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COMMON FIXED POINT WITH CONTRACTIVE MODULUS ON REGULAR CONE METRIC SPACE VIA CONE C-CLASS FUNCTION

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Abstract

In this paper, we discuss the common fixed point theorems with the assumption of weakly compatible and coincidence point of four maps on an upper semi continuous contractive modulus and (φ, ψ) contractive type mappings in complete cone metric space via cone $C-$class functions.

1. Introduction and Mathematical Preliminaries

The notion of cone metric space is initiated by Huang and Zhang [2] and also they discussed some properties of the convergence of sequences and proved the fixed point theorems of a contraction mapping for cone metric spaces; Any mapping T of a complete cone metric space X into itself that satisfies, for some $0 \leq k < 1$, the inequality

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$d(Tx, Ty) \leq kd(x, y), \forall x, y \in X$ has a unique fixed point. Note on (φ, ψ) contractive type mappings and related fixed point are proved by Arslan Hojat Ansari [8]. The common fixed point theorems with the assumption of weakly compatible and coincidence point of four maps on an upper semi continuous contractive modulus in complete cone Banach space are proved by R. Krishnakumar and D. Dhamodharan [4].

In this paper we investigate the common fixed point theorems with the assumption of (φ, ψ) contractive type mappings, weakly compatible and coincidence point of four maps on an upper semi continuous contractive modulus in complete cone metric space via cone $C-$class functions.

Definition 1.1 : Let E be the real Banach space. A subset P of E is called a cone if and only if:

(b_1) P is closed, non empty and $P \neq 0$

(b_2) $ax + by \in P$ for all $x, y \in P$ and non negative real numbers a, b

(b_3) $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y, x \in P$. We will write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while x, y will stand for $y, x \in intP$, where $intP$ denotes the interior of P. The cone P is called normal if there is a number $K > 0$ such that $0 \leq x \leq y$ implies $\|x| \leq K\|y\|$ for all $x, y \in E$. The least positive number satisfying the above is called the normal constant.

Example 1.2 [7] : Let $K > 1.$ be given. Consider the real vector space with

$$E = \{ax + b : a, b \in R; x \in [1, \frac{1}{k}, 1]\}$$

with supremum norm and the cone

$$P = \{ax + b : a \geq 0, b \leq 0\}$$

in E. The cone P is regular and so normal.

Definition 1.3 : Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies

(b_1) $d(x, y) = 0$ if and only if $x = y$, (b$_2$) $d(x, y) = d(y, x)$,

(b$_3$) $d(x, z) \leq d(x, y) + d(y, z)$.

Then (X, d) is called a cone metric space *(CMS)*.

Example 1.4 : Let $E = R^2$

$$P = \{(x, y) : x, y \geq 0\}$$

$X = R$ and $d : X \times X \to E$ such that

$$d(x, y) = (|x, y|, \alpha|x, y|)$$

where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.

Definition 1.5 : Let (X, d) be a CMS, $x \in X$ and $\{x_n\}_{n\geq 0}$ be a sequence in X. Then $\{x_n\}_{n\geq 0}$ converges to x whenever for every $c \in E$ with $0 \ll E$, there is a natural number $N \in N$ such that $d(x_n, x) \ll c$ for all $n \geq N$. It is denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.

Definition 1.6 : Let (X, d) be a CMS, $x \in X$ and $\{x_n\}_{n\geq 0}$ be a sequence in X. $\{x_n\}_{n\geq 0}$ is a Cauchy sequence whenever for every $c \in E$ with $0 \ll c$, there is a natural number $N \in N$, such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$

Definition 1.7 : Let (X, d) be a CMS, $x \in X$ and $\{x_n\}_{n\geq 0}$ be a sequence in X. (X, d) is a complete cone normed space if every Cauchy sequence is convergent. Complete cone normed spaces will be called complete cone metric spaces.

Lemma 1.8 [3] : Let (X, d) be a CMS, P be a normal cone with normal constant K, and $\{x_n\}$ be a sequence in X. Then

(A) the sequence $\{x_n\}$ converges to x if and only if $d(x_n, x) \to 0$ as $n \to \infty$,

(B) the sequence $\{x_n\}$ is Cauchy if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$,

(c) the sequence $\{x_n\}$ converges to x and the sequence $\{y_n\}$ converges to y, then $d(x_n, y_n) \to d(x, y)$.

Definition 1.9 : Let f and g be two self maps defined on a set X maps f and g are said to be commuting of $fgx = gfx$ for all $x \in X$

Definition 1.10 : Let f and g be two self maps defined on a set X maps f and g are said to be weakly compatible if they commute at coincidence points. that is if $fx = gx$ forall $x \in X$ then $fgx = gfx$

Definition 1.11 : Let f and g be two self maps on set X. If $fx = gx$, for some $x \in X$ then x is called coincidence point of f and g

Lemma 1.12 : Let f and g be weakly compatible self mapping of a set X. If f and g have a unique point of coincidence, that is $w = fx = gx$ then w is the unique common fixed point of f and g.

Definition 1.13 : An ultra altering distance function is a function $\varphi : P \to P$ which satisfies

(a) φ is continuous.

(b) $\varphi(0) > 0$.

Definition 1.14 [9]: : A mapping $F : P^2 \to P$ is called cone C, *class* function if it is continuous and satisfies following axioms:

1. $F(s,t) \leq s$;

2. $F(s,t) = s$ implies that either $s = 0$ or $t = 0$; for all $s, t \in P$.

We denote cone C,class functions as $\mathcal{C}$.

Example 1.15 [9] : The following functions $F : P^2 \to P$ are elements of $\mathcal{C}$, for all $s, t \in [0, \infty)$:

1. $F(s,t) = s, t$,

2. $F(s,t) = ks$, where $0<k<1$,

3. $F(s,t) = s\beta(s)$, where $\beta : [0,\infty) \to [0,1)$,

4. $F(s,t) = \Psi(s)$, where $\Psi : P \to P$,$\Psi(0) = 0$,$\Psi(s) > 0$ for all $s \in P$ with $s \neq 0$ and $\Psi(S) \leq s$ for all $s \in P$.,

5. $F(s,t) = s, \varphi(s)$, where $\varphi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\varphi(t) = 0 \Leftrightarrow t = 0$;

6. $F(s,t) = s, h(s,t)$, where $h : [0,\infty) \times [0,\infty) \to [0,\infty)$ is a continuous function such that $h(s,t) = 0 \Leftrightarrow t = 0$ for all $t, s > 0$.

