].$$

As per our sense of knowledge

$$\begin{aligned}
E^2_{aG_3 2_{F_3}}(f) &= \frac{1}{11}[3E^2_{aG_3}(f) + 8E^2_{2_{F_3}}(f)] \\
&= \frac{1}{11}\left[-\frac{2}{45}\{f_{4,0}(\xi_1,0) + f_{0,4}(0,\eta_1)\} + \frac{2}{45}\{f_{4,0}(\xi_2,0) + f_{0,4}(0,\eta_2)\}\right] \\
&= \frac{2}{495}[\{f_{4,0}(\xi_2,0) + f_{0,4}(0,\eta_2)\} - \{f_{4,0}(\xi_1,0) + f_{0,4}(0,\eta_1)\}] \\
&= \frac{2}{495}\int_{\eta_1}^{\eta_2}\int_{\xi_1}^{\xi_2}[f_{5,0}(x,0) + f_{0,5}(0,y)]dxdy \\
&\qquad \text{(supposing } \xi_1 < \xi_2 \text{ and } \eta_1 < \eta_2\text{)}.
\end{aligned}$$

Accordingly

$$\begin{aligned}
|E^2_{aG_3 2_{F_3}}(f)| &\approx \left|\frac{2}{495}\int_{\eta_1}^{\eta_2}\int_{\xi_1}^{\xi_2}[f_{5,0}(x,0) + f_{0,5}(0,y)]dxdy\right| \\
&\le \frac{2}{495}\int_{\eta_1}^{\eta_2}\int_{\xi_1}^{\xi_2}|[f_{5,0}(x,0) + f_{0,5}(0,y)]|dxdy
\end{aligned}$$

$\because$ $f(x,y)$ takes the way over the closed and bounded rectangle $[-1,1] \times [-1,1]$ so it is compact and attains its maximum over the domain $[-1,1] \times [-1,1]$. So

$$|E^2_{aG_3 2_{F_3}}(f)| \le \frac{2}{495}\int_{\eta_1}^{\eta_2}\int_{\xi_1}^{\xi_2} dxdy,$$

where

$$\begin{aligned}
M &= \max_{\substack{-1 \le x < 1 \\ -1 \le y \le 1}} |[f_{5,0}(x,0) + f_{0,5}(0,y)]| \\
&= \frac{2M}{495}|(\xi_2 - \xi_1) \times (\eta_2 - \eta_1)|.
\end{aligned}$$

A theoretical error bound is thus estimated from the above theorem as (ξ_1, η_1) and (ξ_2, η_2) are unknown points in $[-1,1] \times [-1,1]$. Again it drove up the result that the error in the approximation will come under a very less margin as the distance between the points (ξ_1, η_1) and (ξ_2, η_2) goes small.

Corollary 4.1 : The error bound for the truncation error $E^2_{aG_3 2_{F_3}}(f)$ is given by

$$|E^2_{aG_3 2_{F_3}}(f)| \leq \frac{8M}{495}.$$

Proof : From theorem 4.2

$$|E^2_{aG_3 2_{F_3}}(f)| \leq \frac{2M}{495} |(\xi_2 - \xi_1) \times (\eta_2 - \eta_1)| \in [-1,1] \times [-1,1]$$

where

$$M = \max_{\substack{-1 \leq x < 1 \\ -1 \leq y \leq 1}} |[f_{5,0}(x,0) + f_{0,5}(0,y)]|.$$

Choosing $|\xi_2 - \xi_1| \leq 2$ and $|\eta_2 - \eta_1| \leq 2$ we get

$$|E^2_{aG_3 2_{F_3}}(f)| \leq \frac{8M}{495}.$$

□

5. Adaptive Cubature Algorithm for Evaluation of Double Integrals

To evaluate double integrals over any rectangle using adaptive cubature, we adopt the following four steps algorithm.

Input : Function $f : [a,b] \times [c,d] \to \mathbb{R}$ and the prescribed tolerance ϵ.

Output : An approximation $Q(f)$ to the integral $I(f) = \int_a^b \int_c^d f(x,y)dxdy$ such that $|Q(f) - I(f)| \leq \epsilon$.

