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GREATEST COMMON DIVISOR DEGREE ENERGY OF GRAPHS

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Abstract

Let G be a simple graph of order n. We introduce the concept of greatest common divisor degree matrix $M(G)$ of G and greatest common divisor degree energy $E_{GCD}(G)$ of G. Also we compute the greatest common divisor energy of some classes of graphs and regular graphs of order 10.

1. Introduction

In 1736, Euler first introduced the concept of graph theory. Energy of graphs was first introduced by I.G.Gutman in 1978 [6]. Spectral theory has emerged as a potential area of interdisciplinary research and Energy of graph is of recent interest. Chandrasekar

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Key Words : *Energy of graphs, Maximum degree energy of graphs, Greatest common divisor degree energy of a graph.*

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Adiga and Smitha introduced the concept of maximum degree energy and discussed about the maximum degree energy of some standard graphs [3]. These motivates us to introduce an greatest common degree energy of graphs and discuss about the greatest common divisor energy of some standard graphs.

2. Preliminaries

We present some known definitions and results related to energy of graphs and greatest common divisor energy of graphs for ready reference to go through the work presented in this paper.

Definition 2.1 [6] : Let G be a simple graph and let $V(G) = \{v_1, v_2, ..., v_n\}$ be its vertex set. The adjacency matrix $A(G)$ of the graph G is a square matrix of order n whose $(i, j)^{th}$ entry is equal to unity if the vertices v_i and v_j are adjacent and is equal to zero otherwise.

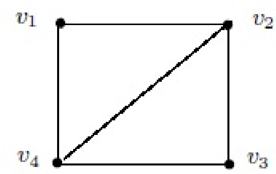
Definition 2.2 [6] : Let G be a simple graph and let $A(G)$ be the adjacency matrix of the graph G. The eigen values $\lambda_1, \lambda_2, ..., \lambda_n$ of $A(G)$, assumed in non-increasing order, are the eigen values of the graph G. Then the energy $E(G)$ of G is defined as the sum of the absolute values of its eigen values.$i.e.E(G) = \sum_{i=1}^{n} |\lambda_i|$.

Definition 2.3 [3] : Let G be a simple graph with n vertices $v_1, v_2, ..., v_n$ and let d_i be the degree of v_i, $i = 1, 2, ..., n$. Then the maximum degree matrix $M(G) = [d_{ij}]$ is defined as

$$d_{ij} = \begin{cases} max\{d_i, d_j\} & \text{if } \boldsymbol{v}_i, v_j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases}.$$

Denote the eigen values of the maximum degree matrix $M(G)$ by $\mu_1, \mu_2, ..., \mu_n$ and label them in non-increasing order. The maximum degree energy of G is defined as $E_M(G) = \sum_{i=1}^{n} |\mu_i|$.

Example 2.4 : Consider a following graph G. Here, $d(v_1) = d(v_3) = 3$ and $d(v_2) = d(v_4) = 2$.



In the above example, the maximum degree matrix of the graph G is

$$M(G) = \begin{pmatrix} 0 & 3 & 3 & 3 \\ 3 & 0 & 3 & 0 \\ 3 & 3 & 0 & 3 \\ 3 & 0 & 3 & 0 \end{pmatrix}.$$

Then the maximum degree eigen values are $\mu_1 = 0, \mu_2 = -1, \mu_3 = \frac{-3\sqrt{17}+3}{2}$ and $\mu_4 = \frac{3\sqrt{17}+3}{2}$ and also the maximum degree energy of the graph G is $E_M(G) \approxeq 13.3693$.

3. Greatest Common Divisor Degree Energy

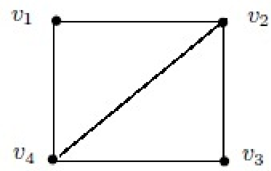
In this section, we define a greatest common divisor matrix and greatest common divisor energy of a simple graph G. Also we find the greatest common divisor energy of complete graph K_n and the star graph S_n.

