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# GENERATING FUNCTIONS FOR THE JACOBI POLYNOMIALS AND LAGUERRE POLYNOMIALS

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**Abstract**

The object of this Paper Presenting a general Class of mixed generating functions for the Jocobi; Polynomials, Laguerre Polynomials and Related to several Polynomials. It is also shown how the main generating function can be below (2:3) suitably applied to yield numerous further results involving Jocobi Polynomials, Laguerre Polynomials and various other Polynomials associated with them.

## 1. Introduction

Put

$$Q_n^{\alpha,\beta}(x) = \sum_{k=0}^{n} \binom{n+a}{n-a} \binom{n+b}{k} \left(\frac{x-1}{2}\right) \left(\frac{x+1}{2}\right) \tag{1.1}$$

$$\sum_{n=0}^{\infty} \binom{m+n}{n} Q_{m+n}^{(\alpha-n,\beta-n)}(X) t^n$$

$$= \left[1+\frac{1}{2}(x+1)t\right]^{\alpha} \left[1+\frac{1}{2}(x-1)t\right]^{\beta} Q_m^{(\alpha,\beta)} \left[x+\frac{1}{2}(x^2-1)t\right] \tag{1.2}$$

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Key Words : *Jocobi Polynomials. Laguerre polynomials non vanishing function.*

which is observed by Singhal and Srivastava [9, P, 759] is the special case Y =1 of a bilinear generating function according to Srivastava [5, P, 465, Equation (21)].
One of the most general applications of the generating relation (1:2) of the type indicated above yields a mixed multilateral generating function for $Q_n^{(\alpha-n,\beta-n)}(x)$ which was given by Srivastava and which has been reproduced in latest book on the subject by Srivastava and Manocha. The Object of the Present note is to develop a substantially more generating class of mixed generating functions for the Jocobi Polynomials as yet another interesting consequence of (1.2). We also show how our main generating function (2.3) can be below (2.3) suitably applied to yield numerous further results involving Jocobi Polynomials Laguerre Polynomials and various Polynomials associated with them.

## 2. The Main Result

Our Main generating function for the Jocobi Polynomials is contained following :
**Theorem** : Corresponding to non-vanishing function $\Omega_\mu(y_1,\cdots,y_s)$ of $s$ variables $y_1,\cdots,y_s$ $(S \geq 1)$ and of (Complex) order $\mu$, let

$$\Lambda_{m,p,q}^{p,\sigma}[x;y,\cdots,y_s;t]=\sum_{n=0}^{\infty}a_n Q_{m+qn}^{(\alpha-pqn,\beta\sigma qn)}(x)\Omega_{\mu+pm}(y_1,\cdots,y_s)t^n,\quad a_n\neq 0 \tag{2.1}$$

where $p$ and $q$ are positive Integers, and $p$ and $\sigma$ are suitable complex parameters also for an integer $m \geq 0$. Let

$$\phi_{n,m,p,q}^{\alpha,\psi,\sigma}[x;y_1,\cdots,y_s;z]=\sum_{k=0}^{[n/q]}a_k Q_{m+n}^{(\alpha-n+pqk,\beta-n+\sigma qk)}(x)\Omega_{\mu+pk}(y_1,\cdots,y_s)z^k \tag{2.2}$$

then

$$\sum_{n=0}^{\infty}\phi_{n,m,p,q}^{\alpha,\beta,p,\sigma}(x;y_1,\cdots,y_s;z)t^n=\left[1+\frac{1}{2}(x+1)t\right]^{\alpha}\left[1+\frac{1}{2}(x-1)t\right]^{\beta}$$

$$\Lambda_{m,p,q}^{(1-p,1-\sigma)}\left[x+\frac{1}{2}(x^2-1)t;y_1,\cdots,y_s;zt^q\left\{1+\frac{1}{2}(x+1)t\right\}^{(p-1)q}\left\{1+\frac{1}{2}(x-1)t\right\}^{(\sigma-1)q}\right]. \tag{2.3}$$

provided that each side exists.
**Proof** : For Convenience, let $\Delta$ denote the left-hand side of the generating function (2.3) substituting for the polynomials.

