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ON FOURIER SERIES OF I-FUNCTIONS OF r-VARIABLES

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Abstract

The object of this paper is to evaluate some integrals involving the I-function of r-variables and employ them to obtain some Fourier Sine Series for I-function of r-variables. These results are useful in solving certain problems of Applied Mathematics and Engineering. On specializing the parameters the result can be reduced to integrals involving H-function of r-variables and in particular H-function of 2-variables. The corresponding Fourier Sine Series for these functions can also be derived which reduces to the result proved by K. L. Choudhary [1, p.53].

1. Introduction

Notations used :

$(a_p) = {}_1(a_j)_p$ stands for $a_1, a_2, \cdots, a_p$

${}_1(a_j; \alpha_j)_p$ stands for $(a_1; \alpha_1), (a_2, ; \alpha_2), \cdots, (a_p; \alpha_p)$.

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${}_1(a_j;\alpha_j,A_j)_p$ stands for $(a_1;\alpha_1,A_1),(a_2;\alpha_2,A_2),\cdots,(a_p;\alpha_p,A_p)$.
${}_1(a_j;\alpha_j,A_j;1)_p$ stands for $(a_1;\alpha_1,A_1;1),(a_2;\alpha_2,A_2;1),\cdots,(a_p;\alpha_p,A_p;1)$.
The generalized Fox's H-function, namely I-function of r-variables introduced by Prathima, et al. [2, p.38] is defined and represented in the following manner:

$$I[z_1,\cdots,z_r]=I^{0,N:m_1,n_1;\cdots;m_r,n_r}_{P,Q:p_1,q_1;\cdots,p_r,q_r}$$
$$\left[\begin{array}{c|c} z_1 & {}_1(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_P:\ {}_1(c_j^{(1)};\gamma_j^{(1)},C_j^{(1)})_{p_1},\cdots;\ {}_1(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r} \\ \cdots & \\ z_r & {}_1(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_Q:\ {}_1(d_j^{(1)},\delta_j^{(1)},D_j^{(1)})_{q_1};\cdots;\ {}_1(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{q_r}\end{array}\right]$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\theta_1(s_1)\cdots\theta_r(s_r)\ \phi(s_1,\cdots,s_r)\ z_1^{s_1}\cdots z_r^{s_r}ds_1\cdots ds_r, \qquad (1.1)$$

where $\phi(s_1,\cdots,s_r)$ and $\theta_i(s_i),i=1,2,\cdots,r$ are given by,

$$\phi(s_1,\cdots,s_r)=\frac{\prod\limits_{j=1}^{N}\Gamma^{A_j}\left(1-a_j+\sum\limits_{i=1}^{r}a_j^{(i)}s_i\right)}{\prod\limits_{j=1}^{Q}\Gamma^{B_j}\left(1-b_j+\sum\limits_{i=1}^{r}\beta_j^{(i)}s_i\right)\prod\limits_{j=N+1}^{P}\Gamma^{A_j}\left(a_j-\sum\limits_{i=1}^{r}\alpha_j^{(i)}s_i\right)}, \qquad (1.2)$$

$$\theta_i(s_i)=\frac{\prod\limits_{j=1}^{m_i}\Gamma^{D_j^{(i)}}(d_j^{(i)}-\delta_j^{(i)}s_i)\prod\limits_{j=1}^{n_i}\Gamma^{C_j^{(i)}}(1-c_j^{(i)}+\gamma_j^{(i)}s_i)}{\prod\limits_{j=m_i+1}^{q_i}\Gamma^{D_j^{(i)}}(1-d_j^{(i)}+\delta_j^{(i)}s_i)\prod\limits_{j=n_i+1}^{p_i}\Gamma^{C_j^{(i)}}(c_j^{(i)}-\gamma_j^{(i)}s_i)}. \qquad (1.3)$$