Definition 3.1 : Let G be a simple graph with n vertices $v_1, v_2, ..., v_n$ and let d_i be the degree of v_i for every $i = 1, 2, ..., n$. Define

$$a_{ij} = \begin{cases} g.c.d.\{d_i, d_j\} & \text{if } \boldsymbol{v}_i, v_j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases}.$$

Then the $n \times n$ matrix $M(G) = [a_{ij}]$ is called a greatest common divisor degree matrix(g.c.d. degree matrix) of G. The g.c.d. degree characteristic polynomial of the g.c.d. degree matrix M(G) is defined by $\phi(G; \lambda) = det(\lambda I - M(G)) = \lambda^n + c_1\lambda^{(n-1)} + ... + c_n$ where I is the unit matrix of order n. The roots $\lambda_1, \lambda_2, ..., \lambda_n$ assumed in non-increasing order of $\phi(G; \lambda) = 0$ are the greatest common divisor degree eigen values(g.c.d. degree eigen values) of G. The greatest common divisor degree energy (g.c.d. degree energy) of a graph G is defined as $E_{GCD}(G) = \sum\limits_{i=1}^{n} |\lambda_i|$.

Example 3.2 : Consider a following graph G.Here, $d(v_1) = d(v_3) = 3$ and $d(v_2) = d(v_4) = 2$.



In the above example, the g.c.d. degree matrix of the graph G is $M(G) = \begin{pmatrix} 0 & 1 & 3 & 1 \\ 1 & 0 & 1 & 0 \\ 3 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$. Then the g.c.d. degree eigen values are $\lambda_1 = 0$,$\lambda_2 = -3$, $\lambda_3 = \dfrac{\sqrt{33}+3}{2}$ and $\lambda_4 = \dfrac{-\sqrt{33}+3}{2}$ and the greatest common divisor degree energy of the graph G is $E_{GCD}(G) \approxeq 8.7446$.

Theorem 3.3 : If G is the complete graph K_n, then $-(n-1)$ and $-(n-1)^2$ are g.c.d. degree eigen values of G with multiplicity $(n-1)$ and 1 respectively and $E_{GCD}(K_n) = 2(n-1)^2$.

Proof : We have

$$
\begin{aligned}
|\lambda I - M(K_n)| &= \begin{vmatrix} \lambda & -(n-1) & -(n-1) & ... & -(n-1) \\ -(n-1) & \lambda & -(n-1) & ... & -(n-1) \\ -(n-1) & -(n-1) & \lambda & ... & -(n-1) \\ ... & ... & ... & ... & ... \\ -(n-1) & -(n-1) & -(n-1) & ... & \lambda \end{vmatrix} \\
&= \begin{vmatrix} \lambda & -(n-1) & -(n-1) & ... & -(n-1) \\ -\lambda-(n-1) & \lambda+(n-1) & 0 & ... & 0 \\ -\lambda-(n-1) & 0 & \lambda+(n-1) & ... & 0 \\ ... & ... & ... & ... & ... \\ -\lambda-(n-1) & 0 & 0 & ... & \lambda+(n-1) \end{vmatrix} \\
&= (\lambda+(n-1))^{n-1} \begin{vmatrix} \lambda & -(n-1) & -(n-1) & ... & -(n-1) \\ -1 & 1 & 0 & ... & 0 \\ -1 & 0 & 1 & ... & 0 \\ ... & ... & ... & ... & ... \\ -1 & 0 & 0 & ... & 1 \end{vmatrix} \\
&= (\lambda+(n-1))^{n-1}(\lambda+(n-1)^2).
\end{aligned}
$$

Therefore $-(n-1)$ and $-(n-1)^2$ are g.c.d. degree eigen values of G with multiplicity

$(n-1)$ and 1 respectively.