7. $F(s,t) = \varphi(s), F(s,t) = s \Rightarrow s = 0$, here $\varphi : [0,\infty) \to [0,\infty)$ is a upper semi continuous function such that $\varphi(0) = 0$ and $\varphi(t) < t$ for $t > 0$.

Lemma 1.16 : Let ψ and φ are altering distance and ultra altering distance functions respectively , $F \in \mathcal{C}$ and $\{s_n\}$ a decreasing sequence in P such that

$$\psi(s_{n+1}) \leq F(\psi(s_n), \varphi(s_n))$$

for all $n \geq 1$. Then $\lim_{n\to\infty} s_n = 0$.

2. Main Result

Theorem 2.1 : Let (X, d) be a complete cone metric space with regular cone P such that $d(x, y) \in intP$. Suppose that the mappings P, Q, S and T are four self maps of (X, d) such that $T(X) \subseteq P(X)$ and $S(X) \subseteq Q(X)$ and satisfying

$$\begin{aligned} & \psi(d(Ty, Sx)) \\ \leq \; & F(\psi(ad(Px, Qy) + b\{d(Px, Sx) + d(Qy, Ty)\} + c\{d(Px, Ty) + d(Qy, Sx)\}), \\ & \varphi(ad(Px, Qy) + b\{d(Px, Sx) + d(Qy, Ty)\} + c\{d(Px, Ty) + d(Qy, Sx)\})) \end{aligned} \tag{2.1}$$

for all $x, y \in X$, where $a, b, c \geq 0$ and $a + 2b + 2c = 1$. ψ and φ are altering distance and ultra altering distance functions respectively ,$F \in \mathcal{C}$ such that $\psi(t + s) \leq \psi(t) + \psi(s)$. Suppose that the pairs $\{P, S\}$ and $\{Q, T\}$ are weakly compatible, then P, Q, S and T have a unique common fixed point.

Proof : Suppose x_0 is an arbitrary initial point of X and define the sequence $\{y_n\}$ in X such that

$$y_{2n} = Sx_{2n} = Qx_{2n+1}$$

$$y_{2n+1} = Tx_{2n+1} = Px_{2n+2}$$

By (2.1) implies that

$$\begin{aligned} \psi(d(y_{2n+1}, y_{2n})) &= \psi(d(Tx_{2n+1}, Sx_{2n})) \\ &\leq F(\psi(ad(Px_{2n}, Qx_{2n+1}) + b\{d(Px_{2n}, Sx_{2n}) + d(Qx_{2n}, Tx_{2n+1})\} \\ &\quad +c\{d(Px_{2n}, Tx_{2n+1}) + d(Qx_{2n+1}, Sx_{2n})\}), \\ &\quad \varphi(ad(Px_{2n}, Qx_{2n+1}) + b\{d(Px_{2n}, Sx_{2n}) + d(Qx_{2n}, Tx_{2n+1})\} \\ &\quad +c\{d(Px_{2n}, Tx_{2n+1}) + d(Qx_{2n+1}, Sx_{2n})\})) \end{aligned}$$

$$
\begin{aligned}
&\leq F(\psi(ad(y_{2n-1},y_{2n})+b\{d(y_{2n-1},y_{2n})+d(y_{2n},y_{2n+1})\}\\
&\quad +c\{d(y_{2n-1},y_{2n+1})+d(y_{2n},y_{2n})\}),\varphi(ad(y_{2n-1},y_{2n})\\
&\quad +b\{d(y_{2n-1},y_{2n})+d(y_{2n},y_{2n+1})\}+c\{d(y_{2n-1},y_{2n+1})+d(y_{2n},y_{2n})\}))\\
&\leq F(\psi(ad(y_{2n-1},y_{2n})+b\{d(y_{2n-1},y_{2n})+d(y_{2n},y_{2n+1})\}\\
&\quad +cd(y_{2n-1},y_{2n+1})).\varphi(ad(y_{2n-1},y_{2n})+bd(y_{2n-1},y_{2n}))\\
&\quad +d(y_{2n},y_{2n+1})\}+cd(y_{2n-1},y_{2n+1})))\\
&\leq F(\psi((a+b+c)d(y_{2n-1},y_{2n})+(b+c)d(y_{2n},y_{2n+1}),\\
&\quad \varphi((a+b+c)d(y_{2n-1},y_{2n})+(b+c)d(y_{2n},y_{2n+1}))\\
&\leq \psi((a+b+c)d(y_{2n-1},y_{2n})+(b+c))d(y_{2n},y_{2n+1})
\end{aligned}
$$

$$
\begin{aligned}
\Rrightarrow d(y_{2n+1},y_{2n}) &\leq d(y_{2n},y_{2n-1})\\
d(y_{2n+1},y_{2n}) &\leq hd(y_{2n},y_{2n-1}) \qquad (2.2)
\end{aligned}
$$

implies that the sequence $\{d(y_{2n+1},y_{2n})\}$ is monotonic decreasing and continuous. There exists a real number, say $r \geq 0$ such that

$$\lim_{n\to\infty} d(y_{2n+1},y_{2n}) = r,$$

as $n \to \infty$ equation (2.2) $\Rightarrow$

$$\psi(r) \leq F(\psi(r),\varphi(r))$$

so, $\psi(r) = 0$ or $\varphi(r) = 0$ which is only possible if $r = 0$. Thus

$$\lim_{n\to\infty} d(y_{2n+1},y_{2n}) = 0.$$

Claim: $\{y_{2n}\}$ is a Cauchy sequence.

Suppose $\{y_{2n}\}$ is not a Cauchy sequence.