$$\phi_{n,m,p,q}^{\alpha,b,p,\sigma}(x;y_1,\cdots,y_s;z).$$

From (2.2) into the left hand side of (2.3), we have

$$\begin{aligned}
\Delta &= \sum_{n=0}^{\infty} t^n \sum_{k=0}^{n/q} \begin{pmatrix} m+n \\ n-qk \end{pmatrix} a_k Q_{m+n}^{(\alpha-n+pqk,\beta-n+\sigma qk)}(x)\Omega_{\mu+pk}(y_1,\cdots,y_s)z^k \\
&= \sum_{k=0}^{\infty} a_k \Omega_{\mu+pk}(y_1,\cdots,y_s)(zt^q)^k \\
&\quad \cdot \sum_{n=0}^{\infty} \begin{pmatrix} m+2n+n \\ n \end{pmatrix} Q_{m+qk+n}^{(\alpha+(\rho-1)qk-n,\beta+(\sigma-1)qk-n)}(x)t^n,
\end{aligned}$$

by interchanging the order of the double summation involved.

The inner series can be summed by applying the generating relation (1.2) with $m$, $\alpha$ and $\beta$ replaced by $m+qk, \alpha+(p-1)qk$ and $\beta+(\sigma-1)2q$ respectively $(k=0,1,2,\cdots)$ and we thus find that

$$\begin{aligned}
\Delta &= \left[1+\frac{1}{2}(x+1)t\right]^{\alpha}\left[1+\frac{1}{2}(x-1)t\right]^{\beta} \\
&\quad \sum_{n=0}^{\infty} a^k Q_{m+qk}^{(\alpha-(\rho-1)qk,\beta+(\sigma-1)qk)}\left[x+\frac{1}{2}(x^2-1)t\right]\Omega_{\mu+pk}(y_1,\cdots,y_s) \\
&\quad \cdot\left(zt^q\left[1+\frac{1}{2}(x-1)t\right]^{(p-1)q}\left[1+\frac{1}{2}(x-1)t\right]^{(-si-1)q}\right)^k
\end{aligned}$$

Interpreting this last infinite series by means of the definition (2.1), we arrive at once at the right hand side of the assertion (2.3).

This evidently completes the proof of the theorem under the assumption that the double series involved in the first two steps of our proof are absolutely convergent. Thus in general, the theorem holds true for those values of the various parameters and variables involved for which each side of the assertion (2.3) exists.

## 3. Applications

At the outset we should remark that, in the special case when $p=\sigma=0$ our main assertion (2.3) reduces immediately to the aforementioned result of Srivastava; ([8, p. 230, Corollary 5]; see also [10, p.423, Corollary 5]. Furthermore, our theorem with $q=p=\sigma=1$ is essentially the same as a recent result of Das [4, p.99] who proved this special case of our assertion (2.3) in a markedly different manner by emplying certain differential operators.

In order to illustrate how our theorem can be applied to derive various generating functions involving Jacobi Polynomials, we recall the familiar result ([8, P, 145, Equation (31)]; See also [10, P.58 Equation (1.11)] for an alternate form:)

$$\sum_{n=0}^{\infty} \frac{\prod_{j=1}^{\mu} (\gamma_j)_n}{\prod_{j=1}^{v} (\delta_j)_n} \ Q_n^{(\alpha,\beta)}(x)t^n$$

$$F_{v:1:1}^{u+2:0;0} \left[ \begin{array}{c} \alpha+1, \beta+1; (\gamma_\mu) \cdots ; \cdots ; \frac{1}{2}(x-1)t, \frac{1}{2}(x-1)t \\ \\ (\delta_{\cdot v}) : \alpha+1; \beta+1; \end{array} \right] \tag{3.1}$$

where $(\lambda)_n = \Gamma(\lambda+n)/\Gamma(\lambda), F_{q:s;v}^{p:r;u}$ denotes a general double hypergeometric series defined by [10, P. 63 Equation (16)].

$$F_{q:s;v}^{p:r;u} \left[ \begin{array}{cc} (a_p) : (a_r); (y_u); & \\ & x, y \\ (b_q) : (\beta_s); (s_v); & \end{array} \right]$$

$$= \sum_{m,n=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_{m+n} \prod_{j=1}^{r} (\alpha_j)_m \prod_{j=1}^{u} (\gamma_j)_n}{\prod_{j=1}^{q} (b_j)_{m+n} \prod_{j=1}^{s} (\beta_j)_m \prod_{j=1}^{v} (\delta_j)_n} \frac{x^m}{\underline{|m}} \frac{y^n}{\underline{|n}} \tag{3.2}$$

and for convenience, $(a_p)$ abbreviates the array of $p$ parameters $a_1, \cdots, a_p$, with similar interpretations for $(b_q)$, $(\alpha_r)$, $(\beta_s)$, et cetera.