Hence $E_{GCD}(K_n) = 2(n-1)^2$. □

Theorem 3.4 : If G is a star graph S_n of order n, then 0 is the g.c.d. degree eigen values of G with multiplicity $n-2$ and also $\sqrt{n-1}$ and $-\sqrt{n-1}$ are g.c.d. degree eigen values of G and each has multiplicity 1 and $E_{GCD}(S_n) = 2\sqrt{n-1}$.

Proof : We have

$$|\lambda I - M(S_n)| = \begin{vmatrix} \lambda & -1 & -1 & ... & -1 & -1 \\ -1 & \lambda & 0 & ... & 0 & 0 \\ -1 & 0 & \lambda & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & \lambda & 0 \\ -1 & 0 & 0 & ... & 0 & \lambda \end{vmatrix}$$

$$= \begin{vmatrix} \lambda & -1 & -1 & ... & -1 & -1 \\ -1 & \lambda & 0 & ... & 0 & 0 \\ 0 & -\lambda & \lambda & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & \lambda & 0 \\ 0 & 0 & 0 & ... & -\lambda & \lambda \end{vmatrix}$$

$$= \lambda^n - (n-1)\lambda^{n-2}.$$

Therefore 0 is the g.c.d. degree eigen values of G with multiplicity $n-2$ and also $\sqrt{n-1}$ and $-\sqrt{n-1}$ are g.c.d. degree eigen values of G and each has multiplicity 1.

Hence $E_{GCD}(S_n) = 2\sqrt{n-1}$.

Remark 3.5 : $E_{GCD}(S_n) \leq E_{GCD}(K_n)$.

4. G. C. D. Degree Energy of 3-Regular Graphs of Order 10

In this section, we define g.c.d. degree equivalence of two graphs G and H and g.c.d. degree unique of a graph G. Also we compute the g.c.d. degree energy of 3-regular graphs of order 10. Finally, we prove that Peterson graph P is not g.c.d. degree unique.

Definition 4.1 : Two graphs G and H are said to be g.c.d. degree equivalent if $E_{GCD}(G) = E_{GCD}(H)$ and written as $G \sim H$.

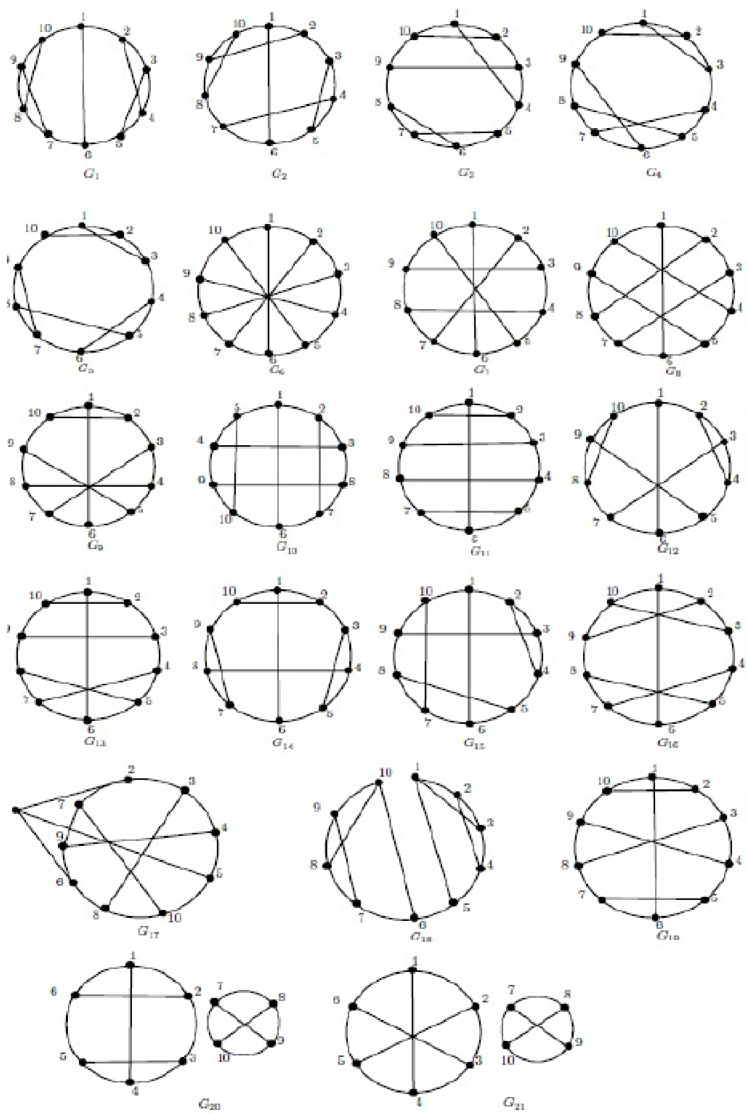
Remark 4.2 : The relation $\sim$ is an equivalence relation on the family $\mathbb{G}$ of graphs.

