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SOME FIXED POINTS THEOREMS USING SEMI CONTINUOUS FUNCTION

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Abstract

A large number of extensions of Banach Contraction Mapping Principle are attempted by many authors in many research papers. Rakotch used a decreasing function on positive real numbers to [0,1) for a contraction type condition and obtained a fixed point theorem. A slight variation of the Rakotch theorem is presented by Geraghty. In the theorem of Geraghty, the function used by Rakotch satisfies the condition that whenever the image sequence under the function converges to 1 then the sequence converges to 0. Boyd and Wong obtained more general fixed point theorem by replacing the decreasing function in the theorem of Rakotch by an upper semi-continuous function. Matkowski in his fixed point theorem further modified the condition on the function of Rakotch by defining the function from positive real numbers to itself to be monotone non-decreasing and satisfying the condition that is given by limit of the n-th composition of that function as n tending to infinity is 0 for all positive real numbers. Browder, Meer and Keeler, Kirk, Suzuki, Alber and Rhoades extended the results further. We have proved two fixed point theorems by taking a function to be an upper semicontinuous function from right. We have also presented two examples to support our theorems. Finally we have generalized a celebrated fixed point theorem by Kannan.

Key Words : *Upper semi continuous function, Fixed point, Complete metric space.*

2000 AMS Subject Classification : 47H10.

Introduction

Stefan Banach, a celebrated Mathematician from Poland, stated and proved the first astonishing fixed point theorem in 1922, known as the "Banach Contraction Mapping Principle" [3]. This theorem is the origin of Metric Fixed Point Theory. Fixed points, Banach Contraction Mapping Principle and Brower Fixed Point theorem has wide applications in many branches of Mathematics. See some of them in [16]. Especially non-linear differential equations can be solved by using Banach Contraction Mapping Principle. Later on this principle has been generalized by many Mathematicians in many different ways. See Baillon [2], Meer and Keeler [9], Kirk [6], Suzuki [15], Alber [1] and Rhoades [13]. In fact vast literature is available regarding the generalization and extension of the noteworthy principle. In this research paper some generalizations of Banach Contraction Mapping Principle are proved. In these results we have modified the conditions of the Banach Contraction Mapping Principle and obtained two fixed point theorems.

1. Preliminaries and Definitions

Definition 1.1 (Metric Space) [7] : A "Metric Space" is a pair (X, d), where X is a set and d is a metric on X (or distance function on X), that is, a function defined on $X \times X$ such that for all $x, y, z \in X$ we have:

(M1) d is real-valued, finite and non-negative.

(M2) $d(x, y) = 0$ if and only if $x = y$.

(M3) $d(x, y) = d(y, x)$ (Symmetry)

(M4) $d(x, y) \leq d(x, z) + d(z, y)$ (Triangle inequality).

Example 1.1 [7] : The set of all real numbers, taken with the usual metric defined by $d(x, y) = |x - y|$ is a metric space.

Note 1.1 [12] : It is important to note that if (X, d) is a metric space and $A \subseteq X$ then (A, d) is also a metric space.

Definition 1.2 [7] : A "Fixed Point" of a mapping $T : X \to X$ of a set X into itself is an $x \in X$ which is mapped onto itself, that is $Tx = x$.

Definition 1.3 [14] : Let (X, d) be a metric space and let T be a mapping on X. Then T is called a "Contraction" if there exists $r \in [0,1)$ such that $d(Tx, Ty) \leq rd(x,y) \; \forall \; x, y \in X$.

The following famous theorem is referred to as the Banach Contraction Mapping Principle.

Theorem 1.1 (Banach) [3] : Let (X, d) be a complete metric space and let T be a contraction on X. Then T has a unique fixed point.

Definition 1.4 : A sequence $\{x_n\}_{n=1}^{\infty}$ in a metric space (X, d) is said to converge or to be convergent if there is an $x \in X$ such that $\lim\limits_{n\to\infty} d(x_n, x) = 0$, x is called the limit of $\{x_n\}_{n=1}^{\infty}$ and we write $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

Definition 1.5 : A sequence $\{x_n\}_{n=1}^{\infty}$ in a metric space (X, d) is said to be a "Cauchy Sequence" is for every $\epsilon > 0$ there is an $N = N(\epsilon)$ such that $d(x_m, x_n) < \epsilon$ for every $m, n > N$.

Theorem 1.2 : Every convergent sequence in a metric space is a Cauchy sequence.

Note 1.2 : The converse of the above theorem is not true in general. That is a Cauchy sequence in a metric space X may or may not converge in X.

Definition 1.6 : A metric space (X, d) is said to be a "Complete Metric Space" if every Cauchy Sequence in X converges in X.

Definition 1.7 : A function $\psi : \mathbb{R} \to [0, \infty)$ is said to be an "Upper Semi-Continuous from right" if for any sequence $\{t_n\}_{n=1}^{\infty}$ converging to $t \geq 0$, $\limsup\limits_{n\to\infty} \psi(t_n) \leq \psi(t)$.