Also, $z_i\neq 0$ $(i=1,\cdots,r)$, $\omega=\sqrt{-1}$, m_j,n_j,p_j,q_j $(j=1,\cdots,r)$, N,P,Q are non-negative integers such that $0\leq N\leq P$, $Q\geq 0$, $0\leq m_j\leq q_j$, $0\leq n_j\leq p_j$ $(j=1,2,\cdots,r)$ (not all zero simultaneously). $\alpha_j^{(i)}$ $(j=1,2,\cdots,P,i=1,2,\cdots,r)$, $\beta_j^{(i)}$ $(j=1,2,\cdots,Q,i=1,2,\cdots,r)$, $\gamma_j^{(i)}$ $(j=1,2,\cdots,p_i,i=1,2,\cdots,r)$ and $\delta_j^{(i)}$ $(j=1,2,\cdots,q_i,i=1,2,\cdots,r)$ are positive numbers. a_j $(j=1,2,\cdots,P)$, b_j $(j=1,2,\cdots,Q)$, $c_j^{(i)}$ $(j=1,2,\cdots,p_i,i=1,2,\cdots,r)$ and $d_j^{(i)}$ $(j=1,2,\cdots,q_i,i=1,2,\cdots,r)$ are complex numbers. The exponents A_j $(j=1,2,\cdots,P)$, B_j $(j=1,2,\cdots,Q)$, $C_j^{(i)}$ $(j=1,2,\cdots,p_i,i=1,2,\cdots,r)$ and $D_j^{(i)}$ $(j=1,2,\cdots,q_i,i=1,2,\cdots,r)$ of various gamma functions may take non integer values. The I-function of r variables is analytic if

$$\Psi_i=\sum_{j=1}^{P}A_j\alpha_j^{(i)}-\sum_{j=1}^{Q}B_j\beta_j^{(i)}+\sum_{j=1}^{p_i}C_j^{(i)}\gamma_j^{(i)}-\sum_{j=1}^{q_i}D_j^{(i)}\delta_j^{(i)}\leq 0,\quad i=1,2,\cdots,r.$$

The integral (1.1) converges absolutely if $|arg(z_i)| < \frac{1}{2}\Delta_i\pi, i = 1, 2, \cdots, r$ where

$$\begin{aligned}\Delta_i &= -\sum_{j=n+1}^{P} A_j\alpha_j^{(i)} - \sum_{j=1}^{Q} B_j\beta_j^{(i)} + \sum_{j=1}^{m_i} D_j^{(i)}\delta_j^{(i)} - \sum_{j=m_i+1}^{q_i} D_j^{(i)}\delta_j^{(i)} \\ &\quad + \sum_{j=1}^{n_i} C_j^{(i)}\gamma_j^{(i)} - \sum_{j=n_i+1}^{p_i} C_j^{(i)}\gamma_j^{(i)} > 0.\end{aligned}$$

Hann D. B. D. E. [2, p. 73]

$$\int_0^{\pi/2} (\sin x)^{u-1}(\cos x)^{v-1} e^{i(u+v)x} dx = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)} e^{iu\frac{\pi}{2}} \tag{1.4}$$

Re $u > 0$, Re $v > 0$.

Lryzhk I. M. and Gradshtejn I. S. (3, p. 490]

$$\int_0^{\pi/2} e^{4mix} \sin 4nx dx = \begin{cases} 0, & \text{if} \quad m \neq n, m = n = 0 \\ \frac{\pi i}{4}, & \text{if} \quad m = n \neq 0. \end{cases} \tag{1.5}$$

2. The Integrals Involving I-functions of r-variables

(1)

$$\int_0^{\pi/2} (\sin x)^{u-1}(\cos x)^{v-1} e^{i(u+v)x} I^{0,N:m_1,n_1;\cdots;m_r,n_r}_{P,Q:p_1,q_1;\cdots,p_r,q_r} \left[\begin{array}{c} z_1(\sin x\cos x)^{h_1}e^{2h_1xi} \\ \vdots \\ z_r(\sin x\cos x)^{h_r}e^{2h_rxi} \end{array}\right.$$
$$\left|\begin{array}{l} {}_1(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_P : {}_1(c_j^{(1)},\gamma_j^{(1)},C_j^{(1)})_{p_1};\cdots; {}_1(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r} \\ {}_1(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_Q : {}_1(d_j^{(1)},\delta_j^{(1)},D_j^{(1)})_{q_1};\cdots; {}_1(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{q_r} \end{array}\right] dx$$
$$= e^{iu\frac{\pi}{2}} I^{0,N+2:m_1,n_1;\cdots;m_r,n_r}_{P+2,Q+1:p_1,q_1;\cdots,p_r,q_r} \left[\begin{array}{c} z_1e^{ih_1\frac{\pi}{2}} \\ \vdots \\ z_re^{ih_r\frac{\pi}{2}} \end{array}\right| \begin{array}{l} (1-u;h_1,\cdots,h_r;1),(1-v;h_1,\cdots,h_r;1), \\ {}_1(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_Q, \end{array}$$
$$\left.\begin{array}{l} {}_1(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_P : {}_1(c_j^{(1)},\gamma_j^{(1)},C_j^{(1)})_{p_1};\cdots; {}_1(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r} \\ (1-u-v;2h_1,\cdots,2h_r;1) : {}_1(d_j^{(1)},\delta_j^{(1)},D_j^{(1)})_{q_1};\cdots; {}_1(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{q_r} \end{array}\right] \tag{2.1}$$

$$Re\left(u + \sum_{i=1}^{r} h_i \frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > 0, \quad Re\left(v + \sum_{i=1}^{r} h_i \frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > 0, \quad j = 1, 2, \cdots, m_i.$$