Then there exists an $\epsilon > 0$ and sub sequence $\{n_i\}$ and $\{m_i\}$ such that $m_i < n_i < m_{i+1}$

$$d(y_{m_i},y_{n_i}) \geq \epsilon \text{ and } d(y_{m_i},y_{n_{i-1}}) \leq \epsilon \qquad (2.3)$$

$$\epsilon \leq d(y_{m_i},y_{n_i}) \leq d(y_{m_i},y_{n_{i-1}}) + d(y_{n_{i-1}},y_{n_i})$$

therefore

$$\lim_{i\to\infty} y_{m_i},y_{n_i}) = \epsilon$$

now

$$\epsilon \leq d(y_{m_{i-1}}, y_{n_{i-1}}) \leq d(y_{m_{i-1}}, y_{m_i}) + d(y_{m_i}, y_{n_{i-1}})$$

by taking limit $i \to \infty$ we get,

$$\lim_{i\to\infty} d(y_{m_{i-1}}, y_{n_{i-1}}) = \epsilon$$

from (2.7) and (2.8)

$$\begin{aligned}\psi(\epsilon) &\leq \psi(d(y_{m_i}, y_{n_i})) = \psi(d(Sx_{m_i}, Tx_{n_i}))\\ &\leq F(\psi(\lambda(x_{m_i}, x_{n_i})), \varphi(\lambda(x_{m_i}, x_{n_i})))\Phi(\lambda(x_{m_i}, x_{n_i}))\end{aligned}$$

where implies

$$\psi(\epsilon) \leq F(\psi(\lambda(x_{m_i}, x_{n_i})), \varphi(\lambda(x_{m_i}, x_{n_i}))) \tag{2.4}$$

$$\begin{aligned}\lambda(x_{m_i}, x_{n_i}) &= ad(Px_{m_i}, Qx_{n_i}) + bd(Px_{m_i}, Sx_{m_i}) + d(Qx_{n_i}, Tx_{n_i}),\\ &\quad + c(d(Px_{m_i}, Tx_{n_i}) + d(Qx_{n_i}, Sx_{m_i}))\\ &= ad(Tx_{m_{i-1}}, Sx_{n_{i-1}}) + b)Tx_{m_{i-1}}, Sx_{m_i}) + d(Sx_{n_{i-1}}, Tx_{n_i}),\\ &\quad + c(d(Tx_{m_{i-1}}, Tx_{n_i}) + d(Sx_{n_{i-1}}, Sx_{m_i}))\\ &= ad(y_{m_{i-1}}, y_{n_{i-1}}) + by_{m_{i-1}}, y_{m_i}) + d(y_{n_{i-1}}, y_{n_i}),\\ &\quad c(d(y_{m_{i-1}}, y_{n_i}) + d(y_{n_{i-1}}, y_{m_i}))\end{aligned}$$

Taking limit as $i \to \infty$, we get

$$\begin{aligned}&\lim_{i\to\infty} \lambda(x_{m_i}, x_{n_i}) = a\epsilon + b0 + 0 + c(\epsilon + \epsilon)\}\\ &\lim_{i\to\infty} \lambda(x_{m_i}, x_{n_i}) \leq \epsilon\end{aligned}$$

Therefore from (2.4) we have,$\psi(\epsilon) \leq F(\psi(\epsilon), \varphi(\epsilon))$
so, $\psi(\epsilon) = 0$ or $\varphi(\epsilon) = 0$ This is a contraction because $\epsilon > 0$.
Therefore $\{y_{2n}\}$ is Cauchy sequence in X

Hence $\{y_n\}$ is a Cauchy sequence.

There exists a point l in (X, d) such that
$\lim_{n\to\infty} \{y_n\} = l$, $\lim_{n\to\infty} S_{2n} = \lim_{n\to\infty} Qx_{2n+1} = l$ and $\lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Px_{2n+2} = l$

that is,

$$\lim_{n\to\infty} S_{2n} = \lim_{n\to\infty} Qx_{2n+1} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Px_{2n+2} = x^*$$

Since $T(X) \subseteq P(X)$, there exists a point z in X Such that $x^* = Pz$ then by (1)

$$\begin{aligned}
\psi(d(Sz,x^*)) &\le F(\psi(d(Sz,x^*)),\varphi(d(Sz,x^*))) \\
&\le F(\psi(d(Sz,Tx_{2n-1}) + d(Tx_{2n-1},x^*)),\varphi(d(Sz,Tx_{2n-1}) + d(Tx_{2n-1},x^*))) \\
&\le F(\psi(ad(Pz,Qx_{2n-1}) + b\{d(Pz,Sz) + d(Qx_{2n-1},Tx_{2n-1})\} \\
&\quad + c\{d(Pz,Tx_{2n-1}) + d(Qx_{2n-1},Sz)\} + d(Tx_{2n-1},x^*)), \\
&\quad \varphi(ad(Pz,Qx_{2n-1}) + b\{d(Pz,Sz) + d(Qx_{2n-1},Tx_{2n-1})\} \\
&\quad + c\{d(Pz,Tx_{2n-1}) + d(Qx_{2n-1},Sz)\} + d(Tx_{2n-1},x^*)))
\end{aligned}$$

Taking the limit as $n \to \infty$

$$\begin{aligned}
\psi(d(Sz,x^*)) &\le F(\psi(ad(x^*,x^*) + b\{d(x^*,x^*) + d(x^*,Sz)\} \\
&\quad + c\{d(x^*,x^*) + d(x^*,Sz)\} + d(x^*,x^*)), \\
&\quad \varphi(ad(x^*,x^*) + b\{d(x^*,x^*) + d(x^*,Sz)\} \\
&\quad + c\{d(x^*,x^*) + d(x^*,Sz)\} + d(x^*,x^*))) \\
&\le F(\psi(0 + b\{d(x^*,Sz) + 0\} + c\{0 + d(x^*,Sz)\} + 0 \\
&\quad + (b+c)d(x^*,Sz)),\varphi(0 + b\{d(x^*,Sz) + 0\} + c\{0 + d(x^*,Sz)\} + 0 \\
&\quad + (b+c)d(x^*,Sz)))
\end{aligned}$$

