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ON CONNECTED RESTRAINED DOMINATING GRAPH OF A GRAPH

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Abstract

Let G be any nontrivial connected graph. The connected restrained dominating graph $D_{cr}(G)$ of G is a graph with $V(D_{cr}(G)) = V(G) \cup \xi(G)$, where $\xi(G)$ is the set of all minimal connected restrained dominating sets of G, in which two vertices $u, v \in V(D_{cr}(G))$ are adjacent if $u \in V$ and $v = \xi(G)$ is a minimal connected restrained dominating set of G containing u. In this paper we initiate this new graph valued function in the field of graph theory and obtained its basic properties viz., connectedness, covering invariants, connectivity, travarsability and planarity.

1. Introduction

All graphs considered here are simple, finite, connected and nontrivial. Let $G = (V(G), E(G))$ be a graph, where $V(G)$ is the *vertex set* and $E(G)$ be the *edge set*

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of G. The vertex $v \in V$ is called a *pendant vertex*, if $deg_G(v) = 1$ and an *isolated vertex* if $deg_G(v) = 0$, where $deg_G(x)$ is the *degree of a vertex* $x \in V(G)$. A vertex which is adjacent to a pendant vertex is called a *support vertex*. We denote $\delta(G)(\Delta(G))$ as the *minimum(maximum) degree* and $n = |V(G)|$, $m = |E(G)|$ the *order* and *size* of G respectively. A *spanning subgraph* is a subgraph containing all the vertices of G. A shortest $u - v$ path is often called a *geodesic*. The *diameter* $diam(G)$ of a connected graph G is the length of any longest geodesic. The *neighborhood* of a vertex u in V is the set $N(u)$ consisting of all vertices v which are adjacent with u. The *closed neighborhood* is $N[u] = N(u) \cup \{u\}$. A subset $S \subseteq V(G)$ is said to be vertex covering set if every edge of G is incident to at least one vertex in S. The minimum cardinality among all vertex covering sets is called vertex covering number. It is denoted by $\alpha_0(G)$. A subset $F \subseteq E(G)$ is said to be edge covering set if every vertex of G is incident to at least one edge in F. The minimum cardinality among all edge covering sets is called edge covering number. It is denoted by $\alpha_1(G)$. The cardinality of maximum independent set of vertices(respectively edges) of a graph G is called vertex(respectively edge) independence number. It is denoted by $\beta_0(G)$ (respectively $\beta_1(G)$). The vertex connectivity is the minimum number of vertices are required to disconnect a graph. It is denoted by $\kappa(G)$. Similarly, the edge connectivity is the minimum number of edges are required to disconnect a graph. It is denoted by $\lambda(G)$.

A set $D \subseteq V$ of a graph $G = (V, E)$ is a *dominating set* if for every vertex $v \in V - D$ there exists a vertex $u \in D$ such that v is adjacent to u. A dominating set D is said to be *minimal* if no proper subset of D is a dominating set. The minimum cardinality of a minimal dominating set of G is called a *domination number* $\gamma(G)$ of G. A dominating set D is said to be *connected dominating set* if $\langle D \rangle$ is connected. The *connected domination number* $\gamma_c(G)$ of G is the minimum cardinality of a minimal connected dominating set of G [16]. A dominating set D is said to be a *cototal dominating set* if $\langle V - D \rangle$ contains no isolated vertices. The *cototal domination number* $\gamma_{cl}(G)$ of G is the minimum cardinality of a minimal cototal dominating set of G [15]. This concept was also studied as restrained domination in graphs by G. S. Domke [5] as follows:

A set $S \subseteq V$ is a *restrained dominating set* if every vertex in $V - S$ is adjacent to a vertex in S and to another vertex in $V - S$. Let $\gamma_r(G)$ denote the size of a smallest restrained dominating set with cardinality $\gamma_r(G)$.