Proof:

$$\int_0^{\pi/2} (\sin x)^{u-1} (\cos x)^{v-1} e^{i(u+v)x} I^{0,N:m_1,n_1;\cdots;m_r,n_r}_{P,Q:p_1,q_1;\cdots,p_r,q_r} \left[\begin{array}{l} z_1(\sin x\cos x)^{h_1} e^{2h_1 xi} \\ \vdots \\ z_r(\sin x\cos x)^{h_r} e^{2h_r xi} \end{array} \right| \begin{array}{l} {}_1(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_P : {}_1(c_j^{(1)},\gamma_j^{(1)},C_j^{(1)})_{p_1};\cdots;\ {}_1(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r} \\ {}_1(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_Q : {}_1(d_j^{(1)},\delta_j^{(1)},D_j^{(1)})_{q_1};\cdots;\ {}_1(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{q_r} \end{array} \right] dx$$

$$= \int_0^{\pi/2} (\sin x)^{u-1} (\cos x)^{v-1} e^{i(u+v)x} \left[\left(\frac{1}{2\pi i}\right)^r \int_{L_1} \cdots \int_{L_r} \phi(s_1,\cdots,s_r)\theta_1(s_1)\cdots,\theta_r(s_r) \right.$$
$$\left. (z_1(\cos x\sin x)^{h_1} e^{2h_1 ix})^{s_1} \cdots (z_r(\cos x\sin x)^{h_r} e^{2h_r ix})^{s_r} ds_1\cdots ds_r \right] dx.$$

Changing the order of integration

$$= \left(\frac{1}{2\pi i}\right)^r \int_{L_1} \cdots \int_{L_r} \phi(s_1,\cdots,s_r)\theta_1(s_1)\cdots\theta_r(s_r) z_1^{s_1}\cdots z_r^{s_r}$$
$$\left[\int_0^{\pi/2} (\sin x)^{u+\sum\limits_{i=1}^r h_i s_i - 1} (\cos x)^{v+\sum\limits_{i=1}^r h_i s_i - 1} e^{i\left(u+v+\sum\limits_{i=1}^r 2h_i s_i\right)x} dx \right] ds_1\cdots ds_r.$$

Solving the inner integral using (1.4),

$$= \left(\frac{1}{2\pi i}\right)^r \int_{L_1} \cdots \int_{L_r} \phi(s_1,\cdots,s_r)\theta_1(s_1)\cdots\theta_r(s_r) z_1^{s_1}\cdots z_r^{s_r}$$
$$\frac{\Gamma\left(u+\sum\limits_{i=1}^r h_i s_i\right)\Gamma\left(v+\sum\limits_{i=1}^r h_i s_i\right)}{\Gamma\left(u+v+\sum\limits_{i=1}^r 2h_i s_i\right)} e^{i\left(u+\sum\limits_{i=1}^r h_i s_i\right)\frac{\pi}{2}} ds_1\cdots ds_r.$$

From which (2.1) is obtained by using (1.1).

(2)

$$\int_0^{\pi/2}(\sin x)^{u-1}(\cos x)^{v-1}e^{i(u+v)x}I^{0,N:m_1,n_1;\cdots;m_r,n_r}_{P,Q:p_1,q_1;\cdots,p_r,q_r}\left[\begin{array}{c} z_1(\cos x)^{h_1}e^{h_i xi} \\ \vdots \\ z_r(\cos x)^{h_r}e^{h_r xi}\end{array}\right. \left|\begin{array}{c} {}_1(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_P:\ {}_1(c_j^{(1)},\gamma_j^{(1)},C_j^{(1)})_{p_1};\cdots;\ {}_1(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r} \\ {}_1(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_Q:\ {}_1(d_j^{(1)},\delta_j^{(1)},D_j^{(1)})_{q_1};\cdots;\ {}_1(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{q_r}\end{array}\right] dx$$

$$= \Gamma(u)e^{iu\frac{\pi}{2}}I^{0,N+1:m_1,n_1;\cdots;m_r,n_r}_{P+1,Q+1:p_1,q_1;\cdots,p_r,q_r}\left[\begin{array}{c} z_1 \\ \vdots \\ z_r\end{array}\right. \left|\begin{array}{l} (1-v;h_1,\cdots,h_r;1), \\ {}_1(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_Q,\end{array}\right.$$

