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A SMOOTH TRANSCENDENTAL APPROXIMATION TO $|x|$

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Abstract

In this note we present a new simple and smooth transcendental approximation to $f(x) = |x|$, with sufficient accuracy. The proposed formula gives better approximation than $\sqrt{x^2 + \mu^2}$ in terms of accuracy.

1. Introduction

In many practical situations, the optimization techniques use the derivative of the objective function and we need to optimize an expression of the type $\Sigma|x_i|$. But, $f(x) = |x|$ is not derivable at zero. So the general question is, are there any good approximations of the absolute value function which are smooth? One simple approximation is $\sqrt{x^2 + \mu^2}$ [4]. In [2], $\sqrt{x^2 + \mu}$ was efficiently used as smooth approximation to $|x|$ and Carlos Ramirez et al. [3] proved it to be the most computationally efficient smooth approximation to $|x|$. S. Voronin et al. [4] proved that -

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$$||x| - \sqrt{x^2 + \mu^2}| \leq \mu \quad \text{where} \quad \mu > 0 \in \mathbb{R}. \tag{1.1}$$

There are also some transcendental approximations to $|x|$ which are smooth. $\log(e^{2x} + 1) - x$ [5] is asymptotically better approximation. Though $\sqrt{x^2 + \mu^2}$ is computationally efficient, but as far as accuracy is concerned better smooth transcendental approximations can be given to $|x|$. One such approximation is $x \cdot erf\left(\frac{x}{\mu}\right)$ [6]. We propose a new smooth approximation to $|x|$ by using hyperbolic function $\tan hx$ [1].

2. The Main Result

Theorem 1 : The approximation $g(x) = x \cdot \tanh(x/\mu); \mu > 0 \in \mathbb{R}$ to $|x|$ satisfies

$$\frac{dg(x)}{dx} = \frac{x}{\mu} \cdot sech^2\left(\frac{x}{\mu}\right) + \tanh\left(\frac{x}{\mu}\right) \tag{2.1}$$

and

$$||x| - x \cdot \tanh\left(\frac{x}{\mu}\right)| < \mu. \tag{2.2}$$

Proof : To establish (2.2), we first show that -

$$x \cdot \tanh\left(\frac{x}{\mu}\right) \leq |x| \tag{2.3}$$

In $[0, \infty)$, we have $|x| - x \cdot \tanh\left(\frac{x}{\mu}\right) = x - x \cdot \tanh\left(\frac{x}{\mu}\right) = x[1 - \tanh\left(\frac{x}{\mu}\right)]$.
As, $x \in [0, \infty)$ and $0 \leq \tanh\left(\frac{x}{\mu}\right) < 1$, therefore $|x| - x \cdot \tanh\left(\frac{x}{\mu}\right) \geq 0$, that means $x \cdot \tanh\left(\frac{x}{\mu}\right) \leq |x|$.
In $(-\infty, 0]$, we have $|x| - x \cdot \tanh\left(\frac{x}{\mu}\right) = -x - x \cdot \tanh\left(\frac{x}{\mu}\right) = -x[1 + \tanh\left(\frac{x}{\mu}\right)]$.
As, $-x \geq 0$ and $-1 < \tanh\left(\frac{x}{\mu}\right) \leq 0$, therefore $|x| - x \cdot \tanh\left(\frac{x}{\mu}\right) \geq 0$, that means $x \cdot \tanh\left(\frac{x}{\mu}\right) \leq |x|$. In fact, in both the situations $0 \leq x \cdot \tanh\left(\frac{x}{\mu}\right) \leq |x|$.
Now, to prove (2.2) there are two cases -
Case 1 : If $x = 0$, then the result is obvious.
Case 2 : If $x \neq 0$, then $0 < x \cdot \tanh\left(\frac{x}{\mu}\right) \leq |x|$ which implies that $|x \cdot \tanh\left(\frac{x}{\mu}\right)| \leq |x|$, giving us

$$|\tanh\left(\frac{x}{\mu}\right)| \leq 1.$$

Thus, 1 is the least upper bound of the set $\{|\tanh\left(\frac{x}{\mu}\right)| : \mu > 0\}$. Therefore

$$1 < |\tanh\left(\frac{x}{\mu}\right)| + \frac{|\mu|}{|x|}, \quad \text{i.e.} \quad 1 - |\tanh\left(\frac{x}{\mu}\right)| < \frac{|\mu|}{|x|}.$$