Which is a contraction since $a + 2b + 2c = 1$.

$$\text{therefore} \quad Sz = Pz = x^*$$

Since $S(X) \subseteq Q(X)$ there exists a point $w \in X$ such that $x^* = Qw$.
By (1)

$$\begin{aligned}
\psi(d(Sz,x^*)) &\le \psi(d(Sz,Tw)) \\
&\le F(\psi(ad(Pz,Qw) + b\{d(Pz,Sz) + d(Qw,Tw)\} \\
&\quad + c\{d(Pz,Tw) + d(Qw,Sw)\}),\varphi(ad(Pz,Qw) \\
&\quad + b\{d(Pz,Sz) + d(Qw,Tw)\} + c\{d(Pz,Tw) + d(Qw,Sw)\}))
\end{aligned}$$

$$
\begin{aligned}
&\leq F(\psi(ad(x^*,x^*)+b\{d(x^*,x^*)+d(x^*,Tw)\}+c\{d(x^*,Tw)\\
&+d(x^*,x^*)\}),\varphi(ad(x^*,x^*)+b\{d(x^*,x^*)+d(x^*,Tw)\}\\
&+c\{d(x^*,Tw)+d(x^*,x^*)\}))\\
&\leq F(\psi(0+b\{0+d(x^*,Tw)\}+c\{d(x^*,Tw)+0\}),\\
&\varphi(0+b\{0+d(x^*,Tw)\}+c\{d(x^*,Tw)+0\}))\\
F(\psi(d(x^*,Tw)) &\leq F(\psi((b+c)d(x^*,Tw)),\varphi((b+c)d(x^*,Tw)))
\end{aligned}
$$

which is a contradiction since $a+2b+2c=1$.

$$\text{therefore} \quad Tw = Qw = x^*$$

Thus $Sz = Pz = Tw = Qw = x^*$

Since P and S are weakly compatible maps,

Then $SP(z) = PS(z)$

$Sx^* = Px^*$

To prove that x^* is a fixed point of S

Suppose $Sx^* \neq x^*$ then by (2.1)

$$
\begin{aligned}
\psi(d(Sx^*,x^*)) &\leq \psi(d(Sx^*,Tx^*))\\
&\leq F(\psi(ad(Px^*,Qw)+b\{d(Px^*,Sx^*)+d(Qw,Tw)\}+\\
&\quad +c\{d(Px^*,Tw)+d(Qw,Sx^*)\}),\\
&\varphi(ad(Px^*,Qw)+b\{d(Px^*,Sx^*)+d(Qw,Tw)\}+\\
&\quad +c\{d(Px^*,Tw)+d(Qw,Sx^*)\}))\\
&\leq F(\psi(ad(Sx^*,x^*)+b\{d(Sx^*,Sx^*)+d(x^*,x^*)\}+\\
&\quad +c\{d(Sx^*,x^*)+d(x^*,Sx^*)\}),\\
&\varphi(ad(Sx^*,x^*)+b\{d(Sx^*,Sx^*)+d(x^*,x^*)\}+\\
&\quad +c\{d(Sx^*,x^*)+d(x^*,Sx^*)\}))\\
&\leq F(\psi(ad(Sx^*,x^*)+b\{0+0\}+2cd(Sx^*,x^*)),\\
&\varphi(ad(Sx^*,x^*)+b\{0+0\}+2cd(Sx^*,x^*)))\\
\psi(d(Sx^*,x^*)) &\leq F(\psi((a+2c)d(Sx^*,x^*)),\varphi((a+2c)d(Sx^*,x^*)))
\end{aligned}
$$

Which is a contradiction, Since $a+2b+2c=1$.

$$Sx^* = x^*$$

Hence $Sx^* = Px^* = x^*$ Similarly, Q and T are weakly compatible maps then $TQw = QTw$, that is $Tx^* = Qx^*$

To prove that x^* is a fixed point of T.
Suppose $Tx^* \neq x^*$ by (2.1)

$$\begin{aligned}
\psi(d(Tx^*,x^*)) &\leq \psi(d(Sx^*,Tx^*)) \\
&\leq F(\psi(ad(Px^*,Qx^*) + b\{d(Px^*,Sx^*) + d(Qx^*,Tx^*)\} + \\
&\quad + c\{d(Px^*,Tx^*) + d(Qx^*,Sx^*)\}), \varphi(ad(Px^*,Qx^*) \\
&\quad + b\{d(Px^*,Sx^*) + d(Qx^*,Tx^*)\} + c\{d(Px^*,Tx^*) + d(Qx^*,Sx^*)\})) \\
&\leq F(\psi(ad(x^*,Tx^*) + b\{d(x^*,x^*) + d(Tx^*,Tx^*)\} + c\{d(x^*,Tx^*) \\
&\quad + d(Tx^*,x^*)\}), \varphi(ad(x^*,Tx^*) + b\{d(x^*,x^*) + d(Tx^*,Tx^*)\} + \\
&\quad + c\{d(x^*,Tx^*) + d(Tx^*,x^*)\})) \\
&\leq F(\psi(ad(Tx^*,x^*) + b\{0+0\} + 2c)Tx^*,x^*)), \\
&\quad \varphi(ad(Tx^*,x^*) + b\{0+0\} + 2c)Tx^*,x^*))) \\
\psi(d(Tx^*,x^*)) &\leq F(\psi((a+2c)d(Tx^*,x^*)), \varphi((a+2c)d(Tx^*,x^*)))
\end{aligned}$$

which is a contradiction since $a+2b+2c=1$.

