International J. of Math. Sci. & Engg. Appls. (IJMSEA)
ISSN 0973-9424, Vol. 11 No. II (August, 2017), pp. 173-179

SOLUTION OF DIFFERENTIAL EQUATIONS BY USING DIFFERETIAL TRANSFORM METHOD

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Abstract

In this article, differential transform method (DTM) is applied to solve initial value problems in ordinary and partial differential equations. If the system considered has a solution in terms of the series expansion of known functions, this powerful method catches the exact solution. So as to show this capability and robustness, some systems of differential equations are solved as numerical examples.

1. Introduction

The differential transform was first introduced by Zhou [1] and it is applied to solve differential equation occurred in electrical circuit analysis. The DTM is the method to determine the coefficients of the Taylor series of the function by solving the induced recursive equation from the given differential equation. The updated version of the Taylor series method which is called the differential transform method. It is possible to obtain exact solution of various Initial value problems using the concept of Differential

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Key Words : *Differential transform method, Differential inverse transform, Ordinary differential equations, Partial differential equation.*

transform method. Initial value problem in the second order differential equation occurs in science and engineering fields. We extend the application of the differential transformation method which is based on Taylor series expansion to obtain analytical approximate solution of the initial value problem.

This paper is organized as, in Section 2 the one-dimensional Differential transform method is described and solved some problems. In section 3 two-dimensional Differential transform method is described with example and conclusion is given in Section 4.

2. Differential Transformation Method

Definition 1 : The differential transformation of the k-th derivative of function $u(x)$ is defined as follows.

$$U(k) = \frac{1}{k!}\left[\frac{d^k u(x)}{dx^k}\right]_{x=x_0} \tag{2.1}$$

and the differential inverse transformation of $U(k)$ is defined as follows:

$$u(x) = \sum_{k=0}^{\infty} U(k)(x-x_0)^k. \tag{2.2}$$

In real applications, function $u(x)$ is expressed by a finite series and equation (2.2) can be written as

$$u(x) = \sum_{k=0}^{n} U(k)(x-x_0)^k. \tag{2.3}$$

Equation (2.3) implies $\sum\limits_{k=n+1}^{\infty} U(k)(x-x_0)^k$ is negligibly small. In fact, n is decided by the convergence of natural frequency in this study. The following theorems that can be deduced from equation (2.1) and (2.2) are given below.

Theorem 1 : If $u(x) = y(x) \pm z(x)$ then $U(k) = Y(k) \pm Z(k)$.

Theorem 2 : If $u(x) = ay(x)$, then $U(k) = aY(k)$, where a is a constant.

Theorem 3 : If $u(x) = \left(\frac{d^m y(x)}{dx^m}\right)$ then $U(k) = \frac{(m+k)!}{k!}Y(k+m)$.

Theorem 4 : If $u(x) = y(x)z(x)$, then $U(k) = \sum\limits_{k_1=0}^{k} Y(k_1)Z(k-k_1)$.

Theorem 5 : If $u(x) = x^n$ then $U(k) = \delta(k-n)$, $\delta(k-n) = \begin{cases} 1 & k=n \\ 0 & k \neq n \end{cases}$

To demonstrate the method introduced in this study, few examples are solved here.

Example 1 : We first start by considering the following equatiion given by

$$u''(x) + \frac{4}{x}u'(x) + u(x) = 10 + 18x + x^2 + x^3, ` \; 0 < x \leq 1 \tag{2.4}$$

with initial conditions

$$u(0) = 0, \quad u'(0) = 0. \tag{2.5}$$

By multiplying both sides of equation (2.4) by x and then taking differential transformation of both sides of the resulting equation using Theorems 1-5, the following recurrence relation is obtained:

$$U(k+1) = \frac{1}{(k+1)(k+4)} \times (10\delta(k-1)+18\delta(k-2)+\delta(k-3)+\delta(k-4)-\sum_{l=0}^{k}\delta(l-1)U(k-l)). \tag{2.6}$$

By using equation (2.1) and (2.5), the following transformed intial conditions at $x_0 = 0$ can be obtained:

$$U(0) = 0 \tag{2.7}$$

$$U(1) = 0. \tag{2.8}$$

Substituting equation (2.7) and (2.8) at $k = 1$ into equation (2.6), we have

$$U(2) = 1 \tag{2.9}$$

Following the same recursive procedure, we find $U(k+1) = 0, k = 3, 4, 5, \cdots$ and listing the computation and result corresponding to $n = 3$, we have

$$U(3) = 1. \tag{2.10}$$

Using equations (2.7) - (2.10) and the inverse transformation rule in equation (2.3), we get the following solution

$$u(x) = x^2 + x^3. \tag{2.11}$$

Note that for $n > 3$ one evaluates the same solution, which is the exact solution of equation (2.4) with the initial conditions in equation (2.5).

