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LINE GRAPH OF $S-$ VALUED GRAPHS

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Abstract

Graph colouring and its generalizations are useful tools in modelling a wide variety of scheduling and assignment problems. In [6], the authors intorduced the notion of semiring valued (in short, $S-$valued) graphs. In this paper we introduce the notion of line graph of a $S-$valued graph G^S and relate it with the edge-chromatic number of G^S.

1. Introduction

In the mathematical discipline of graph theory, the line graph of an undirected graph G is another graph L(G) that represents the adjacencies between edges of G. The name line graph comes from a paper by [4] although both [13] and [5] used the construction before this. Other terms used for the line graph include the covering graph, the derivative, the edge-to-vertex dual, the conjugate, the representative graph, the interchange graph, the adjoint graph, and the derived graph.

Key Words : $S-$*valued graphs.*

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[13] proved that with one exceptional case the structure of a connected graph G can be recovered completely from its line graph. Many other properties of line graphs follow by translating the properties of the underlying graph from vertices into edges, and by Whitney's theorem the same translation can also be done in the other direction.

Line graphs are used to track changes over short and long periods of time. When smaller changes exist, line graphs are better to use than bar graphs. Line graphs can also be used to compare changes over the same period of time for more than one group. A line graph $L(G)$ (also called an adjoint, conjugate, covering, derivative, derived, edge, edge-to-vertex dual, interchange, representative) of a simple graph G is obtained by associating a vertex with each edge of the graph and connecting two vertices with an edge iff the corresponding edges of G have a vertex in common [3].

In [6], the authors introduced the notion of semiring valued graphs. In [7], we have introduced the notion of K-Coloring of S-valued graphs, which partitions the set of vertex set of a $S-$valued graph G^S into classes in such a way that, for each pair of vertives of G^S, whether or not they are allowed to be in the same class. In [9] and [10], we have studied the bounds and the chromatic numbers of some $S-$ valued graphs of K-Colorable S-valued graphs. It dealt with the vertex colouring of the $S-$valued graph G^S. In [11], we have introduced the notion of edge-colouring which has been reformulated in [12] where the edges of $S-$ valued graph G^S are coloured with different colours rather than the vertices of G^S.

In this paper, we introduce the notion of line graphs in G^S and discuss some of its properties.

2. Preliminaries

In this section, we recall some basic definitions that are needed for our work.

Definition 2.1 [1,5] : A $K-$vertex colouring of a graph G is an assignment of $K-$colours to the vertices of G such that no two adjacent vertices receive the same colour.

Definition 2.2 [1,5] : A graph G that required $K-$different colours for its colouring and not less, is called a $K-$chromatic graph and the number K is called the chromatic number of G.

Definition 2.3 [1, 5] : If $\chi(G) = K, G$ is said to be $K-$chromatic.
Definition 2.4 [6] : A semiring $(S, +, \cdot)$ is an algebraic system with a non-empty set Stogether with two binary operations $+$ and $\cdot$ such that

1. $(S, +, 0)$ is a monoid.
2. $(S, \cdot)$ is a semigroup.
3. For all $a, b, c \in S,\ a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$.
4. $0 \cdot x = x \cdot 0 = 0\ \forall\ x \in S$.

Definition 2.5 [6] : Let $(S, +, \cdot)$ be a semiring. $\preceq$ is said to be a Canonical preorder if for $a, b \in S,\ a \preceq b$ if and only if there exists $c \in S$ such that $a + c = b$.
Definition 2.6 [6] : Let $G = (V, E \subset V \times V)$ be the underlying graph with $V, E \neq \phi$. For any semiring $(S, +, \cdot)$, a Semiring-valued graph (or an $S-$valued graph) G^S is defined to be the graph $G^S = (V, E, \sigma, \psi)$ where $\sigma : V \to S$ and $\psi : E \to S$ is defined to be

$$\psi(x, y) \quad = \quad \begin{cases} min\,\{\sigma(x), \sigma(y)\} & if\ \sigma(x) \preceq \sigma(y)\ or\ \sigma(y) \preceq \sigma(x) \\ 0 & otherwise \end{cases}$$