The connected cototal domination number of a graph has been studied by B. Basavanagoud and S. M. Hosamani [1] which is defined as follows:

A dominating set $D \subseteq V$ of a graph $G = (V, E)$ is said to be a *connected cototal dominating set* if $\langle D \rangle$ is connected and $\langle V - D \rangle \neq \phi$, contains no isolated vertices. A connected cototal dominating set is said to be *minimal* if no proper subset of D is connected cototal dominating set. The *connected cototal domination number* $\gamma_{ccl}(G)$ of G is the minimum cardinality of a minimal connected cototal dominating set of G. This concept was also studied as connected restrained domination in graphs by H. Chen et.al., [4] as follows:

Let $G = (V, E)$ be a graph. A *k-connected restrained dominating set* is a set $S \subseteq V$ where S is a restrained dominating set and $G[S]$ has at most k-components. The *k-connected restrained domination number* of G is denoted by $\gamma_r^k(G)$ is the smallest cardinality of a k- connected restrained dominating set of G.

2. Motivation

Interconnection networks are usually modeled by graphs whose vertices represent nodes and whose edges are associated with the communication links between nodes. Then, a path between two vertices in the graph represents a possible communication route between the corresponding nodes in the network. Moreover, the path length gives a measure of the communication delay experienced when communicating through the route it represents. Thus, the diameter of the graph, defined as the maximum distance between any two different nodes, represents the maximum communication delay between any two nodes of the network. As a consequence, when finding interconnection network models it is important for the diameter to be as small as possible.

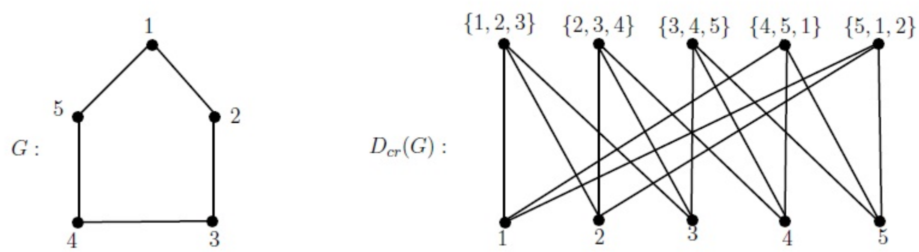
A good interconnection network model should also remain communicating, and with reasonable efficiency, in the presence of faults. The graph connectivity measures the maximum number of faults that a network can tolerate to remain communicating. The efficiency of the communication under a given number of faults is quantified by the wide diameter, which measures the increase in the maximum communication delay between any two nodes in the networks.

Many graphs presenting good properties as interconnection network models can be obtained using graph operators [10]. In particular, the line graph has been widely used to obtain good interconnection network models.

In this paper we explore the use of the *connected restrained dominating graph*, operator as a possible way to build new interconnection network models. For more information on dominating graphs, the interested reader may see [2, 11-14].

Definition 1 : The connected restrained dominating graph $D_{cr}(G)$ of G is a graph with $V(D_{cr}(G)) = V(G) \cup \xi(G)$, where $\xi(G)$ is the set of all minimal connected restrained dominating sets of G, in which two vertices $u, v \in V(D_{cr}(G))$ are adjacent if $u \in V$ and $v = \xi(G)$ is a minimal connected restrained dominating set of G containing u.

In figure 1, a graph G and its connected restrained dominating graph $D_{cr}(G)$ are depicted.



3. Preliminary Results

Let $G = (V, E)$ be any connected graph with $V(G) = \{v_1, v_2, v_3, \cdots, v_n\}$ are the vertices of G and let $\xi(G) = \{D_1, D_2, D_3, \cdots D_k\}$ be the set containing all minimal connected restrained dominating sets of G. Then by the definition of connected restrained dominating graph, $V(D_{cr(G)}) = V(G) \cup \xi(G)$.

We begin with the following already know auxiliary results and straightforward observations.