$$Tx^* = x^*.$$

Hence. $Tx^* = Qx^* = x^*$
Thus $Sx^* = Px^* = Tx^* = Qx^* = x^*$
That is, x^* is a common fixed point of P, Q, S and T
To prove that the uniqueness of x^*
Suppose that x^* and y^*, $x^* \neq y^*$ are common fixed points of P, Q, S and T respectively,

by (2.1) we have,

$$\begin{aligned}
\psi(d(x^*, y^*)) &\leq \psi(d(Sx^*, Ty^*)) \\
&\leq F(\psi(ad(Px^*, Qy^*) + b\{d(Px^*, Sx^*) + d(Qy^*, Ty^*)\} + \\
&\quad + c\{d(Px^*, Ty^*) + d(Qy^*, Sx^*)\}), \varphi(ad(Px^*, Qy^*) \\
&\quad + b\{d(Px^*, Sx^*) + d(Qy^*, Ty^*)\} + \\
&\quad + c\{d(Px^*, Ty^*) + d(Qy^*, Sx^*)\})) \\
&\leq F(\psi(ad(x^*, y^*) + b\{d(x^*, x^*) + d(y^*, y^*)\} \\
&\quad + c\{d(x^*, y^*) + d(y^*, x^*)\}), \varphi(ad(x^*, y^*) \\
&\quad + b\{d(x^*, x^*) + d(y^*, y^*)\} + c\{d(x^*, y^*) + d(y^*, x^*)\})) \\
&\leq F(\psi(ad(x^*, y^*) + b\{0 + 0\} + c\{d(x^*, y^*) + d(y^*, x^*)\}), \\
&\quad \varphi(ad(x^*, y^*) + b\{0 + 0\} + c\{d(x^*, y^*) + d(y^*, x^*)\})) \\
&\leq F(\psi((a + 2c)d(x^*, y^*)), \varphi((a + 2c)d(x^*, y^*)))
\end{aligned}$$

which is a contradiction. Since $a + 2b + 2c = 1$.

$$\text{therefore } \ x^* = y^*.$$

Hence x^* is the unique common fixed point of P, Q, S and T respectively. □

Corollary 2.2 : Let (X, d) be a complete cone metric space with regular cone P such that $d(x, y) \in intP$. Suppose that the mappings P, S and T are three self maps of (X, d) such that $T(X) \subseteq P(X)$ and $S(X) \subseteq P(X)$ and satisfying

$$\begin{aligned}
\psi(d(Sx, Ty)) \quad \leq \quad & F(\psi(ad(Px, Py) + b\{d(Px, Sy) + d(Px, Ty)\} \\
& +c\{d(Px, Ty) + d(Py, Sx)\}), \varphi(ad(Px, Py) \\
& +b\{d(Px, Sy) + d(Px, Ty)\} + c\{d(Px, Ty) + d(Py, Sx)\}))
\end{aligned} \tag{2.5}$$

for all $x, y \in X$, where ψ and φ are altering distance and ultra altering distance functions respectively, $F \in \mathcal{C}$ such that $\psi(t + s) \leq \psi(t) + \psi(s)$, $a, b, c \geq 0$ and $a + 2b + 2c < 1$. suppose that the pairs $\{P, S\}$ and $\{P, T\}$ are weakly compatible, then P, S and T have a unique common fixed point.

Proof : The proof of the corollary immediate by taking $P = Q$ in the above Theorem 2.1. □

Definition 2.3 : A mapping $\Phi : P \cup \{0\} \to P \cup \{0\}$ is said to be contractive modulus if it is continuous and which satisfies

1. $\Phi(t) = 0$ if and only if $t = 0$
2. $\Phi(t) \leq t$ for $t \in P$
3. $\Phi(t+s) \leq \Phi(t) + \Phi(s)$ for $t, s \in P$

Theorem 2.4 : Let (X, d) be a complete cone metric space with regular cone P such that $d(x, y) \in intP$. Suppose that the mappings P, Q, S and T are four self maps of (X, d) such that $T(X) \subseteq P(X)$ and $S(X) \subseteq Q(X)$ satisfying

$$\psi(d(Sx, Ty)) \leq F(\psi(\Phi(\lambda(x, y))), \varphi(\Phi(\lambda(x, y)))), \tag{2.6}$$

for all $x, y \in X$, where ψ and φ are altering distance and ultra altering distance functions respectively, $F \in \mathcal{C}$ such that $\psi(t+s) \leq \psi(t) + \psi(s)$ and Φ is contractive modulus. $\lambda(x, y) = \max\{d(Px, Qy), d(Px, Sx), d(Qy, Ty), \frac{1}{2}\{d(Px, Ty) + d(Qy, Sx)\}\}$. Suppose that the pairs $\{P, S\}$ and $\{Q, T\}$ are weakly compatible, then P, Q, S and T have a unique common fixed point.

Proof : Let us take x_0 is an arbitrary point of X and define a sequence $\{y_{2n}\}$ in X such that

$$y_{2n} = Sx_{2n} = Qx_{2n+1}, \quad y_{2n+1} = Tx_{2n+1} = Px_{2n+2}.$$

By (2.6) implies that

$$\begin{aligned}
\psi(d(y_{2n}, y_{2n+1})) &= \psi(d(Sx_{2n}, Tx_{2n+1})) \\
&\leq F(\psi(\Phi(\lambda(x_{2n}, x_{2n+1}))), \varphi(\Phi(\lambda(x_{2n}, x_{2n+1})))) \\
&\leq F(\psi(\lambda(x_{2n}, x_{2n+1})), \varphi(\lambda(x_{2n}, x_{2n+1}))) \\
&= F(\psi(\max\{d(Px_{2n}, Qx_{2n+1}), d(Px_{2n}, Sx_{2n}), d(Qx_{2n+1}, Tx_{2n+1}), \\
&\quad \frac{1}{2}\{d(Px_{2n}, Tx_{2n+1}) + d(Qx_{2n+1}, Sx_{2n})\}\}), \varphi(\max\{d(Px_{2n}, Qx_{2n+1}), \\
&\quad d(Px_{2n}, Sx_{2n}), d(Qx_{2n+1}, Tx_{2n+1}), \frac{1}{2}\{d(Px_{2n}, Tx_{2n+1}) + d(Qx_{2n+1}, Sx_{2n})\}\})
\end{aligned}$$