Example 1.2 : Define the function $\psi : \mathbb{R}^+ \to [0, \infty$ as follows:

$$\psi(t) = \begin{cases} \sqrt{t}, & \text{if } \; t \in [0,1), \\ \\ \sqrt{t}+1, & \text{if } \; t \in [1,\infty) \end{cases}$$

We see that the function is discontinuous at $t = 1$. We have $\psi(1) = 2$. As $t \to 1$ from right we have $\lim\limits_{t\to 1} \psi(t) = 2 \leq 2$ and from left $\limsup\limits_{t\to 1} \psi(t) = 1 \leq 2$. Thus the function is upper semi-continuous from right.

Theorem 1.3 (Rakotch) [11] : Let (X, d) be a complete metric space and suppose that $T : X \to X$ satisfies $d(Tx, Ty) \leq \psi(d(x, y))d(x, y)$ for all $x, y \in X$, where ψ is a decreasing function on $\mathbb{R}^+$ to $[0, 1)$. Then T has a unique fixed point.

A slight variation of the Rakotch Theorem 1.3 is given by Geraghty as follows.

Theorem 1.4 (Geraghty) [5] : Let (X, d) be a complete metric space and suppose that $T : X \to X$ satisfies $d(Tx, Ty) \le \psi(d(x, y))d(x, y)$ for all $x, y \in X$ where $\psi(t_n) \to 1 \Rightarrow t_n \to 0$. Then T has a unique fixed point.

Theorem 1.5 (Boyd-Wong) [4] : Let (X, d) be a complete metric space and suppose that $T : X \to X$ satisfies $d(Tx, Ty) \le \psi(d(x, y))$ for all $x, y \in X$, where $\psi : \mathbb{R} \to [0, \infty)$ is upper semi continuous from the right and satisfies $0 \le \psi(t) < t$ for all $t > 0$. Then T has a unique fixed point in X.

Matkowski replaced the condition of upper semi-continuity on ψ by the another condition and stated and proved the following theorem.

Theorem 1.6 (Matkowski) [8] : Let (X, d) be a complete metric space and suppose that $T : X \to X$ satisfies $d(Tx, Ty) \le \psi(d(x, y))$ for all $x, y \in X$, where $\psi : (0, \infty) \to (0, \infty)$ is monotone non-decreasing and satisfies $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t > 0$. Then T has a unique fixed point in X.

Meer and Keeler used a diverse approach and extended the theorem of Boyd and Wong as follows.

Theorem 1.7 (Meer and Keeler) [9] : Let (X, d) be a complete metric space and suppose that $T : X \to X$ satisfies the condition: for each $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in X$, $\epsilon \le d(x, y) \le \epsilon + \delta \Rightarrow d(Tx, Ty) \le \epsilon$. Then T has a unique fixed point.

Rhoades extended the Banach Contraction Mapping Principle as follows.

Theorem 1.8 (Rhoades) [13] : Let (X, d) be a complete metric space and suppose that $T : X \to X$ satisfies $d(Tx, Ty) \le d(x, y) - \psi(d(x, y))$ for all $x, y \in X$ where $\psi : (0, \infty) \to (0, \infty)$ is continuous and non-decreasing function such that $\psi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

2. Main Results

Now we state and prove two fixed point theorems in which we use the function $\psi : \mathbb{R} \to [0, \infty)$ which is an upper semi-continuous from right.

Theorem 2.1 : Let (X, d) be a complete metric space and suppose that $T : X \to X$ satisfies $d(Tx, Ty) \le \psi(\alpha d(x, Tx) + \beta d(y, Ty))$ for all $x, y \in X$, where, $\psi : \mathbb{R} \to [0, \infty)$ is upper semi continuous from the right and satisfies $0 \le \psi(t) < t$ for all $t > 0, \psi(0) = 0$. Also $0 < \alpha + \beta < 1, \alpha > 0, \beta > 0$. Then T has a unique fixed point in X.

Proof : Let $x_0 \in X$ be an arbitrary but a fixed element in X. Define a sequence of iterates $\{x_n\}_{n=1}^{\infty}$ in X by

$$x_1 = Tx_0, x_2 = Tx_1 = T^2x_0, x_3 = Tx_2 = T^3x_0, \cdots, x_n = Tx_{n-1} = T^nx_0, \cdots.$$

Now consider,

$$\begin{aligned}
&d(x_{n+1}, x_n < d(Tx_n, Tx_{n-1}) \\
&\leq \psi(\alpha d(x_n, Tx_n) + \beta d(x_{n-1}, Tx_{n-1})) \\
&= \psi(\alpha d(x_n, x_{n+1}) + \beta d(x_{n-1}, Tx_n)) \\
&< \alpha d(x_n, x_{n+1}) + \beta d(x_{n-1}, Tx_n) \qquad (\because \quad \psi(t) < t).
\end{aligned}$$