$$\left.\begin{array}{c} {}_1(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_P:\ {}_1(c_j^{(1)},\gamma_j^{(1)},C_j^{(1)})_{p_1};\cdots;\ {}_1(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r} \\ (1-u-v;h_1,\cdots,h_r;1):\ {}_1(d_j^{(1)},\delta_j^{(1)},D_j^{(1)})_{q_1};\cdots;\ {}_1(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{q_r}\end{array}\right] \tag{2.2}$$

$$Re\left(v+\sum_{i=1}^{r}h_i\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>0,\quad j=1,2,\cdots,m_i.$$

Proof:

$$\int_0^{\pi/2}(\sin x)^{u-1}(\cos x)^{v-1}e^{i(u+v)x}I^{0,N:m_1,n_1;\cdots;m_r,n_r}_{P,Q:p_1,q_1;\cdots,p_r,q_r}\left[\begin{array}{c} z_1(\cos x)^{h_1}e^{h_i xi} \\ \vdots \\ z_r(\cos x)^{h_r}e^{h_r xi}\end{array}\right. \left|\begin{array}{c} {}_1(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_P:\ {}_1(c_j^{(1)},\gamma_j^{(1)},C_j^{(1)})_{p_1};\cdots;\ {}_1(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r} \\ {}_1(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_Q:\ {}_1(d_j^{(1)},\delta_j^{(1)},D_j^{(1)})_{q_1};\cdots;\ {}_1(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{q_r}\end{array}\right] dx$$

$$= \int_0^{\pi/2}(\sin x)^{u-1}(\cos x)^{v-1}e^{i(u+v)x}\left[\left(\frac{1}{2\pi i}\right)^r\int_{L_1}\cdots\int_{L_r}\phi(s_1,\cdots,s_r)\theta_1(s_1)\cdots,\theta_r(s_r)\right.$$
$$\left.(z_1(\cos x)^{h_1}e^{h_1ix})^{s_1}\cdots(z_r(\cos x)^{h_r}e^{h_rix})^{s_r}ds_1\cdots ds_r\right]dx.$$

Changing the order of integration

$$= \left(\frac{1}{2\pi i}\right)^r \int_{L_1} \cdots \int_{L_r} \phi(s_1, \cdots, s_r)\theta_1(s_1)\cdots\theta_r(s_r) z_1^{s_1}\cdots z_r^{s_r}$$
$$\left[\int_0^{\pi/2} (\sin x)^{u-1} (\cos x)^{v+\sum\limits_{i=1}^{r} h_i s_i - 1} e^{i\left(u+v+\sum\limits_{i=1}^{r} h_i s_i\right)x} dx\right] ds_1 \cdots ds_r.$$

Solving the inner integral using (1.4),

$$= \left(\frac{1}{2\pi i}\right)^r \int_{L_1} \cdots \int_{L_r} \phi(s_1, \cdots, s_r)\theta_1(s_1)\cdots\theta_r(s_r) z_1^{s_1}\cdots z_r^{s_r}$$
$$\frac{\Gamma(u)\Gamma\left(v+\sum\limits_{i=1}^{r} h_i s_i\right)}{\Gamma\left(u+v+\sum\limits_{i=1}^{r} h_i s_i\right)} e^{iu\frac{\pi}{2}} ds_1 \cdots ds_r.$$

From which (2.2) is obtained by using (1.1).

