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ON EXPANSION FORMULAE INVOLVING MULTIVARIABLE I-FUNCTIONS

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Abstract

The object of this paper is to derive an Expansion formula involving the I-function of r-variables. As a corollary we derived an Expansion formula involving the I-function of 2-variables from the main result. Expansion formulae involving hyper geometric functions and a number of known results can be deduced from them. An Expansion formula involving H-function of 2-variables derived by Mohamed [2, p. 135] is also obtained by specializing the parameters in our main result.

1. Introduction

Notations used:

$(a_p) = {}_1(a_j)_p$ stands for $a_1, a_2, \cdots, a_P$

${}_1(a_j;\alpha_j)_p$ stands for $(a_1;\alpha_1), (a_2;\alpha_2), \cdots, (a_p;\alpha_p)$

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${}_1(a_j;\alpha_j,A_j)_p$ stands for $(a_1;\alpha_1,A_1),(a_2;\alpha_2,A_2),\cdots,(a_p;\alpha_p,A_p)$

$1(a_j;\alpha_j,A_j;1)_p$ stands for $(a_1;\alpha_1,A_1;1),(a_2;\alpha_2,A_2;1),\cdots,(a_p;\alpha_p,A_p;1)$.

The generalized Fox's H-function, namely I-function of r-variables introduced by Prathima, et al. [3, p.38] is defined and represented in the following manner:

$$I[Z_1,\cdots,Z_r]=I^{0,N:m_1,n_1;\cdots;m_r,n_r}_{P,Q:p_1,q_1;\cdots;p_r,q_r}\left[\begin{array}{c|c} z_1 & {}_1(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_P:\ {}_1(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{p_1};\cdots;\ {}_1(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r} \\ \vdots & \\ z_r & {}_1(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_Q:\ {}_1(d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{q_1};\cdots;\ {}_1(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{q_r}\end{array}\right]$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\theta_1(s_1)\cdots\theta_r(s_r)\phi(s_1,\cdots,s_r)z_1^{s_1}\cdots z_r^{s_r}ds_1\cdots ds_r, \tag{1.1}$$

where, $\phi(s_1,\cdots,s_r)$ and $\theta_i(s_i), i=1,2,\cdots,r$ are given by,

$$\phi(s_1,\cdots,s_r)=\frac{\prod\limits_{j=1}^{N}\Gamma^{A_j}\left(1-a_j+\sum\limits_{i=1}^{r}\alpha_j^{(i)}s_i\right)}{\prod\limits_{j=1}^{Q}\Gamma^{B_j}\left(1-b_j+\sum\limits_{i=1}^{r}\beta_j^{(i)}s_i\right)\prod\limits_{j=N+1}^{P}\Gamma^{A_j}\left(a_j-\sum\limits_{i=1}^{r}\alpha_j^{(i)}s_i\right)}, \tag{1.2}$$

$$\theta_i(s_i)=\frac{\prod\limits_{j=1}^{m_i}\Gamma^{D_j^{(i)}}(d_j^{(i)}-\delta_j^{(i)}s_i)\prod\limits_{j=1}^{n_i}\Gamma^{C_j^{(i)}}(1-c_j^{(i)}+\gamma_j^{(i)}s_i)}{\prod\limits_{j=m_i+1}^{q_i}\Gamma^{D_j^{(i)}}(1-d_j^{(i)}+\delta_j^{(i)}s_i)\prod\limits_{j=n_i+1}^{p_i}\Gamma^{C_j^{(i)}}(c_j^{(i)}-\gamma_j^{(i)}s_i)}. \tag{1.3}$$

Also, $z_i\neq 0$ $(i=1,\cdots,r),\omega=\sqrt{-1},m_j,n_j,p_j,q_j$ $(j=1,\cdots,r),N,P,Q$ are non-negative integers such that $0\leq N\leq P,Q\geq 0,0\leq m_j\leq q_j,0\leq n_j\leq p_j$ $(j=1,2,\cdots,r)$ (not all zero simultaneously).