$$= F(\psi(\max\{d(Tx_{2n-1}, Sx_{2n}), d(Tx_{2n-1}, Sx_{2n}), d(Sx_{2n}, Tx_{2n+1}),$$
$$\frac{1}{2}\{d(Tx_{2n-1}, Tx_{2n+1}) + d(Sx_{2n}, Sx_{2n})\}\}), \varphi(\max\{d(Tx_{2n-1}, Sx_{2n}),$$
$$d(Tx_{2n-1}, Sx_{2n}), d(Sx_{2n}, Tx_{2n+1}), \frac{1}{2}\{d(Tx_{2n-1}, Tx_{2n+1}) + d(Sx_{2n}, Sx_{2n})\}\}))$$
$$= F(\psi(\max\{d(Tx_{2n-1}, Sx_{2n}), d(Tx_{2n-1}, Sx_{2n}), d(Sx_{2n}, Tx_{2n+1}),$$
$$\frac{1}{2}d(Tx_{2n-1}, Tx_{2n+1})\}), \varphi(\max\{d(Tx_{2n-1}, Sx_{2n}),$$
$$d(Tx_{2n-1}, Sx_{2n}), d(Sx_{2n}, Tx_{2n+1}), \frac{1}{2}d(Tx_{2n-1}, Tx_{2n+1})\}))$$
$$= F(\psi(\max\{d(y_{2n}, y_{2n-1}), d(y_{2n}, y_{2n+1}), \frac{1}{2}d(y_{2n-1}, y_{2n+1})\}),$$
$$\varphi(\max\{d(y_{2n}, y_{2n-1}), d(y_{2n}, y_{2n+1}), \frac{1}{2}d(y_{2n-1}, y_{2n+1})\}))$$
$$\leq F(\psi(\max\{d(y_{2n}, y_{2n-1}), d(y_{2n}, y_{2n+1})\}),$$
$$\varphi(\max\{d(y_{2n}, y_{2n-1}), d(y_{2n}, y_{2n+1})\}))$$

Since ψ and φ are altering distance and ultra altering distance functions respectively, $F \in \mathcal{C}$ and Φ is an contractive modulus, $\lambda(x_{2n}, x_{2n+1}) = (y_{2n}, y_{2n+1})$ is not possible. Thus,

$$\psi(d(y_{2n}, y_{2n+1})) \leq F(\psi(\Phi(d(y_{2n-1}, y_{2n}))), \varphi(\Phi(d(y_{2n-1}, y_{2n})))). \tag{2.7}$$

Since Φ is an upper semi continuous, contractive modulus. Equation (2.7) implies that the sequence $\{d(y_{2n+1}, y_{2n})\}$ is monotonic decreasing and continuous. There exists a real number, say $r \geq 0$ such that

$$\lim_{n\to\infty} d(y_{2n+1}, y_{2n}) = r,$$

as $n \to \infty$ equation (**??**) $\Rightarrow$

$$\psi(r) \leq F(\psi(\Phi(r)), \varphi(\Phi(r)))$$

so, $\psi(r) = 0$ or $\varphi(r) = 0$ which is only possible if $r = 0$ and Φ is a contractive modulus. Thus

$$\lim_{n\to\infty} d(y_{2n+1}, y_{2n}) = 0.$$

Claim: $\{y_{2n}\}$ is a Cauchy sequence.

Suppose $\{y_{2n}\}$ is not a Cauchy sequence.

Then there exists an $\epsilon > 0$ and sub sequence $\{n_i\}$ and $\{m_i\}$ such that $m_i < n_i < m_{i+1}$

$$d(y_{m_i}, y_{n_i}) \geq \epsilon \quad \text{and} \quad d(y_{m_i}, y_{n_{i-1}}) \leq \epsilon \tag{2.8}$$

$$\epsilon \leq d(y_{m_i}, y_{n_i}) \leq d(y_{m_i}, y_{n_{i-1}}) + d(y_{n_{i-1}}, y_{n_i})$$

$$\text{therefore} \quad \lim_{i\to\infty} d(y_{m_i}, y_{n_i}) = \epsilon$$

now

$$\epsilon \leq d(y_{m_{i-1}}, y_{n_{i-1}}) \leq d(y_{m_{i-1}}, y_{m_i}) + d(y_{m_i}, y_{n_{i-1}})$$

by taking limit $i \to \infty$ we get,

$$\lim_{i\to\infty} d(y_{m_{i-1}}, y_{n_{i-1}}) = \epsilon$$

from (2.7) and (2.8)

$$\begin{aligned}
\psi(\epsilon) &\leq F(\psi(d(y_{m_i}, y_{n_i})), \varphi(d(y_{m_i}, y_{n_i}))) \\
&= F(\psi(\Phi(d(Sx_{m_i}, Tx_{n_i}))), \varphi(\Phi(d(Sx_{m_i}, Tx_{n_i}))) \qquad (2.9) \\
&\leq F(\psi(\Phi(\lambda(x_{m_i}, x_{n_i}))), \varphi(\Phi(\lambda(x_{m_i}, x_{n_i})))
\end{aligned}$$

where implies

$$\begin{aligned}
\lambda(x_{m_i}, x_{n_i}) &= \max\{d(Px_{m_i}, Qx_{n_i}), d(Px_{m_i}, Sx_{m_i}), d(Qx_{n_i}, Tx_{n_i}), \\
&\quad \frac{1}{2}(d(Px_{m_i}, Tx_{n_i}) + d(Qx_{n_i}, Sx_{m_i}))\} \\
&= \max\{d(Tx_{m_{i-1}}, Sx_{n_{i-1}}), d(Tx_{m_{i-1}}, Sx_{m_i}), d(Sx_{n_{i-1}}, Tx_{n_i}), \\
&\quad \frac{1}{2}(d(Tx_{m_{i-1}}, Tx_{n_i}) + d(Sx_{n_{i-1}}, Sx_{m_i}))\} \\
&= \max\{d(y_{m_{i-1}}, y_{n_{i-1}}), d(y_{m_{i-1}}, y_{m_i}), d(y_{n_{i-1}}, y_{n_i}), \\
&\quad \frac{1}{2}(d(y_{m_{i-1}}, y_{n_i}) + d(y_{n_{i-1}}, y_{m_i}))\}
\end{aligned}$$