for every unordered pair (x, y) of $E \subset V \times V$. We call σ, a $S-$vertex set and ψ, a $S-$edge set of $S-$graph G^S.
Definition 2.7 [6] : The open neighbourhood of v_i in G^S is defined as
$N_S(v_i) = \{(v_j, (\sigma(v_j)), where\ (v_i, v_j) \in E,\ \psi(v_i, v_j) \in S\}$ and the closed neighbourhood of v_i in G^S is defined as the set $N_S[v_i] = N_S(v_i) \cup \{(v_i, \sigma(v_i))\}$.
Definition 2.8 [6] : The degree of a vertex v_i of the $S-$valued graph G^S is defined as $deg_S(v_i) = (\sum_{v_j \in N_S(v_i)} \psi(v_i v_j), d(v_i))$ where $d(v_i)$ is the number of edges incident with v_i.
Definition 2.9 [5] : Consider the $S-$valued graph $G^S = \{V, E, \sigma, \psi\}$. A colouring of G^S is given by a function $f : V \times V \to S \times \mathcal{C}$ such that for all $v \in V$, $f(v, v) = (\sigma(v), c(v)),\ c(v) \in \mathcal{C}$.

3. Line Graphs of G^S

In this section, we introduce the notion of line graph of $S-$valued graphs.

Definition 3.1 : Consider a $S-$ valued graph $G^S = (V_S, E_S)$. The line graph of G^S, denoted by $L(G^S)$, is a graph whose vertices are the edges of G^S such that the edges with a common end point in G^S are adjacent in $L(G^S)$. That is, $e, f \in E(L(G^S))$ if and only if e, f are the edges in G^S- having a common end point.

Alternatively We can define the Line graph of a graph G^S as follows.

Definition 3.2 : Consider the graph $G = (V, E)$ where $V = \{v_1, v_2, \cdots, v_m\}$ so that $\mid V \mid= m$ and $E = \left\{e_i^j = (v_i, v_j)/1 \leq i, j \leq m\right\}$ with $\mid E \mid= n$.

Let $V_S = \{(v_i, \sigma(v_i))/i = 1, 2, \cdots, m\}$ and $E_S = \left\{(e_i^j, \psi(e_i^j))/1 \leq i, j \leq m\right\}$.

Then a $S-$ valued graph corresponding to the given crisp graph G is given by $G^S = (V_S, E_S)$.

The line graph $L(G^S)$ of G^S is defined as follows:

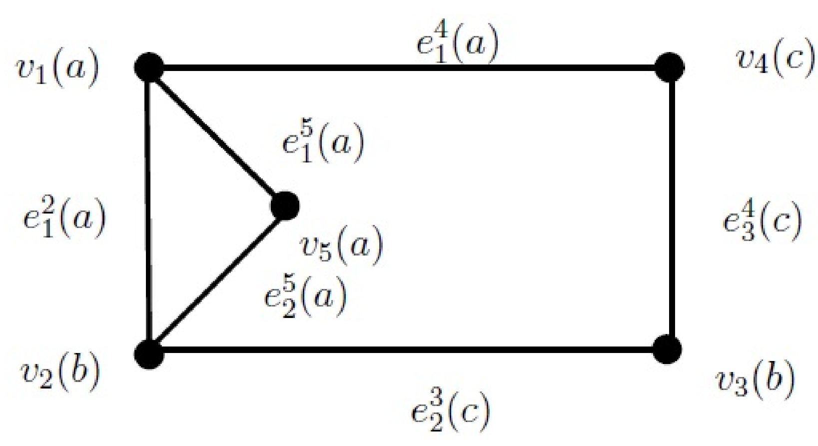
$V_S(L(G^S)) = \left\{e_i^J \in E_S\right\}$ and$E_S(L(G^S)) = \left\{(e_i^j, e_i^k)/e_i^j, e_i^k \in V_S(L(G^S))\right\}$.

Example 3.3 : Consider the Semiring $S = (\{0, a, b, c\}, +, \cdot)$ with the following Cayley table.

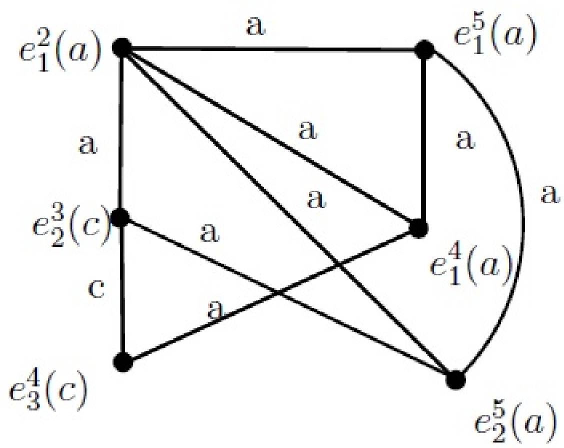
$+$	0	a	b	c
0	0	a	b	c
a	a	a	b	c
b	b	b	b	b
c	c	c	b	c