$\alpha_j^{(i)}$ $(j=1,2,\cdots,P,i=1,2,\cdots,r)$, $\beta_j^{(i)}$ $(j=1,2,\cdots,Q,i=1,2,\cdots,r)$, $\gamma_j^{(i)}$ $(j=1,2,\ldots,p_i,i=1,2,\cdots,r)$, and $\delta_j^{(i)}$ $(j=1,2,\cdots,q_i,i=1,2,\cdots,r)$ are positive numbers.

a_j $(j=1,2,\cdots,P)$, b_j $(j=1,2,\cdots,Q)$, $c_j^{(i)}$ $(j=1,2,\cdots,p_i,i=1,2,\cdots,r)$ and $d_j^{(i)}$ $(j=1,2,\cdots,q_i,i=1,2,\cdots,r)$ are complex numbers.

The exponents A_j $(j=1,2,\cdots,P)$, B_j $(j=1,2,\cdots,Q)$, $C_j^{(i)}$ $(j+1,2,\cdots,p_i,i=1,2,\cdots,r)$ and $D_j^{(i)}$ $(j=1,2,\cdots,q_i,i=1,2,\cdots,r)$ of various gamma functions may take non integer values. The I-function of r variables is analytic if

$$\Psi_i=\sum_{j=1}^{P}A_j\alpha_j^{(i)}-\sum_{j=1}^{Q}B_j\beta_j^{(i)}+\sum_{j=1}^{p_i}C_j^{(i)}\gamma_j^{(i)}-\sum_{j=1}^{q_i}D_j^{(i)}\delta_j^{(i)}\leq 0,\ i=1,2,\cdots,r.$$

The integral (1.1) converges absolutely if $|arg(z_i)|<\frac{1}{2}\Delta_i\pi, i=1,2,\cdots,r$ where

$$\begin{aligned}\Delta_i &= -\sum_{j=N+1}^{P}A_j\alpha_j^{(i)}-\sum_{j=1}^{Q}B_j\beta_j^{(i)}+\sum_{j=1}^{m_i}D_j^{(i)}\delta_j^{(i)}-\sum_{j=m_i+1}^{q_i}D_j^{(i)}\delta_j^{(i)}\\ &+\sum_{j=1}^{n_i}C_j^{(i)}\gamma_j^{(i)}-\sum_{j=n_i+1}^{p_i}C_j^{(i)}\gamma_j^{(i)}>0.\end{aligned}$$

Agrawal and Singhal [1,p.36]

$$\frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(1-\frac{s}{2}\right)}{\Gamma\left(\frac{rs}{2}\right)\Gamma\left(1-\frac{rs}{2}\right)}=\pi e^{\pm i\frac{(r+1)}{2}\pi s}\left[\sum_{k=1}^{r}\left\{\frac{1}{\Gamma\left(\frac{1}{2}+ks\right)\Gamma\left(\frac{1}{2}-ks\right)}\pm i\frac{1}{\Gamma(1+ks)\Gamma(-ks)}\right\}\right] \tag{1.4}$$

2. Expansion Formulae Involving Multivariable I-Functions

Main result :

$$\begin{aligned}&I^{0,N:m_1+1,n_1+1;m_2,n_2;\cdots;m_r,n_r}_{P,Q:p_1+2,q_1+2;p_2,q_2;\cdots;p_r,q_r}\\
&\left[\begin{array}{c|l} z_1 & {}_1(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_P:(1-\frac{w}{2},\frac{d}{2};1),\\ \vdots & \\ z_r & {}_1(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_Q:\ (1-\frac{w}{2},\frac{d}{2};1),\end{array}\right.\\
&\left.\begin{array}{l}{}_1(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{p_1},(1-\frac{tw}{2},\frac{td}{2};1);\ (c_j^{(2)},\gamma_j^{(2)};C_j^{(2)})_{p_2};\cdots;\ {}_1(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{p_r}\\ {}_1(d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{q_1},(1-\frac{tw}{2},\frac{td}{2};1);\ (d_j^{(2)},\delta_j^{(2)};D_j^{(2)})_{q_2};\cdots;\ {}_1(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{q_r}\end{array}\right]\\
&=\pi e^{\pm i\frac{(t+1)}{2}\pi w}\sum_{k=1}^{t}\left\{I^{0,N:m_1,n_1;m_2,n_2;\cdots;m_r,n_r}_{P,Q:p_1+1,q_1+1;p_2q_2;\cdots;p_r,q_r}\right.\\
&\left[\begin{array}{c|l} z_1e^{\pm i\frac{(t+1)}{2}\pi d} & {}_1(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_P:{}_1\ (c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{p_1},\\ \vdots & \\ z_r & {}_1(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_Q:\ {}_1(d_j^{(1)},\delta_j^{(1)},D_j^{(1)})_{q_1},\end{array}\right.\\
&\left.\begin{array}{l}(\frac{1}{2}-kw,kd;1);\ {}_1(c_j^{(2)},\gamma_j^{(2)};C_j^{(2)})_{p_2};\cdots,\ {}_1(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})p_r\\ (\frac{1}{2}-kw,kd;1)\ {}_1(d_j^{(2)},\delta_j^{(2)};D_j^{(2)})_{q_2};\cdots,\ {}_1(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})q_r\end{array}\right]\end{aligned}$$