Theorem A [16] : Suppose $G = C_n; n \geq 3$ then $\gamma_c(G) = n - 1$.

Theorem B [6] : Let G be any connected graph. Then G is eulerian if and only if degree of every vertex is even.

Theorem C [3] : Let G be any connected bipartite graph. Then G is hamiltonian if $V(G) = V_1 \cup V_2$ such that $|V_1(G)| = |V_2(G)|$.

Theorem D [6] : Let G be any nontrivial graph then G is planar if and only if G does not contain subgraphs homeomorphic to K_5 or $K_{3,3}$.

Observation 1 : For any connected graph G, the connected restrained dominating graph D_{cr} is bipartite with partitions $V_1 = V(G)$ and $V_2 = \xi(G)$.

Observation 2 : Let G be a graph of order n and size m. Then

1. $V(D_{cr}(G)) = n + |\xi(G)|$
2. $E(D_{cr}(G)) = \frac{1}{2}\left[\sum\limits_{v\in V(G)}(deg_{D_{cr}(G)}(v)) + \sum\limits_{u\in \xi(G)}(deg_{D_{cr}(G)}(u))\right]$.

Observation 3 : Let G be a graph of order n and size m. Then

1. $deg_{D_{D_{cr(G)}}}(v)$ = no of minimal connected restrained dominating sets containing v if $v \in V(G)$.
2. $deg_{D_{D_{cr(G)}}}(v) = |D_i|$ if $v = D_i \subseteq \xi(G)$.

Proposition 4 : If $G = K_n; n \geq 3$ then $D_{cr}(G) \cong nK_n$.

Proof : Let $G = K_n; n \geq 3$ then one can observe that every vertex of G is a connected restrained dominating set of G. Hence $\xi(G) = \left\{\{v_1\}, \{v_2\}, \{v_3\}, \cdots, \{v_n\}\right\}$. Therefore, by the definition of $D_{cr}(G)$, the result follows. □

Proposition 5 : If $G = C_n; n \geq 4$ then $|\xi(G)| = n$.

Proof :Let $G = C_n; n \geq 4$ with $V(C_n) = \{v_1, v_2, v_3, \cdots, v_n\}$. Let $D \subseteq V(G)$ be any connected dominating set of G. Then by Theorem A, $\gamma_c(G) = n-2$, i.e $|D| = n-2$ and $\langle V - D\rangle \cong K_2$ contains no isolated vertex. Therefore, D is a minimal connected restrained dominating set of G. The following are the possible minimal connected restrained dominating sets of G.

$$\begin{aligned}
D_1 &= \{v_1, v_2, v_3, \cdots, v_{n-2}\} \\
D_2 &= \{v_2, v_3, v_4, \cdots, v_{n-1}\} \\
D_1 &= \{v_3, v_4, v_5, \cdots, v_n\} \\
&\vdots \\
D_{n-1} &= \{v_{n-2}, v_{n-1}, v_n, \cdots, v_{n-4}\} \\
D_n &= \{v_n, v_1, v_2, \cdots, v_{n-3}\}
\end{aligned}$$

Hence $\xi(G) = \{D_1, D_2, D_3, \cdots, D_n\}$. i.e $|\xi(G)| = n$. □

Proposition 6 : If $G = W_n; n \geq 5$ then $|\xi(G)| = n$.

Proof : By Theorem A, $\gamma_c(G) = 1$. Therefore, the result follows from the fact that $W_n = C_{n-1} + K_1$ and using Proposition 5, one can easily verify the result. □

Proposition 7 : If $G = K_{n,m}; 2 \leq n \leq m$ then $|\xi(G)| = nm$.