Thus,

$$\begin{aligned}
&d(x_{n+1}, x_n) < \alpha d(x_n, x_{n+1}) + \beta d(x_{n-1}, x_n) \\
&\therefore \quad (1-\alpha) d(x_{n+1}, x_n) < \beta d(x_{n-1}, x_n) \\
&\therefore \quad d(x_{n+1}, x_n) < \frac{\beta}{1-\alpha} d(x_{n-1}, x_n) \\
&\therefore \quad d(x_{n+1}, x_n) < h d(x_{n-1}, x_n)
\end{aligned}$$

where, $h = \frac{\beta}{1-\alpha}$. Here $0 < h < 1$ because $0 < \alpha + \beta < 1, \alpha > 0, \beta > 0$. Continuing in this way, we get $d(x_{n+1}, x_n) < h^n d(x_0, x_1)$. Taking limit as $n \to \infty$ we get, $d(x_{n+1}, x_n) \to 0$ $(\because \quad 0 < h < 1)$.

Therefore $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. As X is a complete metric space, there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$. We shall show that x is a fixed point of T. As T a is continuous function we have, $x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_{n-1} = T(\lim_{n\to\infty} x_{n-1}) = Tx.$. Therefore $Tx = x$ and x is a fixed point of T. Next we shall show that x is unique fixed point of T. Let $y \in X$ be another fixed point of T. Consider

$$\begin{aligned}
d(x, y) &= d(Tx, Ty) \leq \psi(\alpha d(x, Tx) + \beta d(y, Ty)) \\
&= \psi(\alpha d(x, x) + \beta d(y, y)) \quad (\because \quad Tx = x, Ty = y) \\
&= \psi(0) \qquad\qquad \text{(See M(2) of Definition 1.1)} \\
&= 0 \qquad\qquad (\because \quad \psi(0) = 0).
\end{aligned}$$

$\therefore \quad d(x, y) = 0.$

Thus $x = y$, by M(2) of Definition 1.1, and hence the fixed point of T is unique.

Theorem 2.2 : Let (X, d) be a complete metric space and suppose that $T : X \to X$ satisfies $d(Tx, Ty) \leq \alpha\psi(d(x, Tx)) + \beta\psi(d(y, Ty)))$ for all $x, y \in X$ where, $\psi : \mathbb{R} \to [0, \infty)$ is upper semi continuous from the right and satisfies $0 \leq \psi(t) < T$ for all $t > 0, \psi(0) = 0$. Also $0 < \alpha + \beta < 1, \alpha > 0, \beta > 0$. Then T has a unique fixed point in X.

Proof : Let $x_0 \in X$ be an arbitrary but a fixed element in X. Define a sequence of iterates $\{x_n\}_{n=1}^{\infty}$ in X by

$$x_1 = Tx_0, x_2 = Tx_1 = T^2x_0, x_3 = Tx_2 = T^3x_0, \cdots, x_n = Tx_{n-1} = T^nx_0, \cdots$$

Now consider,

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \leq \alpha\psi(d(x_n, Tx_n)) + \beta\psi(d(x_{n-1}, Tx_{n-1})) \\ &= \alpha\psi(d(x_n, x_{n+1})) + \beta\psi(d(x_{n-1}, x_n)) < \alpha d(x_n, x_{n+1}) + \beta d(x_{n-1}, x_n) \\ &\qquad (\because \quad \psi(t) < t). \end{aligned}$$

Thus

$$\begin{gathered} d(x_{n+1}, x_n) < \alpha d(x_n, x_{n+1}) + \beta d(x_{n-1}, x_n) \\ \therefore \quad (1 - \alpha)d(x_{n+1}, x_n) < \beta d(x_{n-1}, x_n) \\ \therefore \quad (d_{n+1}, x_n) < \frac{\beta}{1 - \alpha} d(x_{n-1}, x_n) \\ \therefore \quad d(x_{n+1}, x_n) < hd(x_{n-1}, x_n) \end{gathered}$$

where, $h = \frac{\beta}{1-\alpha}$. Here $0 < h < 1$ because $0 < \alpha + \beta < 1, \alpha > 0, \beta > 0$.

Continuing in this way, we get $d(x_{n+1}, x_n) < h^n d(x_0, x_1)$. Taking limit as $n \to \infty$ we get, $d(x_{n+1}, x_n) \to 0$ $(\because \quad 0 < h < 1)$.