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	0	a	0
b	0	a	b	c
c	0	0	c	c

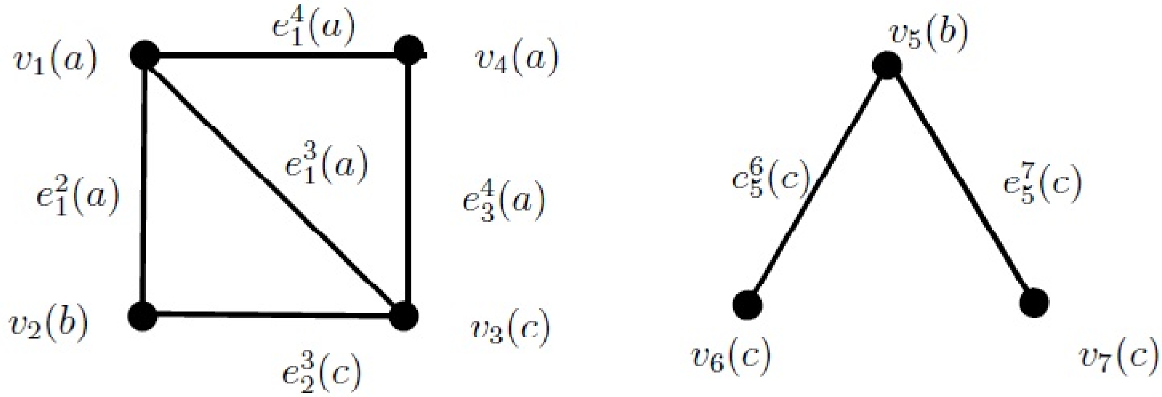
Let $\preceq$ be a canonical pre-order in S, given by $0 \preceq 0$, $0 \preceq a$, $0 \preceq b$, $a \preceq a$, $a \preceq b$, $a \preceq c$, $b \preceq b$, $c \preceq b$, $c \preceq c$. Consider the $S-$valued graph G^S :



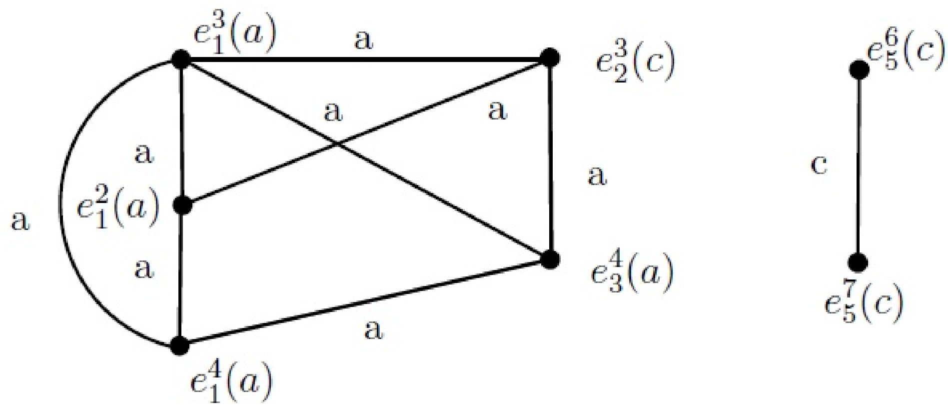
The line graph $L(G^S)$ of the $S-$valued graph G^S is given by:



Example 3.4 : Consider the semi ring $(S, +, \cdot)$ given in Example 3.3. Consider the S valued graph G_1^S on 7 vertices with two components given below:



The line graph $L(G_1^S)$ of G_1^S is given by:



From the above two examples, we observe the following facts connecting a given $S-$valued graph G^S and its line graph $L(G^S)$.

1. The graph G^S- is connected iff its line graph $L(G^S)$ is connected.

2. If H^S is a sub graph of G^S, then $L(H^S)$ is a subgraph of $L(G^S)$.

3. The only connected $S-$valued graph that is isomorphic to its line graph is a $S-$cycle C_n^S for $n \geq 3$.

4. A graph H^S is the line graph of some other $S-$valued graph G^S if and only if it is possible to find a collection of cliques in H^S, partitioning the edges of H^S, so that each vertex of H^S belong to atmost two of the cliques.