Proof : Let $G = K_{n,m}; 2 \leq n \leq m$ then every edge of $K_{n,m}$ is a connected cototal dominating set of G. Hence, $|\xi(G)| = |E(K_{n,m})| = nm$. □

4. Main Results

Theorem 8 : Let $G = (V, E)$ be any connected graph. Then $D_{cr}(G)$ is connected if and only if the following conditions hold:

1. for every $v \in V(G)$ there is some $D_i \in \xi(G)$ such that $v \subseteq D_i$
2. either $D_i \cap D_j \neq \phi$ for all $D_i, D_j \in \xi(G)$ or if $D_i \cap D_j = \phi$ then $N(D_i) \cap N(D_j) \neq \phi$ for all $D_i, D_j \in \xi(G)$, where $N(D_i)$ and $N(D_j)$ are the open neighborhood of the sets D_i and D_j respectively.

Proof : Let G be any connected graph with $V(G) = \{v_1, v_2, v_3, \cdots, v_n\}$. Since by Observation 1, D_{cr} is a bipartite graph with vertex partitions $V_1 = V(G)$ and $V_2 = \xi(G)$ such that $V_1 \cap V_2 = \phi$. Suppose G satisfies the condition (i) and $D_i \cap D_j \neq \phi$ for all $D_i, D_j \in \xi(G)$, then there is an edge joining a vertex of V_1 to a vertex of V_2, that is there exists an $u - v$ path for every pair of vertices in $D_{cr}(G)$. Hence $D_{cr}(G)$ is connected.
Suppose G satisfies condition (i) and if $D_i \cap D_j = \phi$ then $N(D_i) \cap N(D_j) \neq \phi$ for all $D_i, D_j \in \xi(G)$, where $N(D_i)$ and $N(D_j)$ are the open neighborhood of the sets D_i and D_j respectively, then there exists a vertex $v \in N(D_i)$ such that V is adjacent to a vertex $u \in N(D_j)$, that is $D_r(G)$ contains a $u - v$ path between every pair of vertices. Hence $D_{cr}(G)$ is connected.
Conversely, suppose G does not satisfies the hypothesis of the theorem, then the second smallest eigenvalue of $D_{cr}(G)$ is zero. i.e, the algebraic connectivity of $D_{cr}(G)$ is zero. Hence $D_{cr}(G)$ is disconnected. □

5. Covering Invariants for $D_{cr}(G)$

Theorem 9 : Let G be any connected graph. Then

$$\alpha_0(D_{cr}(G)) \;=\; min\{|V|, |\xi|\}.$$

Proof : By the definition of $D_{cr}(G)$, $V(D_{cr}(G)) = \{v_1, v_2, \cdots, v_n\} \cup \{D_1, D_2, D_3, \cdots, D_k\}$ for some positive integer k. In view of Observation 1, $D_{cr}(G)$ is bipartite. Therefore, $max\{|V|, |\xi|\}$ is the vertex independence number of $D_{cr}(G)$. Since any subset of vertex set of a graph G is an independent set if and only if its complement is covering set. Therefore, we have the following cases:

Case 1. Suppose $|V|$ is maximum independent set then $V(D_{cr}(G)) \setminus V(G) = \xi(G)$ is minimum covering set of $D_{cr}(G)$.

Case 2. Suppose $|\xi|$ is maximum independent set then $V(D_{cr}(G)) \setminus \xi(G) = V(G)$ is minimum covering set of $D_{cr}(G)$.

By the virtue of above cases, we conclude that $\alpha_0(D_{cr}(G)) = min\{|V|, |\xi|\}$. □

Corollary 10 : Let G be any connected graph. Then

$$\beta_0(D_{cr}(G)) \;=\; max\{|V|, |\xi|\}.$$

Proof : The proof follows from the following facts:

Fact 1. For any bipartite graph the vertex independence number is the maximum cardinality of any partite set. Since $D_{cr}(G)$ is bipartite therefore the result follows directly from the definition of $D_{cr}(G)$.