$$\pm i I^{0,N:m_1,n_1;m_2,n_2;\cdots;m_r,n_r}_{P,Q:p_1+1,q_1+1;p_2q_2;\cdots;p_r,q_r}\left[\begin{array}{c|l} z_1 e^{\pm i\frac{(t+1)}{2}\pi d} & {}_1(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_P :{}_1(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{p_1}, \\ z_r & {}_1(b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_Q :\ {}_1(d_j^{(1)},\delta_j^{(1)},D_j^{(1)})_{q_1}, \end{array}\right.$$
$$\left.\left.\begin{array}{l}(-kw,kd;1);\ {}_1(c_j^{(2)},\gamma_j^{(2)};C_j^{(2)})_{p_2};\cdots,\ {}_1(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})p_r \\ (-kw,kd;1)\ {}_1(d_j^{(2)},\delta_j^{(2)};D_j^{(2)})_{q_2};\cdots,\ {}_1(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})q_r\end{array}\right]\right\} \tag{2.1}$$

provided every z_i is complex, $\frac{d}{2}>0, \frac{td}{2}>0, kd>0$, $\left|arg\ z_1 \pm (t+1)\frac{\pi d}{2}\right| < \frac{\pi}{2}(\Delta_1 - 2dk)$, $|arg\ z_1| < \frac{\pi}{2}(\Delta_1 + d - dk), |arg\ z_i| < \frac{\pi}{2}\Delta_i, i = 1,2,\cdots,r.$ where

$$\begin{aligned}\Delta_i &= -\sum_{j=n+1}^{P} A_j\alpha_j^{(i)} - \sum_{j=1}^{Q} B_j\beta_j^{(i)} + \sum_{j=1}^{m_i} D_j^{(i)}\delta_j^{(i)} - \sum_{j=m_i+1}^{q_i} D_j^{(i)}\delta_j^{(i)} \\ &\quad + \sum_{j=1}^{n_i} C_j^{(i)}\gamma_j^{(i)} - \sum_{j=n_i+1}^{p_i} C_j^{(i)}\gamma_j^{(i)} \quad \text{for} \quad i = 1,2,\cdots,r.\end{aligned}$$

Proof : Replace r by t and s by $(w+ds_1)$, in (1.4),

$$\begin{aligned}&\frac{\Gamma\left(\frac{w+ds_1}{2}\right)\Gamma\left(1-\frac{w+ds_1}{2}\right)}{\Gamma\left(\frac{t(w+ds_1)}{2}\right)\Gamma\left(1-\frac{t(w+ds_1)}{2}\right)} = \pi e^{\pm i\frac{(t+1)}{2}\pi(w+ds_1)} \\ &\times\left[\sum_{k=1}^{t}\left\{\frac{1}{\Gamma\left(\frac{1}{2}+k(w+ds_1)\right)\Gamma\left(\frac{1}{2}-k(w+ds_1)\right)} \pm i\frac{1}{\Gamma(1+k(w+ds_1))\Gamma(-k(w+ds_1))}\right\}\right]\end{aligned} \tag{2.2}$$