Notation 4.3 : The g.c.d. degree energy equivalence class determined by G is denoted by $[G]$. That is for given graph $G \in \mathbb{G}$, $[G] = \{H \in \mathbb{G} : H \sim G\}$.

Definition 4.4 : A graph G is said to be g.c.d degree energy unique if $[G] = \{G\}$.

Let us consider the g.c.d. degree characteristic polynomial of 3-regular graphs of order 10. Also, we shall compute g.c.d. degree energy of this class of graphs. There are exactly 21 cubic graphs of order 10 given in figure 4.1 [7] and these 21 graphs are denoted by $G_1, G_2, ..., G_{21}$ given in figure 4.1. We prove that Peterson graph $P$is not g.c.d. degree unique but can be determined by its g.c.d. degree energy and its g.c.d degree eigen values.

Figure 4.1: Cubic graphs of order 10.



We computed the g.c.d degree characteristic polynomials of 3-regular graphs of order 10 in table 4.1.

Table 4.1 : g.c.d. degree characteristic polynomial of cubic graphs of order 10.

G_i	$\phi(G_i;\lambda)$
G_1	$\lambda^{10} - 135\lambda^8 - 216\lambda^7 + 5171\lambda^6 + 15552\lambda^5 - 73629\lambda^4 - 227448\lambda^3 + 288684\lambda^2 + 944784\lambda$
G_2	$\lambda^{10} - 135\lambda^8 - 108\lambda^7 + 5751\lambda^6 + 6804\lambda^5 - 88209\lambda^4 - 104976\lambda^3 + 419904\lambda^2 + 472392\lambda$
G_3	$\lambda^{10} - 135\lambda^8 - 162\lambda^7 + 5589\lambda^6 + 11664\lambda^5 - 69984\lambda^4 - 166212\lambda^3 + 196830\lambda^2 + 517758\lambda + 177147$
G_4	$\lambda^{10} - 135\lambda^8 - 108\lambda^7 + 5103\lambda^6 + 8748\lambda^5 - 44469\lambda^4 - 122472\lambda^3 - 78732\lambda^2$
G_5	$\lambda^{10} - 135\lambda^8 - 216\lambda^7 + 5751\lambda^6 + 16524\lambda^5 - 67797\lambda^4 - 288684\lambda^3 - 236196\lambda^2$
G_6	$\lambda^{10} - 135\lambda^8 - 108\lambda^7 + 5751\lambda^6 + 6804\lambda^5 - 88209\lambda^4 - 104976\lambda^3 + 419904\lambda^2 + 472392\lambda$
G_7	$\lambda^{10} - 135\lambda^8 + 5589\lambda^6 - 2916\lambda^5 - 85293\lambda^4 + 78732\lambda^3 + 387099\lambda^2 - 236196\lambda - 531441$
G_8	$\lambda^{10} - 135\lambda^8 + 5751\lambda^6 - 3888\lambda^5 - 96957\lambda^4 + 139968\lambda^3 + 498636\lambda^2 - 944784\lambda$
G_9	$\lambda^{10} - 135\lambda^8 - 54\lambda^7 + 5751\lambda^6 + 1944\lambda^5 - 962888\lambda^4 - 4374\lambda^3 + 597051\lambda^2 - 157464\lambda - 708588$
G_{10}	$\lambda^{10} - 135\lambda^8 + 5265\lambda^6 - 972\lambda^5 - 61965\lambda^4 - 43740\lambda^3 + 229635\lambda^2 + 393660\lambda + 177147$