Fact 2. Since any subset of vertex set of a graph G is an independent set if and only if its complement is covering set. Therefore, the result follows from Theorem 9. □

Theorem 11 : Let G be any connected graph. Then

$$\beta_1(D_{cr}(G)) \;=\; min\{|V|, |\xi|\}.$$

Proof : Let G be any connected graph. Then by the definition of $D_{cr}(G)$ is bipartite. Since for any bipartite graph G, $\alpha_0(G) = \beta_1(G)$. Therefore the result follows from Theorem 9. □

Corollary 12 : Let G be any connected graph. Then

$$\alpha_1(D_{cr}(G)) = |V| + |\xi(G)| - min\{|V|, |\xi|\}.$$

Proof : Let G be any connected graph. By the definition of $D_{cr}(G)$, $V(D_{cr}(G)) = V(G) \cup \xi(G)$, that is $|V(D_{cr}(G))| = |V| + |\xi(G)|$. Therefore the theorem follows from Theorem 11 and the fact that $\alpha_1(G) + \beta_1(G) = |V(G)|$. Further, we have the following useful observations:

(i) . Suppose $|V(G)| < |\xi(G)|$, then $\alpha_1(D_{cr}(G)) = |\xi(G)|$.

(ii) . Suppose $|V(G)| > |\xi(G)|$, then $\alpha_1(D_{cr}(G)) = |V(G)|$.

6. Connectivity for $D_{cr}(G)$

Theorem 13 : For any connected graph G,

$$\kappa(D_{cr}(G)) = min\left\{ \min_{v \in V_1 \subseteq V(D_{cr}(G))}(deg(v)), \quad \min_{D_i \in V_2 \subseteq V(D_{cr}(G))}(|D_i|) \right\}$$

where $\kappa(D_{cr}(G))$ is the vertex connectivity of $D_{cr}(G)$.

Proof : Let $G = (V(G), E(G))$ be any connected graph. By Observation 1, the connected restrained dominating graph $D_{cr}(G)$ of a graph G is bipartite with vertex partitions $V_1 = V(G)$ and $V_2 = \xi(G)$. Therefore, by famous Menger's theorem, the maximum number of internally disjoint path between either $v_i, v_j \in V_1$ or $u_i, u_j \in V_2$ is equal to the minimum number of vertices in a vertex cut which leaves either v_i and v_j disconnected or u_i and u_j disconnected. Therefore, we consider the following cases:

Case 1. Let for some vertex $v \in V_1$ where $\deg_{D_{cr}(G)}(v) = k = \delta(D_{cr}(G))$ for some positive integer k. Then deleting those vertices which are adjacent to v in $D_{cr}(G)$ results in a disconnected graph.

Case 2. Let for some vertex $u \in V_2$ where $\deg_{D_{cr}(G)}(u) = k = \delta(D_{cr}(G))$ for some positive integer k. Then deleting those vertices which are adjacent to u in $D_{cr}(G)$ results in a disconnected graph. □

Theorem 14 : For any connected graph G,

$$\lambda(D_{cr}(G)) = min\left\{ \min_{v \in V_1 \subseteq V(D_{cr}(G))}(deg(v)), \quad \min_{D_i \in V_2 \subseteq V(D_{cr}(G))}(|D_i|) \right\}$$

where $\kappa(D_{cr}(G))$ is the edge connectivity of $D_{cr}(G)$.

Pproof: Let $G = (V(G), E(G))$ be any connected graph with $V(G) = \{v_1, v_2, v_3, \cdots, v_n\}$. Let $\xi = \{D_i\}_{i=1}^{k}$ be the set containing all connected restrained dominating sets of G. Let $E(v_i)$ be the set of edges incident to $v_i \in V_1 \subseteq V(D_{cr}(G))$ and $E(u_j)$ be the set of edges incident to $u_j \in V_2 = \xi(G) \subseteq V(D_{cr}(G))$ for each $i = 1, 2, 3, \cdots, n$ and $j = 1, 2, 3, \cdots, k$. We define the following disjoint sets:

$$\begin{aligned} A_i &= \{a_i^e : e \in E_1(v_i)\} \qquad \text{and} \\ B_i &= \{b_i^k : k = 1, 2, 3, \cdots, deg_{D_{cr}(G)}(v_i)\} \\ A_j &= \{a_j^e : e \in E_2(u_i)\} \qquad \text{and} \\ B_j &= \{b_j^k : k = 1, 2, 3, \cdots, deg_{D_{cr}(G)}(u_j)\} \end{aligned}$$