Therefore $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. As X is a complete metric space, there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$. We shall show that x is a fixed point of T. As T is a continuous function we have, $x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_{n-1} = T(\lim_{n\to\infty} x_{n-1}) = Tx$. Therefore $Tx = x$ and x is a fixed point of T. Next we shall show that x is unique fixed point of T. Let $y \in X$ be another fixed point of T. Consider

$$\begin{aligned} d(x, y) &= d(Tx, Ty) \leq \alpha\psi(d(x, Tx) + \beta\psi(d(y, Ty)) \\ &= \alpha\psi(d(x, x)) + \beta\psi(d(y, y)) \qquad (\because \quad Tx = x, Ty = y) \\ &= \alpha\psi(0) + \beta\psi(0) \qquad \text{(See M(2) of Definition 1.1)} \\ &= 0 \qquad (\because \quad \psi(0) = 0). \end{aligned}$$

$\therefore \quad d(x,y)=0.$

Thus $x=y$ and hence the fixed point of T is unique.

Example 2.1 : Consider the metric space $(\mathbb{R}^+, |\ |)$, that is the metric space of nonnegative real numbers with the absolute value metric. See example 1.1 and note 1.1. This metric space is a complete metric space. Define the function $T:\mathbb{R}^+\to\mathbb{R}^+$ by $Tx=\frac{x}{13}$. Define the function $\psi:\mathbb{R}^+\to[0,\infty)$ by

$$\psi(t)=\begin{cases} \frac{t}{4}, & \text{if } \ 0\le t<2 \\ \\ \frac{t}{2}, & \text{if } \ 2\le t<\infty. \end{cases}$$

The graph of the function $\psi(t)$ is as follows.

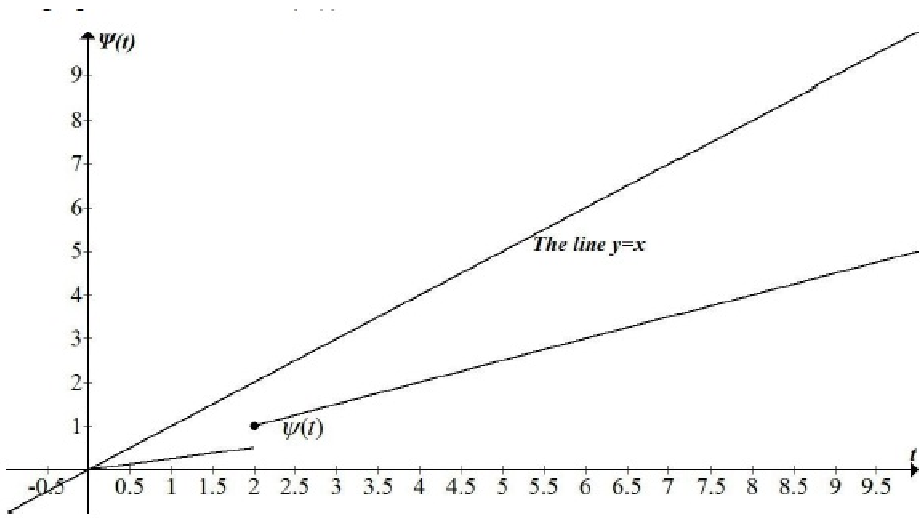

Figure 2.1: The graph of the function $\psi(t)$ shows that it is upper semi-continuous from right and $\psi(t)<t$ and also $\psi(0)=0$.

It is straight forward to see that the function $\psi(t)$ defined above is continuous at every point except at $t=2$. As $t\to 2$ from right we have $\lim\limits_{t\to 2}\psi(t)=1\le 1$ and from left we have $\limsup\limits_{t\to 2}\psi(t)=\frac{1}{2}\le 1$, where $\psi(2)=1$.

Thus clearly the function $\psi(t)$ is upper semi-continuous from right. It also satisfies $0<\psi(t)<t$ for $t>0$, because the $\frac{t}{4}<t, \frac{t}{2}<t\ \forall\ \ t\in\mathbb{R}^+$. This fact can also be easily seen from the graph of ψ. The graph of $\psi(t)$ lies below the line $y=x$. Also $\psi(0)=0$. Choose $\alpha=\beta=\frac{1}{3}$. Then we can verify that the condition in the Theorem 2.1, that is $d(Tx,Ty)\le\psi(\alpha d(x,Tx)+\beta d(y,Ty))$for all $x,y\in\mathbb{R}^+$ is satisfied.

We observe that $d(Tx,Ty) = d\left(\frac{x}{13},\frac{y}{13}\right) = \frac{|x-y|}{13}$.
Also

$$\begin{aligned}\psi(\alpha d(x,Tx)+\beta d(y,Ty)) &= \psi\left(\frac{1}{3}d\left(x,\frac{x}{13}\right)+\frac{1}{3}d\left(y,\frac{y}{13}\right)\right)\\ &= \psi\left(\frac{1}{3}\left(\frac{12x}{13}\right)+\frac{1}{3}\left(\frac{12y}{13}\right)\right) = \psi\left(\frac{4(x+y)}{13}\right).\end{aligned}$$

Now we consider all the three cases of values of $\frac{4(x+y)}{13}$ as shown in the following figure.
Case 1 : $\frac{4(x+y)}{13} = 0$.
Case 2 : $0 < \frac{4(x+y)}{13} < 2$.
Case 3 : $2 \le \frac{4(x+y)}{13} < \infty$.