G_{11}	$\lambda^{10} - 135\lambda^8 - 107\lambda^7 + 5589\lambda^6 + 7776\lambda^5 - 76545\lambda^4 - 139968\lambda^3 + 150903\lambda^2 + 393660\lambda + 177147$
G_{12}	$\lambda^{10} - 135\lambda^8 - 107\lambda^7 + 6075\lambda^6 + 5852\lambda^5 - 114453\lambda^4 - 78732\lambda^3 + 944784\lambda^2 + 314928\lambda - 2834352$
G_{13}	$\lambda^{10} - 135\lambda^8 - 54\lambda^7 + 5427\lambda^6 + 2916\lambda^5 - 69984\lambda^4 - 48114\lambda^3 + 229635\lambda^2 + 236196\lambda + 708544$
G_{14}	$\lambda^{10} - 135\lambda^8 - 162\lambda^7 + 6075\lambda^6 + 11664\lambda^5 - 104976\lambda^4 - 249318\lambda^3 + 492075\lambda^2 + 133844\lambda + 708544$
G_{15}	$\lambda^{10} - 135\lambda^8 - 54\lambda^7 + 5589\lambda^6 + 2916\lambda^5 - 84564\lambda^4 - 52488\lambda^3 + 354294\lambda^2 + 511758\lambda + 177147$
G_{16}	$\lambda^{10} - 135\lambda^8 + 5103\lambda^6 - 61965\lambda^4 + 236196\lambda^2$
G_{17}	$\lambda^{10} - 135\lambda^8 + 6075\lambda^6 - 5832\lambda^5 - 120285\lambda^4 + 262440\lambda^3 + 787320\lambda^2 - 3149280\lambda + 2834352$
G_{18}	$\lambda^{10} - 135\lambda^8 - 216\lambda^7 + 5103\lambda^6 + 15552\lambda^5 - 26973\lambda^4 - 122472\lambda^3 - 78732\lambda^2$
G_{19}	$\lambda^{10} - 135\lambda^8 - 108\lambda^7 + 5913\lambda^6 + 6804\lambda^5 - 102789\lambda^4 - 113724\lambda^3 + 649539\lambda^2 + 314928\lambda - 1240029$
G_{20}	$\lambda^{10} - 135\lambda^8 - 324\lambda^7 + 5103\lambda^6 + 23328\lambda^5 - 9477\lambda^4 - 183708\lambda^3 - 236196\lambda^2$
G_{21}	$\lambda^{10} - 135\lambda^8 - 216\lambda^7 + 4131\lambda^6 + 17496\lambda^5 + 19683\lambda^4$

By comparing the roots of g.c.d. degree characteristic polynomial of cubic graphs of order 10, we can find out the g.c.d. degree energy of these graphs. We compute them to four decimal place in table 4.2.

Table 4.2 : g.c.d. degree energy of cubic graphs of order 10.

G_i	$E_{GCD}(G_i)$	G_i	$E_{GCD}(G_i)$	G_i	$E_{GCD}(G_i)$
G_1	45.3694	G_8	45.3694	G_{15}	44.3840
G_2	44.5790	G_9	45.9480	G_{16}	42.0000
G_3	44.4360	G_{10}	43.4160	G_{17}	48.0000
G_4	40.5440	G_{11}	44.1050	G_{18}	40.6710
G_5	42.8780	G_{12}	48.0000	G_{19}	46.7370
G_6	44.8320	G_{13}	43.1430	G_{20}	42.0000
G_7	45.2320	G_{14}	46.5530	G_{21}	36.0000

Theorem 4.5 : Six cubic graphs of order 10 are not g.c.d. degree energy unique. The g.c.d. degree energy eigen values of two cubic graphs of order 10 are different in exactly 3 values if they have equal g.c.d. degree energy.