Since, $D_{cr}(G)$ is bipartite therefore a_i^e is adjacent to a_k^e if and only if v_i and v_j are the ends of an edge $e \in E(D_{cr}(G))$. Similarly, a_j^e is adjacent to a_l^e if and only if u_j and u_l are the ends of $e \in E(D_{cr}(G))$. then it is not difficult to check that $D_{cr}(G)$ is k-edge connected if and only if $D_{cr}(G)$ is k-vertex connected. Therefore, we consider the following cases.

Case 1. Let for some vertex $v \in V_1$ where $\underset{D_{cr}(G)}{deg}(v) = k = \delta(D_{cr}(G))$ for some positive integer k. Then deleting those edges which are incident to v in $D_{cr}(G)$ results in a disconnected graph.

Case 2. Let for some vertex $u \in V_2$ where $\underset{D_{cr}(G)}{deg}(u) = k = \delta(D_{cr}(G))$ for some positive integer k. Then deleting those edges which are incident to u in $D_{cr}(G)$ results in a disconnected graph. □

7. Traversability

Theorem 15 : Let G be any connected graph. The $D_{cr}(G)$ is eulerian if and only if the following conditions hold:

1. for every $v \in V(G)$ there is some $D_i \in \xi(G)$ such that $v \subseteq D_i$
2. either $D_i \cap D_j \neq \phi$ for all $D_i, D_j \in \xi(G)$ or if $D_i \cap D_j = \phi$ then $N(D_i) \cap N(D_j) \neq \phi$ for all $D_i, D_j \in \xi(G)$, where $N(D_i)$ and $N(D_j)$ are the open neighborhood of the sets D_i and D_j respectively.

3. for every $v \in D_i \subseteq \xi(G)$; $1 \leq i \leq k$ such that $\bigcup_{i=1}^{k} D_i$ is even

4. for every $D_i \in \xi(G)$; $1 \leq i \leq k$ such that $|D_i|$ is even.

Proof : Let $G = (V(G), E(G))$ be any connected graph with $V(G) = \{v_1, v_2, v_3, \cdots, v_n\}$. Let $\xi = \{D_i\}_{i=1}^{k}$ be the set containing all connected restrained dominating sets of G. Suppose G satisfies the conditions (1) and (2) then by Theorem 8, $D_{cr}(G)$ is connected. Further if (3) and (4) holds, then every vertex $v \in V_1 \subseteq V(D_{cr}(G))$ and $u \in V_2 = \xi(G) \subseteq V(D_{cr}(G))$ has even degree. Therefore, by Theorem B, $D_{cr}(G)$ is eulerian.

Conversely, let $D_{cr}(G)$ be eulerian, then $D_{cr}(G)$ must be connected. Therefore, by Theorem 8, conditions (1) and (2) holds. Since, $D_{cr}(G)$ is eulerian, therefore by Theorem B, degree of every vertex must be even. By Observation 1, $D_{cr}(G)$ is bipartite, therefore the vertex set of G can be partitioned into two disjoint sets $V(G) = V_1$ and $\xi(G) = V_2$, which implies that every vertex $v \in D_i$ must be present in even number of connected restrained dominating sets of G, and every $D_i \in \xi(G)$ must have even cardinality. hence (3) and (4) holds. □

Theorem 16 : Let G be any connected graph. The $D_{cr}(G)$ is hamiltonian if G satisfies the following conditions:

1. for every $v \in V(G)$ there is some $D_i \in \xi(G)$ such that $v \subseteq D_i$

2. either $D_i \cap D_j \neq \phi$ for all $D_i, D_j \in \xi(G)$ or if $D_i \cap D_j = \phi$ then $N(D_i) \cap N(D_j) \neq \phi$ for all $D_i, D_j \in \xi(G)$, where $N(D_i)$ and $N(D_j)$ are the open neighborhood of the sets D_i and D_j respectively.