Example 2 : We next consider the following Lane-Emden equation given in [6]

$$u''(x) + \frac{8}{x}u'(x) + xu(x) = x^5 - x^4 + 44x^2 - 30x, \quad 0 < x \leq 1 \tag{2.12}$$

with initial conditions

$$u(0) = 0, \quad u'(0) = 0. \tag{2.13}$$

By multiplying both sides of equatiion (2.12) by x and then taking differential transformation of both sides of the resulting equation using Theorems 1-5, we obtain the following recurrence relation

$$U(k+1) = \frac{1}{(k+1)(k+8)} \times (\delta(k-6) - \delta(k-5) + 44\delta(k-3) - 30\delta(k-2) - \sum_{l=0}^{k} \delta(l-2)U(k-l). \tag{2.14}$$

We apply the differential transformation at $x_0 = 0$, therefore, the initial conditions given in equation (2.13) are transformed as follows:

$$U(0) = 0, \tag{2.15}$$

$$U(1) = 0. \tag{2.16}$$

Substituting equatiion (2.15) and (2.16) at $k = 1$ into equation (2.14), we have

$$U(2) = 0. \tag{2.17}$$

Following the same recurswive procedure, we find $U(k+1) = 0, k = 4, 5, \cdots$ and listing the computation and result corresponding to $n = 4$, we have

$$U(3) = -1, \tag{2.18}$$

$$U(4) = 1. \tag{2.19}$$

Using equation (2.15) - (2.19) and the inverse transformation rule in equation (2.3), we get the following solution

$$u(x) = x^3 + x^4. \tag{2.20}$$

For $n > 4$, one evaluates that the solution (2.20), which is the exact solution of equation (2.12) under the initial conditions in equatiion (2.13).

3. Two-dimensional Differential Transform Method

The theory to solve partial differential equation by two-dimensional differential transform is given in [4] this section.

Definition 2 : The two-dimensional differential transform of function $w(x, y)$ is defined as follows:

$$W(k, h) = \frac{1}{k!h!} \left[\frac{\partial^{k+h}}{\partial x^k \partial y^h} w(x, y) \right]_{\substack{x = x_0 \\ y = y_0}} \tag{3.1}$$

$w(x,y)$ is the original function and $W(k,h)$ is the transformed function.

Definition 3 : The differential inverse transformation of $W(k,h)$ is defined as follows:

$$w(x,y) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} W(k,h)x^k y^h. \tag{3.2}$$

From above equation we obtain

$$w(x,y) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} \frac{1}{k!h!}\left[\frac{\partial^{k+h}}{\partial x^k \partial y h^h} w(x,y)\right]_{\substack{x=x_0\\ y=y_0}} x^k y^h. \tag{3.3}$$

Equation (3.3) implies that the concept of two-dimensional differential transform is derived from two-dimensional Taylor series expansion. The lower case letters represent the original functions and upper case letters stands for the transformed functions. The transformed function follows basic mathematical operations [4].

Theorem 6 : If $w(x,y) = u(x,y) \pm v(x,y)$ then $W(h,k) = U(h,k) \pm V(h,k)$.

Theorem 7 : If $w(x,y) = \lambda u(x,y)$ then $W(h,k) = \lambda U(h,k)$.

Theorem 8 : If $w(x,y) = \frac{\partial u(x,y)}{\partial x}$ then $W(h,k) = (k+1)U(k+1,h)$.

Theorem 9 : If $w(x,y) = \frac{\partial u(x,y)}{\partial y}$ then $W(h,k) = (h+1)U(k+1,h)$.

Theorem 10 : If $w(x,y) = \frac{\partial^{r+s} u(x,y)}{\partial x^r \partial y^s}$ then

$$W(k,h) = (k+1)(k+2),\cdots(k+r)(h+1)(h+2)\cdots(h+s)\cup(k+r,h+s)$$

Theorem 11 : If $w(x,y) = u(x,y)v(x,y)$ then $W(k,h) = \sum_{r=0}^{k}\sum_{s=0}^{h} U(r,h-s)V(k-r,s)$.

Theorem 12 : If $w(x,y) = x^m y^n$ then $w(k,h) = \delta(k-m,h-n) = \delta(k-m)\delta(h-n)$ where

$$\delta(k-m) = \begin{cases} 1, & k=m \\ 0, & k \neq m \end{cases} \qquad \delta(h-n) = \begin{cases} 1, & h=n \\ 0, & h \neq n \end{cases}.$$

Example 3 : Consider the partial differential equation [2]

$$\frac{\partial w(x,t)}{\partial t}\left[\frac{\partial w(x,t)}{\partial x}\right]^2 - 1 = 0 \tag{3.4}$$

with initial condition

$$W(x,0) = x \tag{3.5}$$

whose solution [2] is $w(x,y) = x + t$.

On taking two dimensional differential transform of (3.4) and by using the Theorems 6-12, we have

$$\sum_{r=0}^{k}\sum_{s=0}^{h}(k-r-1)^2(h-s+1)(W(k-r+1,s))^2W(r,h-s+1)-\delta(k,h)=0. \quad (3.6)$$

Using equation (3.2) and initial condition (3.5) we have

$$W(i,0)=0, \quad i=0,2,3,\cdots,m \quad (3.7)$$

$$W(1,0)=1. \quad (3.8)$$

Substituting equation (3.7) and (3.8) into (3.6) and by recursive method [3, 4] the results corresponding to $m\to\infty$ are listed

$$W(0,1)=1 \quad (3.9)$$

and all others are zero, substituting all $W(k,h)$ into (3.3) we obtain series solution as follows $w(x,t)=x+t$.

4. Conclusion

In this study, the differential transformation method is implemented to the Ordinary Differential Equations and partial differential equatiions as singular initial value problems. Differential equations are solved and exact solutions are obtained by using differential transformation method. It is shown that differential transformation method is a very fast convergent, precise and cost efficient tool for solving the ordinary differential equations and partial differential equations.

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