Taking limit as $i \to \infty$, we get

$$\begin{aligned}
&\lim_{i\to\infty} \lambda(x_{m_i}, x_{n_i}) = \max\{\epsilon, 0, 0, \frac{1}{2}(\epsilon + \epsilon)\} \\
&\lim_{i\to\infty} \lambda(x_{m_i}, x_{n_i}) = \epsilon
\end{aligned}$$

Therefore from (2,9) we have, $\psi(\epsilon) \leq F(\psi(\Phi(\epsilon)), \varphi(\Phi(\epsilon)))$

This is a contraction because $\epsilon > 0$, ψ and φ are altering distance and ultra altering

distance functions respectively, $F \in \mathcal{C}$ and Φ is contractive modulus.
Therefore $\{y_{2n}\}$ is Cauchy sequence in X
There exits a point z in X such that $\lim_{n\to\infty} y_{2n} = z$
Thus,

$$\lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Qx_{2n+1} = z \quad \text{and}$$
$$\lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Px_{2n+2} = z$$
$$(i.e) \quad \lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Qx_{2n+1} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Px_{2n+2} = z$$

$T(X) \subseteq P(X)$, there exists a point $u \in X$ such that $z = Pu$

$$\begin{aligned}\psi(d(Su, z)) &= \psi(d(Su, Tx_{2n+1}) + d(Tx_{2n+1}, z))\\ &\leq \psi(d(Su, Tx_{2n+1})) + \psi d(Tx_{2n+1}, z))\\ &\leq F(\psi(\Phi(\lambda(u, x_{2n+1}))), \varphi(\Phi(\lambda(u, x_{2n+1}))) + \psi(d(Tx_{2n+1}, z)))\end{aligned}$$

where

$$\begin{aligned}\lambda(u, x_{2n+1}) &= \max\{d(Pu, Qx_{2n+1}), d(Pu, Su), d(Qx_{2n+1}, Tx_{2n+1}),\\ &\quad \frac{1}{2}(d(Pu, Tx_{2n+1}) + d(Qx_{2n+1}, Su))\}\\ &= \max\{d(z, Sx_{2n}), d(z, Su), d(Sx_{2n}, Tx_{2n+1}),\\ &\quad \frac{1}{2}(d(z, Tx_{2n+1}) + d(Sx_{2n}, Su))\}.\end{aligned}$$

Now taking the limit as $n \to \infty$ we have,

$$\begin{aligned}\lambda(u, x_{2n+1}) &= \max\{d(z, Su), d(z, Su), d(Su, Tu), \frac{1}{2}(d(z, Tu) + d(z, Su))\}\\ &= \max\{d(z, Su), d(z, Su), d(Su, z), \frac{1}{2}(d(z, z) + d(z, Su))\}\\ &= d(z, Su)\end{aligned}$$

Thus

$$\begin{aligned}\psi(d(Su, z)) &\leq F(\psi(\Phi(d(Su, z))), \varphi(\Phi(d(Su, z))) + \psi(d(z, z)))\\ &= F(\psi(\Phi(d(Su, z))), \varphi(\Phi(d(Su, z))))\end{aligned}$$

If $Su \neq z$ then $d(Su, z) > 0$ and hence as Φ is contracive modulus $F(\psi(\Phi(d(Su, z))), \varphi(\Phi(d(Su, z))) < F(\psi(d(Su, z)), \varphi(d(Su, z)))$ Which is a contradiction, $Su = z$ so, $Pu = Su = z$

So u is a coincidence point if P and S. The pair of maps S and P are weakly compatible $SPu = PSu$ that is $Sz = Pz$.
$S(X) \subseteq Q(X)$, there exists a point $v \in X$ such that $z = Qv$.
Then we have

$$\begin{aligned}
\psi(d(z,Tv)) &= \psi(d(Su,Tv)) \\
&\leq F(\psi(\Phi(\lambda(u,v))), \varphi(\Phi(\lambda(u,v))) \\
&\leq F(\psi(\lambda(u,v)), \varphi(\lambda(u,v)) \\
&= F(\psi(\max\{d(Pu,Qv), d(Pu,Su), d(Qv,Tv), \\
&\quad \frac{1}{2}(d(Pu,Tv) + d(Qv,Su))\}), \varphi(\max\{d(Pu,Qv), \\
&\quad d(Pu,Su), d(Qv,Tv), \frac{1}{2}(d(Pu,Tv) + d(Qv,Su))\}) \\
&= F(\psi(\max\{d(z,z), d(z,z), d(z,Tv), \\
&\quad \frac{1}{2}(d(z,Tv) + d(z,z))\}, \varphi(\max\{d(z,z), d(z,z), d(z,Tv), \\
&\quad \frac{1}{2}(d(z,Tv) + d(z,z))\})) \\
&= F(\psi(d(z,Tv)), \varphi(d(z,Tv))
\end{aligned}$$

Thus $\psi(d(z,Tv)) \leq F(\psi(\Phi(d(z,Tv))), \varphi(\Phi(d(z,Tv))))$.
If $Tv \neq z$ then $d(z,Tv) \geq 0$ and hence as Φ is contractive modulus

$$F(\psi(\Phi(d(z,Tv))), \varphi(\Phi(d(z,Tv))) < F(\psi(d(z,Tv))\varphi(d(z,Tv)))$$

which is a contradiction. Therefore $Tv = Qv = z$
So, v is a coincidence point of Q and T.
Since the pair of maps Q and T are weakly compatible, $QTv = TQv$
(i.e) $Qz = Tz$.
Now show that z is a fixed point of S.