Proof : Using table 4.2, we have $[G_1] = \{G_1, G_8\}, [G_{13}] = \{G_{13}, G_{16}\}$ and $[G_{12}] = \{G_{12}, G_{17}\}$.
From table 4.1, we have

$$\phi(G_1;\lambda) = \lambda(\lambda+3)^2(\lambda-3)^2(\lambda+6)^2(\lambda-9)\left(\lambda - \frac{3\sqrt{17}+3}{2}\right)\left(\lambda + \frac{3\sqrt{17}-3}{2}\right) and$$
$$\phi(G_8;\lambda) = \lambda(\lambda+3)^2(\lambda-3)^2(\lambda+6)^2(\lambda-9)\left(\lambda + \frac{3\sqrt{17}+3}{2}\right)\left(\lambda + \frac{-3\sqrt{17}+3}{2}\right).$$

Also,

$$\phi(G_{16};\lambda) = \lambda^2(\lambda-3)(\lambda+3)^3(\lambda+6)(\lambda-6)(\lambda+9)(\lambda-9) and$$
$$\phi(G_{20};\lambda) = \lambda^2(\lambda+3)^3(\lambda-3)(\lambda-6)^2(\lambda-9)^2.$$

Similarly,

$$\phi(G_{12};\lambda) = (\lambda+3)^2(\lambda-3)^3(\lambda+6)^3(\lambda-6)(\lambda-9) and$$
$$\phi(G_{17};\lambda) = (\lambda-3)^5(\lambda+6)^4(\lambda-9).$$

□

Theorem 4.6 : Let $\mathbb{G}$ be the family of 3-regular graphs of order 10. For the Peterson graph P, we have the following properties:

(i) P is not g.c.d. degree energy unique in $\mathbb{G}$.

(ii) P has the maximum g.c.d. degree energy in $\mathbb{G}$.

(iii) P can be identify by its g.c.d. degree energy and its g.c.d. degree eigen values in $\mathbb{G}$.

Proof : (i)The g.c.d. degree matrix of P is

$$M(G) = \begin{bmatrix} 0 & 3 & 0 & 0 & 3 & 3 & 0 & 0 & 0 & 0 \\ 3 & 0 & 3 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 3 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 & 3 & 0 & 0 & 0 & 3 & 0 \\ 3 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 3 \\ 3 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 3 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 3 & 0 & 0 & 3 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 3 & 0 & 3 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 3 & 3 & 0 & 0 \end{bmatrix}$$

Then the characteristic polynomial is

$$\begin{aligned}|\lambda I - M(P)| &= \lambda^{10} - 135\lambda^8 + 6075\lambda^6 - 5832\lambda^5 - 120285\lambda^4 + 262440\lambda^3 \\ &\quad + 787320\lambda^2 - 3149280\lambda + 2834352 \\ &= (\lambda - 3)^5(\lambda + 6)^4(\lambda - 9).\end{aligned}$$

Then we have

$$\lambda_1 = 9, \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = 3, \lambda_7 = \lambda_8 = \lambda_9 = \lambda_{10} = -6.$$

Therefore $E_{GCD}(P) = 48$.

By table 4.1, we have $P \in \{G_{12}, G_{17}\}$.

Hence P is not g.c.d. degree unique in $\mathbb{G}$.

(ii) From the table 4.2, P has the maximum g.c.d. degree energy in $\mathbb{G}$.

(iii) From the theorem 4.5, $\phi(G_{12}; \lambda) = \phi(P; \lambda)$.

Hence G_{17} is the Peterson graph. □

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