(3)
$$\int_0^{\pi/2} (\sin x)^{u-1} (\cos x)^{v-1} e^{i(u+v)x} I^{0,N:m_1,n_1;\cdots;m_r,n_r}_{P,Q:p_1,q_1;\cdots,p_r,q_r}$$
$$\left[\begin{array}{l} z_1(\sin x)^{h_1} e^{h_1 x i} \\ \quad\vdots \\ z_r(\sin x)^{h_r} e^{h_r x i} \end{array}\right.$$
$$\left|\begin{array}{l} {}_1(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_P : \ {}_1(c_j^{(1)},\gamma_j^{(1)},C_j^{(1)})_{p_1};\cdots;\ {}_1(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r} \\ \\ {}_1(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_Q : \ {}_1(d_j^{(1)},\delta_j^{(1)},D_j^{(1)})_{q_1};\cdots;\ {}_1(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{q_r} \end{array}\right] dx$$
$$= \Gamma(v) e^{iu\frac{\pi}{2}} I^{0,N+1:m_1,n_1;\cdots;m_r,n_r}_{P+1,Q+1:p_1,q_1;\cdots,p_r,q_r}$$
$$\left[\begin{array}{l} z_1 e^{ih_1\frac{\pi}{2}} \\ \vdots \\ z_r e^{ih_r\frac{\pi}{2}} \end{array}\right| \begin{array}{l} (1-u;h_1,\cdots,h_r;1), \\ \\ {}_1(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_Q, \end{array}$$
$$\left.\begin{array}{l} {}_1(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_P : \ {}_1(c_j^{(1)},\gamma_j^{(1)},C_j^{(1)})_{p_1};\cdots;\ {}_1(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r} \\ \\ (1-u-v;h_1,\cdots,h_r;1) : \ {}_1(d_j^{(1)},\delta_j^{(1)},D_j^{(1)})_{q_1};\cdots;\ {}_1(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{q_r} \end{array}\right] \tag{2.3}$$

$$Re\left(u+\sum_{i=1}^{r} h_i \frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > 0, \quad j = 1, 2, \cdots, m_i.$$

Proof:

$$\int_0^{\pi/2}(\sin x)^{u-1}(\cos x)^{v-1}e^{i(u+v)x}I^{0,N:m_1,n_1;\cdots;m_r,n_r}_{P,Q:p_1,q_1;\cdots,p_r,q_r}$$
$$\left[\begin{array}{l} z_1(\sin x)^{h_1}e^{h_i x i} \\ \vdots \\ z_r(\sin x)^{h_r}e^{h_r x i}\end{array}\right.$$
$$\left|\begin{array}{l} {}_1(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_P:\ {}_1(c_j^{(1)},\gamma_j^{(1)},C_j^{(1)})_{p_1};\cdots;\ {}_1(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r} \\ \\ {}_1(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_Q:\ {}_1(d_j^{(1)},\delta_j^{(1)},D_j^{(1)})_{q_1};\cdots;\ {}_1(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{q_r}\end{array}\right] dx$$
$$=\int_0^{\pi/2}(\sin x)^{u-1}(\cos x)^{v-1}e^{i(u+v)x}\left[\left(\frac{1}{2\pi i}\right)^r\int_{L_1}\cdots\int_{L_r}\phi(s_1,\cdots,s_r)\theta_1(s_1)\cdots,\theta_r(s_r)\right.$$
$$\left.(z_1(\sin x)^{h_1}e^{h_1 ix})^{s_1}\cdots(z_r(\sin x)^{h_r}e^{h_r ix})^{s_r}ds_1\cdots ds_r\right]dx.$$

Changing the order of integration

$$=\left(\frac{1}{2\pi i}\right)^r\int_{L_1}\cdots\int_{L_r}\phi(s_1,\cdots,s_r)\theta_1(s_1)\cdots\theta_r(s_r)z_1^{s_1}\cdots z_r^{s_r}$$
$$\left[\int_0^{\pi/2}(\sin x)^{u+\sum\limits_{i=1}^r h_i s_i-1}(\cos x)^{v-1}e^{i\left(u+v+\sum\limits_{i=1}^r h_i s_i\right)x}dx\right]ds_1\cdots ds_r.$$

Solving the inner integral using (1.4),

$$=\left(\frac{1}{2\pi i}\right)^r\int_{L_1}\cdots\int_{L_r}\phi(s_1,\cdots,s_r)\theta_1(s_1)\cdots\theta_r(s_r)z_1^{s_1}\cdots z_r^{s_r}$$
$$\frac{\Gamma\left(u+\sum\limits_{i=1}^r h_i s_i\right)\Gamma(v)}{\Gamma\left(u+v+\sum\limits_{i=1}^r h_i s_i\right)}e^{i\left(u+\sum\limits_{i=1}^r h_i s_i\right)\frac{\pi}{2}}ds_1\cdots ds_r.$$

From which (2.3) is obtained by using (1.1).