Consider,

$$\begin{aligned} ||x| - x \cdot \tanh\left(\frac{x}{\mu}\right)| &= \||x| - |x \cdot \tanh\left(\frac{x}{\mu}\right)|| = ||x| - |x| \cdot |\tanh\left(\frac{x}{\mu}\right)|| \\ &= |x| \cdot \{1 - \tanh\left(\frac{x}{\mu}\right)|\} \\ &< |x| \cdot \frac{|\mu|}{|x|} = |\mu| = \mu. \end{aligned}$$

□

Note : $x \cdot \tanh\left(\frac{x}{\mu}\right) \leq |x| < x \cdot \tanh\left(\frac{x}{\mu}\right) + \mu$ and $\lim_{\mu \to 0} x \cdot \tanh\left(\frac{x}{\mu}\right) = |x|$.

The following figures show how fast $x \cdot \tanh\left(\frac{x}{\mu}\right)$ and $\sqrt{x^2 + \mu^2}$ approach to $|x|$ for $\mu = 0.1$ and $\mu = 0.01$.

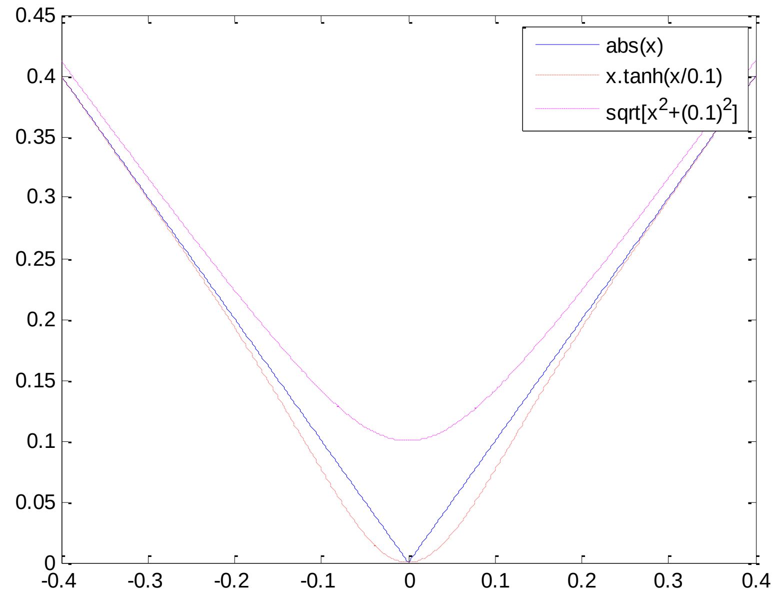

Fig . 1 $\mu = 0.1$

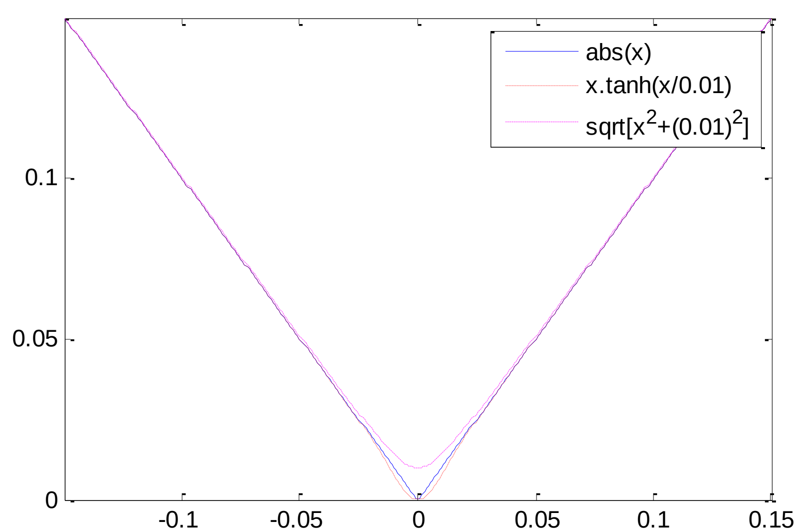

Fig. 2 $\mu = 0.01$

3. Conclusion

$x \cdot \tanh\left(\frac{x}{\mu}\right); \mu \to 0$ is investigated as a smooth transcendental approximation to $|x|$ and it has the following properties -

1. The formula is easy to remember.
2. It is a better smooth approximation than $\sqrt{x^2+\mu}$ in terms of accuracy.
3. It is superior transcendental approximation than $\log(e^{2x}+1)-x$.
4. It is as good as $x \cdot erf\left(\frac{x}{\mu}\right)$.

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