Step 1 : The mixed cubature rule $R^2_{aG_3 2_{F_3}}(f)$ is implemented over the rectangle $[a,b] \times [c,d]$ which has four vertices $\{(a,c),(b,c),(b,d)$ and $(a,d)\}$ to get the approximation of the double integral $I(f) = \int_a^b \int_c^d f(x,y)dxdy$. This approximation is designated as $R^2_{aG_3 2_{F_3}}(f_{[a,b]\times[c,d]})$.

Step 2 : The rectangle of integration $[a,b] \times [c,d]$ is split into four equal pieces of rectangles A_1, A_2, A_3, A_4 having corner points $\{(a,c),(m_1,c),(m_1,m_2),(a,m_2)\}$, $\{(m_1,c),(b,c),(b,m_2),(m_1,m_2)\}$, $\{(m_1,m_2),(b,m_2),(b,d),(m_1,d)\}$ and

$\{(a, m_2), (m_1, m_2), (m_1, d), (a, d)\}$ respectively, where $m_1 = \frac{a+b}{2}$ and $m_2 = \frac{c+d}{2}$. The mixed cubature rule $(R^2_{aG_32_{F_3}}(f)$ is applied over each small rectangle to approximate the double integrals $I_1(f) = \int_a^{m_1}\int_c^{m_2} f(x, y)dxdy,$, $I_2(f) = \int_b^{m_1}\int_c^{m_2} f(x, y)dxdy$, $I_3(f) = \int_{m_1}^b\int_{m_2}^d f(x, y)dxdy$, $I_4(f) = \int_a^{m_1}\int_{m_2}^d f(x, y)dxdy$, respectively. The approximated values are denoted by $R^2_{aG_32_{F_3}}(f \uparrow_{[a,m_1]\times[c,m_2]})$, $R^2_{aG_32_{F_3}}(f \uparrow_{[m_1,b]\times[c,m_2]})$, $R^2_{aG_32_{F_3}}(f \uparrow_{[m_1,b]\times[m_2,d]})$ and $R^2_{aG_32_{F_3}}(f \uparrow_{[a,m_1]\times[m_2,d]})$ respectively.

Step 3 : $R^2_{aG_32_{F_3}}(f \uparrow_{[a,m_1]\times[c,m_2]}) + R^2_{aG_32_{F_3}}(f \uparrow_{[m_1,b]\times[c,m_2]}) + R^2_{aG_32_{F_3}}(f \uparrow_{[m_1,b]\times[m_2,d]}) + R^2_{aG_32_{F_3}}(f \uparrow_{[a,m_1]\times[m_2,d]})$ is compared with $R^2_{aG_32_{F_3}}(f_{[a,b]\times[c,d]})$ to make the estimate of the magnitude of the error in $R^2_{aG_32_{F_3}}(f \uparrow_{[a,m_1]\times[c,m_2]}) + R^2_{aG_32_{F_3}}(f \uparrow_{[m_1,b]\times[c,m_2]}) + R^2_{aG_32_{F_3}}(f \uparrow_{[m_1,b]\times[m_2,d]}) + R^2_{aG_32_{F_3}}(f \uparrow_{[a,m_1]\times[m_2,d]})$.

Step 4 : If $|\text{estimated error}| \le \frac{\epsilon}{2}$ (termination criterion) then $R^2_{aG_32_{F_3}}(f \uparrow_{[a,m_1]\times[c,m_2]}) + R^2_{aG_32_{F_3}}(f \uparrow_{[m_1,b]\times[c,m_2]}) + R^2_{aG_32_{F_3}}(f \uparrow_{[m_1,b]\times[m_2,d]}) + R^2_{aG_32_{F_3}}(f \uparrow_{[a,m_1]\times[m_2,d]})$ is under taken as an approximation to the double integral $I(f) = \int_a^b\int_c^d f(x, y)dxdy$. Otherwise the same strategy is put into action to each of the four rectangles allowing each piece of rectangles a tolerance $\frac{\epsilon}{2}$. If the termination criterion does not meet its goal on one or more of the rectangles, then let us go on splitting the rectangles further into four sub-rectangles and also we repeat the entire process again. Upon acheiving the given accuracy the process stops and the addition of all accepted values produces the required approximation $Q(f)$ to the double integral $I(f)$ so as to get $|Q(f) - I(f)| \le \epsilon$.