3. The Fourier Sine Series Involving I-functions of r-variables

(1)
$$(\sin x)^{4u-1}(\cos x)^{4v-1} I^{0,N:m_1,n_1;\cdots;m_r,n_r}_{P,Q:p_1,q_1;\cdots,p_r,q_r}\left[\begin{matrix} z_1(\sin x\cos x)^{h_1}e^{2h_1xi}\\ \vdots \\ z_r(\sin x\cos x)^{h_r}e^{2h_rxi}\end{matrix}\right.\left|\begin{matrix} {}_1(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_P:\ {}_1(c_j^{(1)};\gamma_j^{(1)},C_j^{(1)})_{p_1};\cdots;\ {}_1(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r}\\ {}_1(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_Q:\ {}_1(d_j^{(1)};\delta_j^{(1)},D_j^{(1)})_{q_1};\cdots;\ {}_1(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{q_r}\end{matrix}\right]$$
$$= \frac{4}{\pi i}\sum_{v=0}^{\infty}\left\{ I^{0,N+2:m_1,n_1;\cdots;m_r,n_r}_{P+2,Q+1:p_1,q_1;\cdots,p_r,q_r}\left[\begin{matrix} z_1e^{ih_1\frac{\pi}{2}}\\ \vdots \\ z_re^{ih_r\frac{\pi}{2}}\end{matrix}\right.\left|\begin{matrix}(1-4u;h_1,\cdots,h_r;1),(1-4v;h_1,\cdots,h_r;1),\\ {}_1(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_Q,\end{matrix}\right.\right.$$
$$\left.\left.\begin{matrix} {}_1(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_P:\ {}_1(c_j^{(1)},\gamma_j^{(1)},C_j^{(1)})_{p_1};\cdots;\ {}_1(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r}\\ (1-4u-4v;2h_1,\cdots,2h_r;1):\ {}_1(d_j^{(1)},\delta_j^{(1)},D_j^{(1)})_{q_1};\cdots;\ {}_1(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{q_r}\end{matrix}\right]\right\}\sin 4(u+v)x \tag{3.1}$$

Proof : Consider

$$f(x) = (\sin x)^{4u-1}(\cos x)^{4v-1} I^{0,N:m_1,n_1;\cdots;m_r,n_r}_{P,Q:p_1,q_1;\cdots,p_r,q_r}\left[\begin{matrix} z_1(\sin x\cos x)^{h_1}e^{2h_1xi}\\ \vdots \\ z_r(\sin x\cos x)^{h_r}e^{2h_rxi}\end{matrix}\right.\left|\begin{matrix} {}_1(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_P:\ {}_1(c_j^{(1)},\gamma_j^{(1)},C_j^{(1)})_{p_1};\cdots;\ {}_1(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r}\\ {}_1(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_Q:\ {}_1(d_j^{(1)},\delta_j^{(1)},D_j^{(1)})_{q_1};\cdots;\ {}_1(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{q_r}\end{matrix}\right]$$

$$= \sum_{n=0}^{\infty} c_n \sin 4(u+n)x, \quad 0 < x < \frac{\pi}{2}. \tag{3.2}$$

Multiplying both sides by $e^{4(u+v)xi}$ and integrating with respect to x from 0 to $\frac{\pi}{2}$,

$$\int_0^{\pi/2} (\sin x)^{4u-1} (\cos x)^{4v-1} e^{4(u+v)xi} I^{0,N:m_1,n_1;\cdots;m_r,n_r}_{P,Q:p_1,q_1;\cdots,p_r,q_r} \left[\begin{array}{c} z_1(\sin x\cos x)^{h_1} e^{2h_1 xi} \\ \vdots \\ z_r(\sin x\cos x)^{h_r} e^{2h_r xi} \end{array}\right| \left.\begin{array}{l} {}_1(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_P:\ {}_1(c_j^{(1)},\gamma_j^{(1)},C_j^{(1)})_{p_1};\cdots;\ {}_1(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r} \\ \\ {}_1(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_Q:\ {}_1(d_j^{(1)},\delta_j^{(1)},D_j^{(1)})_{q_1};\cdots;\ {}_1(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{q_r} \end{array}\right] dx$$
$$= \int_0^{\pi/2} e^{4(u+v)xi} \sum_{n=0}^{\infty} c_n \sin 4(u+n)x dx.$$

Changing the order of summation and integration the integral is equal to

$$\sum_{n=0}^{\infty} c_n \int_0^{\pi/2} e^{4(u+v)xi} \sin 4(u+n)x dx.$$

On evaluating the integral using (1.5) it takes the value $c_v \frac{\pi}{4} i$ from which (3.1) is obtained by using (2.1) and (3.2).