Multiply both sides of (2.2) by $\left(\frac{1}{2\pi\omega}\right)^r\ \theta_1(s_1)\cdots\theta_r(s_r)\ \phi(s_1,\cdots,s_r)z_1^{s_1}\cdots z_r^{s_r}$ where $\omega=\sqrt{-1}$ and integrate over $L_1, L_2,\cdots,L_r$ to get

$$\begin{aligned}&\int_{L_1}\cdots\int_{L_r}\frac{1}{(2\pi\omega)^r}\ \theta_1(s_1)\cdots\theta_r(s_r)\ \phi\ (s_1,\cdots,s_r) \\ &\times\frac{\Gamma\left(\frac{w+ds_1}{2}\right)\Gamma\left(1-\frac{w+ds_1}{2}\right)}{\Gamma\left(\frac{t(w+ds_1)}{2}\right)\Gamma\left(1-\frac{t(w+ds_1)}{2}\right)}z_1^{s_1}\cdots z_r^{s_r}ds_1\cdots ds_r \\ &=\int_{L_1}\cdots\int_{L_r}\frac{1}{(2\pi\omega)^r}\theta_1(s_1)\cdots\theta_r(s_r)\ \phi\ (s_1,\cdots,s_r)\ \pi e^{\pm i\frac{(t+1)}{2}\pi(w+ds_1)} \\ &\times\sum_{k=1}^{t}\left\{\frac{1}{\Gamma(\frac{1}{2}+k(w+ds_1))\Gamma(\frac{1}{2}-k(w+ds_1))}\pm i\frac{1}{\Gamma(1+k(w+ds_1))\Gamma(-k(w+ds_1))}\right\} \\ &z_1^{s_1}\cdots z_r^{s_r}ds_1\cdots ds_r.\end{aligned}$$

Changing the order of summation and integration on right hand side

$$\frac{1}{(2\pi\ \omega)^r}\int_{L_1}\cdots\int_{L_r}\theta_1(s_1)\cdots\theta_r(s_r)\ \phi(s_1,\cdots,s_r)$$
$$\times\frac{\Gamma\left(\frac{w+ds_1}{2}\right)\Gamma\left(1-\frac{w+ds_1}{2}\right)}{\Gamma\left(\frac{t(w+ds_1)}{2}\right)\Gamma\left(1-\frac{t(w+ds_1)}{2}\right)}z_1^{s_1}\cdots z_r^{s_r}ds_1\cdots ds_r$$
$$=\sum_{k=1}^{t}\left\{\frac{1}{(2\pi\omega)^r}\pi e^{\pm i\frac{(t+1)}{2}\pi w}\int_{L_1}\cdots\int_{L_r}\theta_1(s_1)\cdots\theta_r(s_r)\ \phi\ (s_1,\cdots,s_r)\ \pi e^{\pm i\frac{(t+1)}{2}\pi(w+ds_1)}\right.$$
$$\times\ \frac{1}{\Gamma(\frac{1}{2}+k(w+ds_1))\Gamma(\frac{1}{2}-k(w+ds_1))}z_1^{s_1}\cdots z_r^{s_r}ds_1\cdots ds_r$$
$$\pm i\frac{1}{(2\pi\omega)^r}\pi e^{\pm i\frac{(t+1)}{2}\pi w}\int_{L_1}\cdots\int_{L_r}\theta_1(s_1)\cdots\theta_r(s_r)\ \phi\ (s_1,\cdots,s_r)\ \pi e^{\pm i\frac{(t+1)}{2}\pi(w+ds_1)}$$
$$\left.\times\frac{1}{\Gamma(1+k(w+ds_1))\Gamma(-k(w+ds_1))}z_1^{s_1}\cdots z_r^{s_r}ds_1\cdots ds_r\right\}$$

from which (2.1) is obtained by using (1.1). The change of order of summation and integration is justified when the given conditions are satisfied, because of the absolute convergence of the summation and the integrals involved.

Special cases :

When $r = 2$, (2.1) reduces to an expansion formula for I-function of 2-variables, defined by Shanthakumari, Nambisan and Rathie [5].