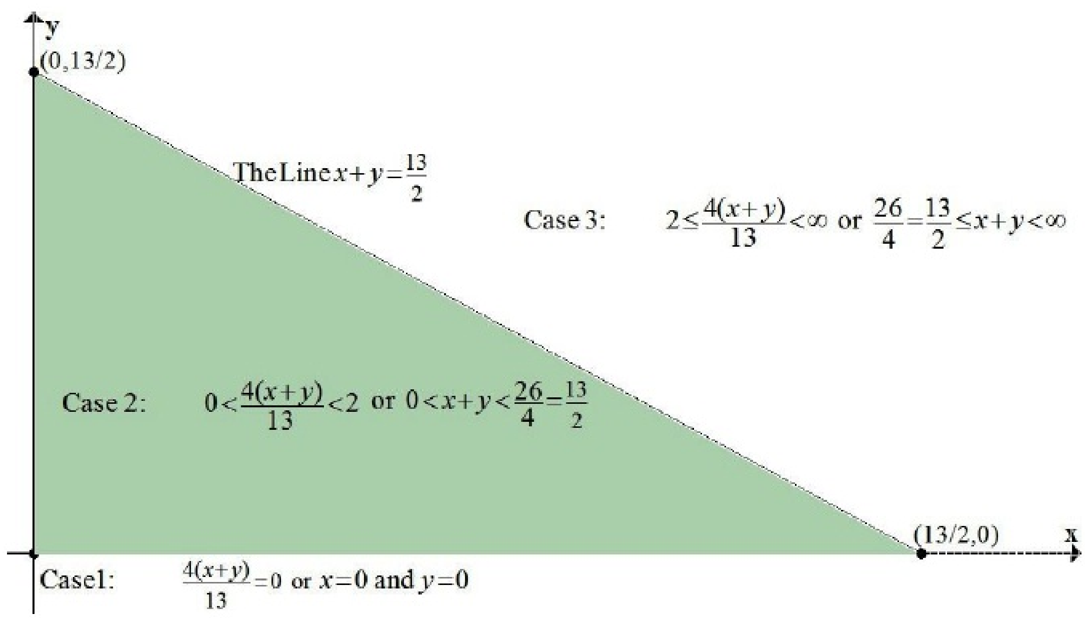

Figure 2.2: Regions showing different values of $\frac{4(x+y)}{13}$

Case 1 : $\frac{4(x+y)}{13} = 0$. That is $x = 0, y = 0$, because $x \ge 0, y \ge 0$.
Then $d(Tx,Ty) = \frac{|x-y|}{13} = \frac{|0-0|}{13} = \frac{0}{13} = 0$ and $\psi\left(\frac{4(x+y)}{13}\right) = \psi(0) = 0$.
Thus $d(Tx,Ty) = 0 \le 0 = \psi(\alpha d(x,Tx)+\beta d(y,Ty))$.
Case 2 ; $0 < \frac{4(x+y)}{13} < 2$. That is $0 < x+y < \frac{26}{4} = \frac{13}{2}$.
Then $d(Tx,Ty) = \frac{|x-y|}{13}$ and $\psi\left(\frac{4(x+y)}{13}\right) = \frac{\frac{4(x+y)}{13}}{4} = \frac{x+y}{13}$.

Clearly $\frac{|x-y|}{13} \leq \frac{x+y}{13}$ for all x, y satisfying $0 < x+y < \frac{26}{4} = \frac{13}{2}$.
Hence $d(Tx, Ty) = \frac{|x-y|}{13} \leq \frac{x+y}{13} = \psi(\alpha d(x, Tx) + \beta d(y, Ty))$.
Case 3 : $2 \leq \frac{4(x+y)}{13} < \infty$. That is $\frac{13}{2} = \frac{26}{4} \leq x+y < \infty$. Then $d(Tx, Ty) = \frac{|x-y|}{13}$.

$$\psi\left(\frac{4(x+y)}{13}\right) = \frac{\frac{4(x+y)}{13}}{2} = \frac{2(x+y)}{13}.$$

Clearly $\frac{|x-y|}{13} \leq \frac{2(x+y)}{13}$ for all x, y satisfying $\frac{26}{4} \leq x+y < \infty$.
Hence $d(Tx, Ty) = \frac{|x-y|}{13} \leq \frac{2(x+y)}{13} = \psi(\alpha d(x, Tx) + \beta d(y, Ty))$.
Thus in all the cases we have $d(Tx, Ty) \leq \psi(\alpha d(x, Tx) + \beta d(y, Ty))$.
Therefore the condition of the theorem 2.1 is satisfied. We observe that $x = 0$ is the unique fixed point of T.