(2) $$(\sin x)^{4u-1} (\cos x)^{4v-1} I^{0,N:m_1,n_1;\cdots;m_r,n_r}_{P,Q:p_1,q_1;\cdots,p_r,q_r} \left[\begin{array}{c} z_1(\cos x)^{h_1} e^{2h_1 xi} \\ \vdots \\ z_r(\cos x)^{h_r} e^{2h_r xi} \end{array}\right| \left.\begin{array}{l} {}_1(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_P:\ {}_1(c_j^{(1)},\gamma_j^{(1)},C_j^{(1)})_{p_1};\cdots;\ {}_1(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r} \\ \\ {}_1(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_Q:\ {}_1(d_j^{(1)},\delta_j^{(1)},D_j^{(1)})_{q_1};\cdots;\ {}_1(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{q_r} \end{array}\right]$$
$$= \frac{4}{\pi i}\Gamma(4u) \sum_{v=0}^{\infty} \Bigg\{ I^{0,N+1:m_1,n_1;\cdots;m_r,n_r}_{P+1,Q+1:p_1,q_1;\cdots,p_r,q_r} \left[\begin{array}{c} z_1 \\ \vdots \\ z_r \end{array}\right| \begin{array}{l} (1-4v;h_1,\cdots,h_r;1), \\ \\ {}_1(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_Q, \end{array}$$
$$\left.\begin{array}{l} {}_1(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_P:\ {}_1(c_j^{(1)},\gamma_j^{(1)},C_j^{(1)})_{p_1};\cdots;\ {}_1(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r} \\ (1-4u-4v;h_1,\cdots,h_r;1):\ {}_1(d_j^{(1)},\delta_j^{(1)},D_j^{(1)})_{q_1};\cdots;\ {}_1(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{q_r} \end{array}\right]\Bigg\} \sin 4(u+v)x. \tag{3.3}$$

Proof : Let

$$f(x) = (\sin x)^{4u-1}(\cos x)^{4v-1} I^{0,N:m_1,n_1;\cdots;m_r,n_r}_{P,Q:p_1,q_1;\cdots,p_r,q_r} \left[\begin{array}{c} z_1(x\cos x)^{h_1}e^{2h_1xi} \\ \vdots \\ z_r(\cos x)^{h_r}e^{2h_rxi} \end{array} \left| \begin{array}{l} {}_1(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_P : \ {}_1(c_j^{(1)},\gamma_j^{(1)},C_j^{(1)})_{p_1};\cdots;\ {}_1(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r} \\ \\ {}_1(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_Q : \ {}_1(d_j^{(1)},\delta_j^{(1)},D_j^{(1)})_{q_1};\cdots;\ {}_1(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{q_r} \end{array} \right. \right]$$

$$= \sum_{n=0}^{\infty} c_n \sin 4(u+n)x, \quad 0 < x < \frac{\pi}{2}. \tag{3.4}$$

Multiplying both sides by $e^{4(u+v)xi}$ and integrating with respect to x from 0 to $\frac{\pi}{2}$,

$$\int_0^{\pi/2} (\sin x)^{4u-1}(\cos x)^{4v-1} e^{4(u+v)xi} I^{0,N:m_1,n_1;\cdots;m_r,n_r}_{P,Q:p_1,q_1;\cdots,p_r,q_r} \left[\begin{array}{c} z_1(\cos x)^{h_1}e^{2h_1xi} \\ \vdots \\ z_r(x\cos x)^{h_r}e^{2h_rxi} \end{array} \left| \begin{array}{l} {}_1(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_P : \ {}_1(c_j^{(1)},\gamma_j^{(1)},C_j^{(1)})_{p_1};\cdots;\ {}_1(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r} \\ \\ {}_1(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_Q : \ {}_1(d_j^{(1)},\delta_j^{(1)},D_j^{(1)})_{q_1};\cdots;\ {}_1(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{q_r} \end{array} \right. \right] dx$$

$$= \int_0^{\pi/2} e^{4(u+v)xi} \sum_{n=0}^{\infty} c_n \sin 4(u+n)x dx.$$

Changing the order of summation and integration the integral is equal to

$$\sum_{n=0}^{\infty} c_n \int_0^{\pi/2} e^{4(u+v)xi} \sin 4(u+n)x dx.$$

Evaluating the integral using (1.5) it takes the value $c_v \frac{\pi}{4} i$ from which (3.3) is obtained by using (2.2) and (3.4).