Corollary :

$$I^{0,N:m_1+1,n_1+1;\cdots;m_2,n_2}_{P,Q:p_1+2,q_1+2;\cdots;p_2,q_2}\left[\begin{matrix} z_1 \\ \vdots \\ z_r\end{matrix}\ \middle|\ \begin{matrix} {}_1(a_j;\alpha_j^{(1)},\alpha_j^{(2)};A_j)_P:(1-\frac{w}{2},\frac{d}{2};1), \\ {}_1(b_j;\beta_j^{(1)},\beta_j^{(2)};B_j)_Q:(1-\frac{w}{2},\frac{d}{2};1),\end{matrix}\right.$$
$$\left.\begin{matrix} {}_1(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{p_1},(1-\frac{tw}{2},\frac{td}{2};1);\ {}_1(c_j^{(2)},\gamma_j^{(2)};C_j^{(2)})_{p_2} \\ {}_1(d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{q_1},(1-\frac{tw}{2},\frac{td}{2};1);\ {}_1(d_j^{(2)},\delta_j^{(2)};D_j^{(2)})_{q_r}\end{matrix}\right]$$

$$= \pi e^{\pm i\frac{(t+1)}{2}\pi w} \sum_{k=1}^{t} \Big\{ I^{0,N:m_1,n_1:m_2,n_2}_{P,Q:p_1+1,q_1+1;p_2,q_2}$$

$$\left[\begin{array}{c|l} z_1 e^{\pm i\frac{(t+1)}{2}\pi d} & {}_1(a_j;\alpha_j^{(1)},\alpha_j^{(2)};A_j)_P:\ {}_1(c_j;\gamma_j^{(1)},C_j^{(1)})_{p_1}, \\ z_2 & {}_1(b_j^{(1)},\beta_j^{(2)};B_j)_Q,\ {}_1(d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{q_1} \end{array} \right.$$

$$\left. \begin{array}{l} (\tfrac{1}{2}-kw,kd;1);\ {}_1(c_j^{(2)},\gamma_j^{(2)};C_j^{(2)})_{p_2} \\ (\tfrac{1}{2}-kw,kd;1);\ {}_2(d_j^{(2)},\delta_j^{(2)};D_j^{(2)})_{q_2} \end{array} \right]$$

$$\pm i I^{0,N:m_1,n_1;m_2,n_2}_{P,Q:p_1+1,q_1+1;p_2,q_2}$$

$$\left[\begin{array}{c|l} z_1 e^{\pm i\frac{(t+1)}{2}\pi d} & {}_1(a_j;\alpha_j^{(1)},\alpha_j^{(2)};A_j)_P : 1(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{p_1}, \\ z_2 & {}_1(b_j;\beta_j^{(1)},\beta_j^{(2)};B_j)_Q:\ {}_1(d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{q_1} \end{array} \right.$$

$$\left. \begin{array}{l} (-kw,kd;1);\ {}_1(c_j^{(2)},\gamma_j^{(2)};C_j^{(2)})_{p_2} \\ (-kw,kd;1);\ {}_1(d_j^{(2)};\delta_j^{(2)};D_j^{(2)})_{q_2} \end{array} \right| \Bigg\} \tag{2.3}$$

provided every z_i is complex, $\frac{d}{2} > 0$, $\frac{td}{2} > 0$, $kd > 0$, $\left|arg\ z_1 \pm (t+1)\frac{\pi}{2}\frac{d}{2}\right| < \frac{\pi}{2}(\Delta_1 - 2dk)$, $|arg\ z_1| < \frac{\pi}{2}(\Delta_1 + d - dk)$, $|arg\ z_2| < \frac{\pi}{2}\Delta_2$ where

$$\begin{aligned} \Delta_i \quad = \quad & -\sum_{j=N+1}^{P} A_j\alpha_j^{(i)} - \sum_{j=1}^{Q} B_j\beta_j^{(i)} + \sum_{j=1}^{m_i} D_j^{(i)}\delta_j^{(i)} \\ & -\sum_{1}^{q}{}_{j=m_i+1} D_j^{(i)}\delta_j^{(i)} + \sum_{j=1}^{n_i} C_j^{(i)}\gamma_j^{(i)} - \sum_{j=n_i+1}^{p_i} C_j^{(i)}\gamma_j^{(i)}, \quad i = 1, 2. \end{aligned}$$

On specializing the parameters for the H-function of r-variables, the result reduces to the Expansion formula involving H-function of r-variables as proved by Mohamed [2, p.135].

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