Example 2.2 : Consider the metric space $(\mathbb{R}^+, |\ |)$, that is the metric space of non-negative real numbers with the absolute value metric. This metric space is a complete metric space. Define the function $T : \mathbb{R}^+ \to \mathbb{R}^+$ and the function $\psi : \mathbb{R}^+ \to [0, \infty$ as in Example 2.1 above. Choose $\alpha = \beta = \frac{1}{3}$. Then we can verify that the condition in the Theorem 2.2, that is $d(Tx, Ty) \leq \alpha\psi(d(x, Tx)) + \beta\psi(d(y, Ty))$ for all $x, y \in \mathbb{R}^+$ is satisfied.

We observe that $d(Tx, Ty) = d\left(\frac{x}{13}, \frac{y}{13}\right) = \frac{|x-y|}{13}$. Also

$$\begin{aligned} \alpha\psi(d(x, Tx)) + \beta\psi(d(y, Ty)) &= \alpha\psi\left(d\left(x, \frac{x}{13}\right)\right) + \beta\psi\left(d\left(y, \frac{y}{13}\right)\right) \\ &= \frac{1}{3}\psi\left(\frac{12}{x}\right) + \frac{1}{3}\psi\left(\frac{12y}{13}\right). \end{aligned}$$

Now we consider the all the four cases of values of $\frac{12x}{13}$ and $\frac{12y}{13}$.
Case 1 : $\frac{12}{x} = 0, \frac{12y}{13} = 0$.
Case 2 : $\frac{12x}{13} = 0, \frac{12y}{13} \neq 0$.
  Sub-case 1 : $0 < \frac{12y}{13} < 2$.
  Sub-case 2 : $2 \leq \frac{12y}{13} < \infty$.
Case 3 : $\frac{12x}{13} \neq 0, \quad \frac{12y}{13} = 0$.
  Sub-case 1 : $0 < \frac{12x}{13} < 2$.
  Sub-case 2 : $2 \leq \frac{12x}{13} < \infty$.
Case 4 : $\frac{12x}{13} \neq 0, \quad \frac{12y}{13} \neq 0$.
  Sub-case 1 : $0 < \frac{12x}{13} < 2, \quad 0 < \frac{12y}{13} < 2$.
  Sub-case 2 : $0 < \frac{12x}{13} < 2, \quad 2 \leq \frac{12y}{13} < \infty$.

Sub-case 3 : $2 \leq \frac{12x}{13} < \infty, \quad 0 < \frac{12y}{13} < 2.$

Sub-case 4 : $2 \leq \frac{12x}{13} < \infty, \quad 2 \leq \frac{12y}{13} < \infty.$

These cases are shown in the following figure.

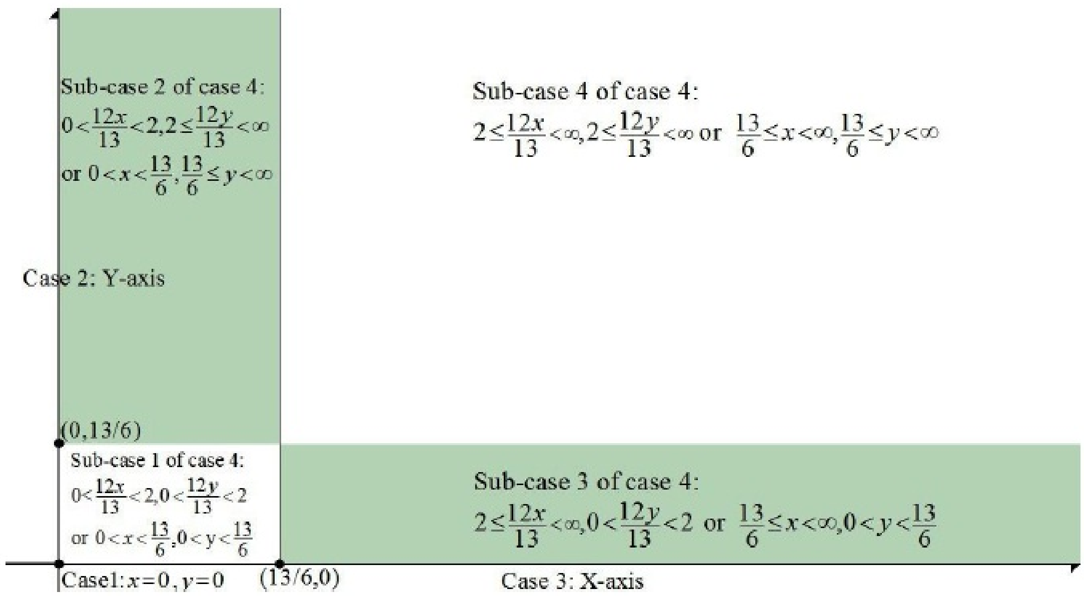

Figure 2.3: Regions showing different values of $\frac{12x}{13}$ and $\frac{12y}{13}$.