(3) $$(\sin x)^{4u-1}(\cos x)^{4v-1} I^{0,N:m_1,n_1;\cdots;m_r,n_r}_{P,Q:p_1,q_1;\cdots,p_r,q_r} \left[\begin{array}{c} z_1(x\cos x)^{h_1}e^{2h_1xi} \\ \vdots \\ z_r(x\cos x)^{h_r}e^{2h_rxi} \end{array} \right. \left| \begin{array}{l} {}_1(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_P:\ {}_1(c_j^{(1)},\gamma_j^{(1)},C_j^{(1)})_{p_1};\cdots;\ {}_1(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r} \\ \\ {}_1(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_Q:\ {}_1(d_j^{(1)},\delta_j^{(1)},D_j^{(1)})_{q_1};\cdots;\ {}_1(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{q_r} \end{array} \right]$$

$$= \frac{4}{\pi i}\Gamma(4v)\sum_{v=0}^{\infty}\Bigg\{ I^{0,N+1:m_1,n_1;\cdots;m_r,n_r}_{P+1,Q+1:p_1,q_1;\cdots,p_r,q_r} \left[\begin{array}{c} z_1e^{ih_1\frac{\pi}{2}} \\ \vdots \\ z_re^{ih_r\frac{\pi}{2}} \end{array} \right| \begin{array}{l} (1-4u;h_1,\cdots,h_r;1), \\ \\ {}_1(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_Q, \end{array}$$

$$\left. \begin{array}{l} {}_1(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_P:\ {}_1(c_j^{(1)},\gamma_j^{(1)},C_j^{(1)})_{p_1};\cdots;\ {}_1(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r} \\ (1-4u-4v;h_1,\cdots,h_r;1):\ {}_1(d_j^{(1)},\delta_j^{(1)},D_j^{(1)})_{q_1};\cdots;\ {}_1(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{q_r} \end{array} \right]\Bigg\}\sin 4(u+v)x. \tag{3.5}$$

Proof : Let

$$f(x) = (\sin x)^{4u-1}(\cos x)^{4v-1} I^{0,N:m_1,n_1;\cdots;m_r,n_r}_{P,Q:p_1,q_1;\cdots,p_r,q_r} \left[\begin{array}{c} z_1(\sin x)^{h_1}e^{2h_1xi} \\ \vdots \\ z_r(\sin x)^{h_r}e^{2h_rxi} \end{array} \right. \left| \begin{array}{l} {}_1(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_P:\ {}_1(c_j^{(1)},\gamma_j^{(1)},C_j^{(1)})_{p_1};\cdots;\ {}_1(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r} \\ \\ {}_1(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_Q:\ {}_1(d_j^{(1)},\delta_j^{(1)},D_j^{(1)})_{q_1};\cdots;\ {}_1(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{q_r} \end{array} \right]$$

$$= \sum_{n=0}^{\infty} c_n \sin 4(u+n)x, \quad 0 < x < \frac{\pi}{2}. \tag{3.6}$$

Multiplying both sides by $e^{4(u+v)xi}$ and integrating with respect to x from 0 to $\frac{\pi}{2}$,

$$\int_0^{\pi/2} (\sin x)^{4u-1} (\cos x)^{4v-1} e^{4(u+v)xi} I^{0,N:m_1,n_1;\cdots;m_r,n_r}_{P,Q:p_1,q_1;\cdots,p_r,q_r}$$

$$\left[\begin{array}{l} z_1(\sin x)^{h_1} e^{2h_1 xi} \\ \vdots \\ z_r(\sin x)^{h_r} e^{2h_r xi} \end{array}\right.$$

$$\left|\begin{array}{l} {}_1(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_P:\ {}_1(c_j^{(1)},\gamma_j^{(1)},C_j^{(1)})_{p_1};\cdots;\ {}_1(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r} \\ \\ {}_1(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_Q:\ {}_1(d_j^{(1)},\delta_j^{(1)},D_j^{(1)})_{q_1};\cdots;\ {}_1(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{q_r} \end{array}\right] dx$$

$$= \int_0^{\pi/2} e^{4(u+v)xi} \sum_{n=0}^{\infty} c_n \sin 4(u+n)x dx.$$

Changing the order of summation and integration the integral is equal to

$$\sum_{n=0}^{\infty} c_n \int_0^{\pi/2} e^{4(u+v)xi} \sin 4(u+n)x dx.$$

Evaluating the integral using (1.5) it takes the value $c_v \frac{\pi}{4} i$ from which (3.3) is obtained by using (2.3) and (3.6).

Special Cases :

1. In the results, when the third variable equal to 1, it reduces to H-function of r-variables.

2. When $r = 2$, all results will reduce to I-function of 2-variables.

3. When $r = 2$ and the third parameter equal to 1, it reduces to H-function of 2-variables.

4. Put $N = P = Q = 0$ and take $r = 1$, then all results reduce to product of r I-functions of 1- variables.

5. When $r = 1$, all results will reduce to I-function of 1-variable.

6. When $r = 1$ and the third parameter equal to 1, it reduces to the result proved by K. L. Choudhary [1].

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