3. $|V(G)| = |\xi(G)|$

Proof : Let $G = (V(G), E(G))$ be any connected graph with $V(G) = \{v_1, v_2, v_3, \cdots, v_n\}$. Let $\xi = \{D_i\}_{i=1}^{k}$ be the set containing all connected restrained dominating sets of G. Suppose G satisfies the conditions (i) and (ii) then by Theorem 8, $D_{cr}(G)$ is connected. Since $D_{cr}(G)$ is bipartite, therefore the vertex set of G can be partitioned into two disjoint sets $V(G) = V_1$ and $\xi(G) = V_2$ therefore if (iii) holds then by Theorem C, $D_{C_r}(G)$ is hamiltonian. □

8. Planarity

Theorem 17 : Let G be any connected graph. The $D_{cr}(G)$ is planar if and only if G satisfies the following conditions:

1. for some $u, v, w \in V(G)$ there is no connected restrained dominating sets $D_1, D_2, D_3 \in \xi(G)$ such that $\{u, v, w\} \subseteq D_i$; $i = 1, 2, 3$ and
2. $D_1 \cap D_2 \cap D_3 \neq \{u, v, w\}$.

Proof : Let G be any nontrivial connected graph with at least three vertices. Suppose on the contrary, G does not satisfies the conditions mentioned in the hypothesis then there exist at least three vertices $u, v, w \in V(G)$ such that $\{u, v, w\} \subseteq D_i$; $i = 1, 2, 3$ and $D_1 \cap D_2 \cap D_3 \neq \{u, v, w\}$. Which implies that $deg_{D_{cr}(G)}(u) \geq 3$, $deg_{D_{cr}(G)}(v) \geq 3$ and $deg_{D_{cr}(G)}(w) \geq 3$. Further, $deg_{D_{cr}(G)}(D_1) \geq 3$, $deg_{D_{cr}(G)}(D_2) \geq 3$ and $deg_{D_{cr}(G)}(D_3) \geq 3$. By Observation 1, $D_{cr}(G)$ is bipartite, therefore $K_{3,3}$ is an induced subgraph of $D_{cr}(G)$. Hence by Theorem D, $D_{cr}(G)$ is nonplanar. Which is a contradiction to our assumption. Conversely, suppose G satisfies the conditions (1) and (2) then $D_{cr}(G)$ contains no subgraph homeomorphic to $K_{3,3}$. Therefore by Theorem D, $D_{cr}(G)$ is planar. □

Theorem 18 : Let G be any connected graph. The $D_{cr}(G)$ is outerplanar if and only if G satisfies the following conditions:

1. for some $u, v, w \in V(G)$ there is no connected restrained dominating sets $D_1, D_2 \in \xi(G)$ such that $\{u, v, w\} \subseteq D_i$; $i = 1, 2$ and
2. $D_1 \cap D_2 \neq \{u, v, w\}$.

Proof : Let G be any nontrivial connected graph with at least three vertices. Suppose on the contrary, G does not satisfies the conditions mentioned in the hypothesis then there exist at least three vertices $u, v, w \in V(G)$ such that $\{u, v, w\} \subseteq D_i$; $i = 1, 2$ and $D_1 \cap D_2 \neq \{u, v, w\}$. Which implies that $deg_{D_{cr}(G)}(u) \geq 3$, $deg_{D_{cr}(G)}(v) \geq 3$ and $deg_{D_{cr}(G)}(w) \geq 3$. Further, $deg_{D_{cr}(G)}(D_1) \geq 2$ and $deg_{D_{cr}(G)}(D_2) \geq$. By Observation 1, $D_{cr}(G)$ is bipartite, therefore $K_{2,3}$ is an induced subgraph of $D_{cr}(G)$. Since any nontrivial graph G is outerplanar if and only if G does not contain subgraph homeomorphic to K_4 or $K_{2,3}$. Therefore, $D_{cr}(G)$ is nonouterplanar. Which is a contradiction to our assumption.