Case 1 : $\frac{12x}{13} = \frac{12y}{13} = 0$, that is $x = y = 0$.

Then $d(Tx, Ty) = \frac{|x-y|}{13} = \frac{|0-0|}{13} = \frac{0}{13} = 0$ and

$$\alpha\psi(d(x, Tx)) + \beta\psi(d(y, Ty)) = \frac{1}{3}\psi(0) + \frac{1}{3}\psi(0) = 0 + 0 = 0.$$

Thus $d(Tx, Ty) = 0 \leq 0 = \alpha\psi(d(x, Tx)) + \beta\psi(d(y, Ty))$.

Case 2 : $\frac{12x}{13} = 0, \frac{12y}{13} \neq 0$. That is $x = 0$ and $y \neq 0$.

Sub-case 1 of Case 2 : $0 < \frac{12y}{13} < 2$. That is $0 < y < \frac{26}{12} = \frac{13}{6}$.

Then $d(Tx, Ty) = \frac{|x-y|}{13} = \frac{|0-y|}{13} = \frac{y}{13} = 0$ and

$$\alpha\psi(d(x, Tx)) + \beta\psi(d(y, Ty)) = \frac{1}{3}\psi(0) + \frac{1}{3}\psi\left(\frac{12y}{13}\right) = 0 + \frac{1}{3}\frac{\left(\frac{12y}{13}\right)}{4} = \frac{y}{13}.$$

Thus $d(Tx, Ty) = \frac{y}{13} \leq \frac{y}{13} = \alpha\psi(d(x, Tx)) + \beta\psi(d(y, Ty))$.

Sub-case 2 of Case 2 : $2 \leq \frac{12y}{13}$. That is $\frac{26}{12} = \frac{13}{6} \leq y < \infty$.
Then $d(Tx, Ty) = \frac{|x-y|}{13} = \frac{|0-y|}{13} = \frac{y}{13}$ and

$$\alpha\psi(d(x, Tx)) + \beta\psi(d(y, Ty)) = \frac{1}{3}\psi(0) + \frac{1}{3}\psi\left(\frac{12y}{13}\right) = 0 + \frac{1}{3}\frac{\left(\frac{12y}{13}\right)}{2} = \frac{2y}{13}.$$

Thus $d(Tx, Ty) = \frac{y}{13} \leq \frac{2y}{13} = \alpha\psi(d(x, Tx)) + \beta\psi(d(y, Ty))$.
Case 3 : $\frac{12x}{13} \neq 0, \quad \frac{12y}{13} = 0$. That is $x \neq 0$ and $y = 0$.
This case is similar to the Case 2, with x and y interchanged.
Case 4 : $\frac{12x}{13} \neq 0, \quad \frac{12y}{13} \neq 0$. Thatis $x \neq 0$ and $y \neq 0$.
Sub-case 1 of Case 4 : $0 < \frac{12x}{13} < 2, \quad 0 < \frac{12y}{13} < 2$. Then $d(Tx, Ty) = \frac{|x-y|}{13}$ and

$$\alpha\psi(d(x, Tx)) + \beta\psi(d(y, Ty)) = \frac{1}{3}\psi\left(\frac{12x}{13}\right) + \frac{1}{3}\psi\left(\frac{12y}{13}\right) = \frac{1}{3}\frac{\left(\frac{12x}{13}\right)}{4} + \frac{1}{3}\frac{\left(\frac{12y}{13}\right)}{4} = \frac{x+y}{13}.$$

Thus $d(Tx, Ty) = \frac{|x-y|}{13} \leq \frac{x+y}{13} = \alpha\psi(d(x, Tx)) + \beta\psi(d(y, Ty))$.
Sub-case 2 of Case 4 : $0 < \frac{12x}{13} < 2, \quad 2 \leq \frac{12y}{13} < \infty$. Then $d(Tx, Ty) = \frac{|x-y|}{13}$ and

$$\alpha\psi(d(x, Tx)) + \beta\psi(d(y, Ty)) = \frac{1}{3}\psi\left(\frac{12x}{13}\right) + \frac{1}{3}\psi\left(\frac{12y}{13}\right) = \frac{1}{3}\frac{\left(\frac{12x}{13}\right)}{4} + \frac{1}{3}\frac{\left(\frac{12y}{13}\right)}{2} = \frac{x+2y}{13}.$$

$$d(Tx, Ty) = \frac{|x-y|}{13} \leq \frac{x+2y}{13} = \alpha\psi(d(x, Tx)) + \beta\psi(d(y, Ty)).$$

Sub-case 3 of Case 4 : $2 \leq \frac{12x}{13} < \infty, \quad 0 < \frac{12y}{13} < 2$.
This case is similar to the Case 2 above, with x and y interchanged and we conclude that

$$d(Tx, Ty) = \frac{|x-y|}{13} \leq \frac{2x+y}{13} = \alpha\psi(d(x, Tx)) + \beta\psi(d(y, Ty)).$$