N.B/ : This algorithm is pretty useful for any cubature rule to evaluate real definite integrals in two dimensions in adaptive integration scheme.

6. Numerical Verification

Table 1 : Mixed Cubature Rule ($R^2_{aG_3 2_{F_3}}(f)$) and anti-Gauss 3-point Rule ($R^2_{aG_3}(f)$) and Fejer's second 3-point Rule ($R^2_{2_{F_3}}(f)$) in approximating some real Definite integrals in non-adaptive environment: A numerical comparison

Integrals	**Exact Value**	**Approximate Value** ($Q(f)$)		
		$R^2_{aG_3}(f)$	$R^2_{2_{F_3}}(f)$	$R^2_{aG_3 2_{F_3}}(f)$
$\int_{-1}^{1}\int_{-1}^{1} e^{x+y} dxdy$	5.52439138	5.56070044	5.51054864	5.52422636
$\int_{-1}^{1}\int_{-1}^{1} e^{-(x^+y^2)} dxdy$	2.23098514	2.41527554	2.17672906	2.24178719
$\int_{0}^{1}\int_{0}^{1} \frac{\sin^2(x+y)}{(x+y)} dxdy$	0.61326603	0.61448059	0.61280975	0.61326544
$\int_{0}^{1}\int_{1}^{2} x^2 y dxdy$	0.40546510	0.40635451	0.40502277	0.40538597
$\int_{0}^{1}\int_{1}^{2} \frac{x}{x^2+y^2} dxdy$	0.19832051	0.19946490	0.19790083	0.19832740
$\int_{0}^{1}\int_{0}^{1} \frac{1}{(x+y+1)^2} dxdy$	0.28768207	0.28928138	0.28706102	0.28766657

Table 2 : Mixed Cubature Rule ($R^2_{aG_3 2_{F_3}}(f)$) and anti-Gauss 3-point Rule ($R^2_{aG_3}(f)$) and Fejer's second 3-point Rule ($R^2_{2_{F_3}}(f)$) in approximating some real definite integrals (given in Table 1) in adaptive environment: A numerical comparison

Integrals	**Approximate Value** ($Q(f)$)					
	$R^2_{aG_3}(f)$	# Steps	$R^2_{2_{F_3}}(f)$	# Steps	$R^2_{aG_3 2_{F_3}}(f)$	# Steps
$\int_{-1}^{1}\int_{-1}^{1} e^{x+y} dxdy$	5.524401353	21	5.524387641	21	5.524391330	05
$\int_{-1}^{1}\int_{-1}^{1} e^{-(x^2+y^2)} dxdy$	2.230980663	37	2.230982171	21	2.230985009	05
$\int_{0}^{1}\int_{0}^{1} \frac{\sin^2(x+y)}{(x+y)} dxdy$	0.613264736	05	0.613258734	05	0.6123260437	01
$\int_{0}^{1}\int_{1}^{2} x^y dxdy$	0.405478892	09	0.405449591	05	0.405462313	0.5
$\int_{0}^{1}\int_{1}^{2} \frac{x}{x^2+y^2} dxdy$	0.198324646	05	0.198318966	05	0.198320646	0.1
$\int_{0}^{1}\int_{0}^{1} \frac{1}{(x+y+1)^2} dxdy$	0.287684852	09	0.287678586	05	0.287682438	0.1

Note : Here the prescribed tolerance $\epsilon = 0.0001$.
All the computations are done using 1C' program.

7. Conclusion

(i) The numerical verification in Table-1 shows the dominance of the mixed cubature rule $R^2_{aG_3 2_{F_3}}(f)$ over the constituent rules $R^2_{aG_3}(f)$ and $R^2_{2_{F_3}}(f)$ in non-adaptive environment.

(ii) Table-2 shows that our adaptive algorithm based on the mixed cubature rule $R^2_{aG_3 2_{F_3}}(f)$ is converging much faster in order to accomplish our approximation with the desired accuracy so far the number of steps is concerned.

(iii) Much better result is attained on the foundation of this mixed rule in adaptive as well as in non-adaptive integration environment than the mixed rule [9] derived previously, which is the basic notion behind this paper.

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