Conversely, suppose G satisfies the conditions (1) and (2) then $D_{cr}(G)$ contains no subgraph homeomorphic to $K_{2,3}$. Therefore, $D_{cr}(G)$ is outerplanar.

□

Theorem 19 : Let G be any connected graph. The $D_{cr}(G)$ is maximal planar if and only if G satisfies the following conditions:

1. for some $u, v \in V(G)$ there is no connected restrained dominating sets $D_1, D_2, D_3 \in \xi(G)$ such that $\{u, v\} \subseteq D_i$; $i = 1, 2, 3$
2. there is no $w \in V(G)$ such that $w \in D_i; i = 1, 2$ and
3. $D_1 \cap D_2 \cap D_3 \neq \{u, v, w\}$.

Proof : Let G be any nontrivial connected graph with at least three vertices. Suppose on the contrary, G does not satisfies the conditions mentioned in the hypothesis then there exist at least three vertices $u, v, w \in V(G)$ such that $\{u, v, w\} \subseteq D_i$; $i = 1, 2, 3$ and $D_1 \cap D_2 \cap D_3 \neq \{u, v, w\}$. Which implies that $deg_{D_{cr}(G)}(u) \geq 3$, $deg_{D_{cr}(G)}(v) \geq 3$ and $deg_{D_{cr}(G)}(w) \geq 2$. Further, $deg_{D_{cr}(G)}(D_1) \geq 3$, $deg_{D_{cr}(G)}(D_2) \geq 3$ and $deg_{D_{cr}(G)}(D_3) \geq 2$. By Observation 1, $D_{cr}(G)$ is bipartite, therefore $K_{3,3} - \{wD_3\}$ is an induced subgraph of $D_{cr}(G)$ where wD_3 is an edge of $K_{3,3}$. Since any nontrivial graph G is maximal planar if and only if G does not contain subgraph homeomorphic to K_{5-x} or $K_{3,3} - x$, where x is any edge of K_5 or $K_{3,3}$. Therefore, $D_{cr}(G)$ is not a maximal planar. Which is a contradiction to our assumption.
Conversely, suppose G satisfies the conditions (1), (2) and (3) then $D_{cr}(G)$ contains no subgraph homeomorphic to $K_{3,3} - x$ is an induced subgraph of $D_{cr}(G)$ where x is an edge of $K_{3,3}$. Therefore, $D_{cr}(G)$ is maximal Planar. □

Theorem 20 : Let G be any connected graph. The $cr(D_{cr}(G)) = 1$ if and only if G satisfies the following conditions:

1. for some $u, v, w \in V(G)$ there exists connected restrained dominating sets $D_1, D_2, D_3 \in \xi(G)$ such that $\{u, v, w\} \subseteq D_i$; $i = 1, 2, 3$
2. $D_1 \cap D_2 \cap D_3 = \{u, v, w\}$

Proof : Let G be any nontrivial connected graph with at least three vertices. Suppose G satisfies the conditions mentioned in the hypothesis then by Theorem 17, $K_{3,3}$ is an induced subgraph of $D_{cr}(G)$. Since any nontrivial graph G has $cr(G) = 1$ if and only if G does not contain subgraph homeomorphic to K_5 or $K_{3,3}$, Therefore, $cr(D_{cr}(G)) = 1$. Conversely, suppose G does not satisfies the conditions then By Theorem 13, $D_{cr}(G)$ is planar. Hence $cr(D_{cr}(G)) = 0$. □

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