Sub-case 4 of Case 4 : $2 \leq \frac{12x}{13} < \infty, \quad 2 \leq \frac{12y}{13} < \infty$. Then $d(Tx, Ty) = \frac{|x-y|}{13}$ and

$$\alpha\psi(d(x, Tx)) + \beta\psi(d(y, Ty)) = \frac{1}{3}\psi\left(\frac{12x}{13}\right) + \frac{1}{3}\psi\left(\frac{12y}{13}\right) = \frac{1}{3}\frac{\left(\frac{12x}{13}\right)}{2} + \frac{1}{3}\frac{\left(\frac{12y}{13}\right)}{2} = \frac{2(x+y)}{13}.$$

Thus $d(Tx, Ty) = \frac{|x-y|}{13} \leq \frac{2(x+y)}{13} = \alpha\psi(d(x, Tx)) + \beta\psi(d(y, Ty))$.
Thus in all cases $d(Tx, Ty) \leq \alpha\psi(d(x, Tx)) + \beta\psi(d(y, Ty))$.
Thus the condition of the Theorem 2.2 is satisfied. We observe that $x = 0$ is the unique fixed point of T.

Remark 2.1 : If we take ψ to be an identity mapping and $\alpha = \beta = \frac{1}{3}$ in the Theorems 2.1 and 2.2, then we get the Kannan Fixed Point Theorem [10] for $r = \frac{1}{3}$ which states as follows:
Let (X, d) be a complete metric space and let $T : X \to X$ be satisfy the condition $d(Tx, Ty) \leq r\{d(x, Tx) + d(y, Ty)\}, \quad \forall\, x, y \in X$, where $r \in \left[0, \frac{1}{2}\right)$, then T has unique fixed point in X.

3. Conclusion

The Theorems 2.1 and 2.2 are the new theorems in the series of fixed point theorems that use a semi-continuous function to impose the contractive type condition on T. We have also given two examples and explained them in detail to prove that our fixed point theorems indeed exists. We conclude that the famous Kannan fixed point theorem [10] is obtained by replacing a semi-continuous function by the identity function that is continuous and taking the Kannan constant 1/3 in our theorems.

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References

[1] Alber Ya. I., Guerre-Delabriere, S., Principles of weakly contractive maps in Hilbert spaces. In: Goldberg, I., Lyubich, Yu. (eds.) New Results in Operator Theory. Advances and Applications, 98 (1997), 7-22. Birkhauser, Basel.

[2] Baillon J. B., Non-expansive mappings and hyperconvex spaces, Fixed Point Theory and Its Applications (R. F. Brown, ed.), Contemp. Maths., Amer, Math. Soc., Providence, R. I. 72 (1988), 11-19.

[3] Banach, S., Sur les operations dans les ensembles abstraits et leur applications aux equations integrales. Fund. Math. 3 (1922), 133-181.

[4] Boyd D. W., Wong J. S. W., Nonlinear contractions. Proc. Am. Math. Soc., 20 (1969), 458-464.

[5] Geraghty M., On contraction mappings. Proc. Am. Math. Soc., 40 (1973), 604-608.

[6] Kirk W. A., Fixed points of asymptotic contractions. J. Math. Anal. Appl., 277 (2003), 645-650.

[7] Kreyszig E., Introductory Functional Analysis with Applications. Wiley Classics Library. Wiley, New York (1989).

[8] Matkowski J., Integrable solutions of functional equations. Diss. Math., 127 (1975).

[9] Meir A., Keeler E., A theorem on contraction mappings. J. Math. Anal. Appl., 28 (1969), 326-329.

[10] Kannan R., Some results on fixed points - II, Amer. Math. Monthly, 76 (1969), 405-408. MR0257838 (41:2487).

[11] Rakotch, E., A note on contractive mappings. Proc. Am. Math. Soc., 13 (1962), 459-465.

[12] Richard R. Goldberg, Methods of Real Analysis. Oxford and IBH Publishing Co. Pvt. Ltd., New Delhi (1970).

[13] Rhoades B. E., Some theorems on weakly contractive maps. Nonlinear Anal., 47 (2001), 2683-2693.

[14] Saleh Almezel, Qamrul Hasan Ansari, Mohamed Amine Khamsi: Topics in Metric Fixed Point Theory. Springer International Publishing, Switzerland, Chapter 2, (2014).

[15] Suzuki T., A definitive result on asymptotic contractions. J. Math. Anal. Appl. 335 (2007), 707-715.

[16] Nalawade Vidyadhar V., Dolhare Uttam P., Importance of Fixed Points in Mathematics: A Survey. International Journal of Applied and Pure Science and Agriculture (IJAPSA), 2(12) [December- 2016], 131-140.