We have

$$
\begin{aligned}
\psi(d(Sz,z)) &= \psi(d(Sz,Tv)) \\
&\le F(\psi(\Phi(\lambda(z,v))),\varphi(\Phi(\lambda(z,v)))) \\
&\le F(\psi(\lambda(z,v)),\varphi(\lambda(z,v))) \\
&= F(\psi(\max\{d(Pz,Qv),d(Pz,Sz),d(Qv,Tv),\frac{1}{2}(d(Pz,Tv)+d(Qv,Sz))\}), \\
&\varphi(\max\{d(Pz,Qv),d(Pz,Sz),d(Qv,Tv),\frac{1}{2}(d(Pz,Tv)+d(Qv,Sz))\})) \\
&= F(\psi(\max\{d(Sz,z),d(Sz,Sz),d(z,z),\frac{1}{2}(d(Sz,z)+d(z,Sz))\}), \\
&\varphi(\max\{d(Sz,z),d(Sz,Sz),d(z,z),\frac{1}{2}(d(Sz,z)+d(z,Sz))\})) \\
&= F(\psi(d(Sz,z)),\varphi(d(Sz,z)))
\end{aligned}
$$

Thus $\psi(d(Sz,z)) \le F(\psi(\Phi(d(Sz,z))),\varphi(\Phi(d(Sz,z))))$.
If $Sz \neq z$ then $d(Sz,z) > 0$ and hence as Φ is contractive modulus
$F(\psi(\Phi(d(Sz,z))),\varphi(\Phi(d(Sz,z))) < F(\psi(d(Sz,z)),\varphi(d(Sz,z)))$
which is a contradiction. There exits $Sz = z$. Hence $Sz = Pz = z$
Show that z is a fixed point of T.
We have

$$
\begin{aligned}
\psi(d(z,Tz)) &= \psi(d(Sz,Tz)) \\
&\le F(\psi(\Phi(\lambda(z,z))),\varphi(\Phi(\lambda(z,z))) \\
&\le F(\psi(\lambda(z,z)),\varphi(\lambda(z,z)) \\
&= F(\psi(\max\{d(Pz,Qz),d(Pz,Sz),d(Qz,Tz),\frac{1}{2}(d(Pz,Tz)+d(Qz,Sz))\}), \\
&\varphi(\max\{d(Pz,Qz),d(Pz,Sz),d(Qz,Tz),\frac{1}{2}(d(Pz,Tz)+d(Qz,Sz))\}) \\
&= F(\psi(\max\{d(z,Tz),d(z,z),d(Tz,Tz),\frac{1}{2}(d(z,Tz)+d(Tz,z))\}) \\
&\varphi(\max\{d(z,Tz),d(z,z),d(Tz,Tz),\frac{1}{2}(d(z,Tz)+d(Tz,z))\}) \\
&= F(\psi(d(z,Tz)),\varphi(d(z,Tz))
\end{aligned}
$$

Thus $\psi(d(z,Tz)) \le F(\psi(\Phi(d(z,Tz))),\varphi(\Phi(d(z,Tz))))$.
If $z \neq Tz$ then $d(z,Tz) > 0$ and hence as Φ is contractive modulus

$$F(\psi(\Phi(d(z,Tz))),\varphi(\Phi(d(z,Tz))) < F(\psi(d(z,Tz)),\varphi(d(z,Tz))).$$

which is a contradiction. Hence $z = Tz$.
Therefore $Tz = Qz = z$.
Therefore $Sz = Pz = Tz = Qz = z$.
That is z is common fixed point of P, Q, S and T.

Uniqueness

Suppose, z and w is $(z \neq w)$ are common fixed point of P, Q, S and T.
we have

$$\begin{aligned}\psi(d(z,w)) &= \psi(d(Sz,Tw))\\ &\leq F(\psi(\Phi(\lambda(z,w))),\varphi(\Phi(\lambda(z,w)))\\ &\leq F(\psi(\lambda(z,w)),\varphi(\lambda(z,w))\\ &= F(\psi(\max\{d(Pz,Qw),d(Pz,Sz),d(Qw,Tw),\frac{1}{2}(d(Pz,Tw)+d(Qw,Sz))\})),\\ &\varphi(\max\{d(Pz,Qw),d(Pz,Sz),d(Qw,Tw),\frac{1}{2}(d(Pz,Tw)+d(Qw,Sz))\})\\ &= F(\psi(\max\{d(z,w),d(z,z),d(w,w),\frac{1}{2}(d(z,w)+d(w,z))\}),\\ &\varphi(\max\{d(z,w),d(z,z),d(w,w),\frac{1}{2}(d(z,w)+d(w,z))\})\\ &= F(\psi(d(z,w)),\varphi(d(z,w))\end{aligned}$$

Thus, $\psi(d(z,w)) \leq F(\psi(\Phi(d(z,w))),\varphi(\Phi(d(z,w))))$
Since $z \neq w$, then $d(z, wd(> 0$ and hence as Φ is contractive modulus.

$$F(\psi(\Phi(d(z,w))),\varphi(\Phi(d(z,w)))) < F(\psi(d(z,w)),\varphi(d(z,w)))$$

$$\text{therefore}\quad F(\psi(d(z,w)),\varphi(d(z,w)) < F(\psi(d(z,w)),\varphi(d(z,w))$$

which is a contradiction,

$$z = w$$

Thus z is the unique common fixed point of P, Q, S and T. □

Corollary 2.5 : Let (X, d) be a complete cone metric space with regular cone P such that $d(x, y) \in intP$. Suppose that the mappings P, S and T are three self maps of (X, d) such that $T(X) \subseteq P(X)$ and $S(X) \subseteq P(X)$ satisfying

$$\psi(d(Sx,Ty)) \leq F(\psi(\Phi(\lambda(x,y))),\varphi(\Phi(\lambda(x,y)))), \tag{2.10}$$

for all $x, y \in X$, where ψ and φ are altering distance and ultra altering distance functions respectively, $F \in \mathcal{C}$ such that $\psi(t+s) \leq \psi(t) + \psi(s)$ and Φ is contractive modulus.
$\lambda(x,y) = \max\{d(Px,Py), d(Px,Sx), d(Py,Ty), \frac{1}{2}\{d(Px,Ty) + d(Py,Sx)\}\}$. The pair $\{S,P\}$ and $\{T,P\}$ are weakly compatible. Then P, S and T have a unique common fixed point.

Proof : The proof of the corollary immediate by taking $P = Q$ in the above Theorem 2.4.

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