In view of (3.1), we now set

$$a_n = \frac{\prod_{j=1}^{u} g_j)_n}{\prod_{j=1}^{v} (\delta_j)_n}, \quad \Omega(y_1, \cdots, y_s) \equiv 1, \ m=0$$

and (for simplicity) let $q = \rho = \sigma = 1$ we then find from our theorem that

$$\sum_{n=0}^{\infty} \psi_n^{\alpha,\beta}(x;z)t^n = \left[1+\frac{1}{2}(x+1)t\right]^{\alpha}] \left[1+\frac{1}{2}(x-1)t\right]^{\beta}$$

$$F_{v:1;1}^{\mu+2:0;0} \left[ \begin{array}{l} \alpha+1, \beta+1(\gamma_u); \cdots ; \cdots ; z\xi(x,t), z\eta(x,t) \\ \\ (\delta_v) : \alpha+1; \beta+1; \end{array} \right], \tag{3.3}$$

where

$$\psi_n^{\alpha,\beta}(x;z)=\sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix}\frac{\prod\limits_{j=1}^{v}(\gamma_j)_k}{\prod\limits_{j=1}^{p}(\delta_j)_k}Q_n^{(\alpha-n+k,\beta-n+k)}(x)z^k \tag{3.4}$$

and

$$\xi(x,t)=\frac{1}{2}(x-1)t\left[1+\frac{1}{2}(x+1)t\right],n(x,t)=\frac{1}{2}(x+1)t\left[1+\frac{1}{2}(x-1)t\right]. \tag{3.5}$$

Some very specialized cases of the generating function (3.3) where considered earlier by Das [12, pp. 102-104].

For such choices of the co-efficients $a_n$ as illustrated above, if the multivariable function

$$\Omega_\mu(y_1,\cdots,y_s),\quad s>1$$

is expressed as a suitable product of several simple functions, our theorem would yield various mixed multilateral generating relations for the Jacobi Polynomials. Also, since [4, p. 64, Equation (4.22.1)]

$$Q_n^{\alpha,\beta-n}(x)=\left(\frac{1-x}{2}\right)^n Q_n^{(-\alpha-\beta-1-n,\beta-n)}\left(\frac{x+3}{x-1}\right),$$

and since [9, P 59, En (4.1.3)]

$$Q_n^{\alpha-n,\beta}(x)=(-1)^n Q_n^{(\beta,\alpha)}(-x). \tag{3.7}$$

Our theorem can easily by restated in terms of the modified Jacobi Polynomials

$$Q_n^{(\alpha-n,\beta)}\quad\text{and}\quad Q_n^{\alpha,n-\beta)}(x)$$

as was done (in special case $p=\sigma=0$) by Srivastava ([6, P, 230, Corollaries 9 and 7]; also [8, P. 423, Corollary 9; P. 424, Corollary 7]).

Finally, since [4, P. 103, equation (5.3.4)]

$$L_n^{\alpha}(x)=\lim_{\beta\to\infty}Q(\alpha,\beta)\left(1-\frac{2x}{\beta}\right)^n, \tag{3.8}$$

analogous results for the Languerre Polynomials

$$L_n^{\alpha}(x)\quad\text{and}\quad L_n^{(\alpha-n)}(x),$$

or for various Polynomials associated with them, can be deduced as appropriate limiting cases of our theorem or of its variations using (2.6) and (3.7). We omit the details involved in deriving, in this manner the indicated of generalizations of corollaries 6,7,8 and 10 of Srivastava ([6, PP, 231-233]; see also [8, PP. 424-426]).

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