Example 3.5 : In example 3.3, $V_S = \{v_1(a), v_2(b), v_3(c), v_4(b), v_5(a)\}$ so that $\mid V_S \mid = 5$ and $\mid E_S \mid = 6$. Now,

$$\mid V_S \mid_S = \sum_{v_i \in V_S} \sigma(v_i) = b$$

Thus,$(\mid V_S \mid_S, \mid V_S \mid) = (b, 5)$.

$$\mid E_S \mid_S = \sum_{e_i^j \in E_S} \psi(e_i^j) = c$$

Therfore $(\mid E_S \mid_S, \mid E_S \mid) = (c, 6)$.

$$\mid V(L(G^S)) \mid_S = \sum_{e_i^j \in E_S} \psi(e_i^j) = c$$

Also, $\mid V(L(G^S)) \mid = 6$ so that,

$$\big(\mid V(L(G^S)) \mid_S, \mid V(L(G^S)) \mid\big) = (\mid E_S \mid_S, \mid E_S \mid) = (c, 6).$$

$$\mid E(L(G^S)) \mid_S = \sum_{e_i^j \in E_S} min\left(\psi(e_i^j), \psi(e_j^k)\right) = c$$

Since, $\mid E(L(G^S)) \mid = 9$, we have $\big(\mid E(L(G^S)) \mid_S, \mid E(L(G^S)) \mid\big) = (c, 9)$.
For $i = 1, 2, 3, 4, 5$ we now calculate $|N_S(v_i)|_S$ of the vertices in G^S as follows:

$$|N_S(v_1)|_S = a; |N_S(v_2)|_S = c; |N_S(v_3)|_S = c; |N_S(v_4)|_S = c; |N_S(v_5)|_S = a.$$

Then, $\sum_{v_i \in V_S} |N_S(v_i)|_S = a + c + c + c + a = c$

Now, for $i = 1, 2, 3, 4, 5$ $\begin{pmatrix} d(v_1) \\ 2 \end{pmatrix} = 3$; $\begin{pmatrix} d(v_2) \\ 2 \end{pmatrix} = 3$

$\begin{pmatrix} d(v_3) \\ 2 \end{pmatrix} = 1$; $\begin{pmatrix} d(v_4) \\ 2 \end{pmatrix} = 1$; $\begin{pmatrix} d(v_5) \\ 2 \end{pmatrix} = 1$. Thus, $\sum_{v_i \in V_S} \begin{pmatrix} d(v_i) \\ 2 \end{pmatrix} = 9$.

Hence, $\left(\sum_{v_i \in V_S} |N_S(v_i)|_S, \sum_{v_i \in V_S} \begin{pmatrix} d(v_i) \\ 2 \end{pmatrix}\right) = (c, 9)$.

This gives us the number of vertices and edges in $L(G^S)$ corresponding to G^S as follows:

$$(\mid V(L(G^S)) \mid_S, \mid V(L(G^S)) \mid) = (\mid E(G^S) \mid_S, \mid E(G) \mid))$$

and

$$(\mid E(L(G^S)) \mid_S, \mid E(L(G^S)) \mid) = \left(\sum_{e_i^j \in E_S} \psi(e_i^j), \sum_{v_i \in V_S} \begin{pmatrix} d(v_i) \\ 2 \end{pmatrix}\right)$$

The above discussion leads to the following theorem.

Theorem 3.6 : The number of vertices and edges in $L(G^S)$ are given by

$$(\mid V(L(G^S)) \mid_S, \mid V(L(G^S)) \mid) = (\mid E(G^S) \mid_S, \mid E(G) \mid))$$

and

$$(\mid E(L(G^S)) \mid_S, \mid E(L(G^S)) \mid) = \left(\sum_{e_i^j \in E_S} \psi(e_i^j), \sum_{v_i \in V_S} \begin{pmatrix} d(v_i) \\ 2 \end{pmatrix}\right)$$

Proof : The vertices of a line graph $L(G^S)$ correspond to the edges of graph G^S, and two distinct edges of G^S are adjacent in $L(G^S)$ if and only if they share a common

point. Formally, we consider the graph $G = (V, E)$ where $V = \{v_1, v_2, \cdots, v_m\}$ so that $\mid V \mid = m$ and $E = \left\{e_i^j = (v_i, v_j)/1 \leq i, j \leq m\right\}$ with $\mid E \mid = n$.
Let $V_S = \{(v_i, \sigma(v_i))/i = 1, 2, \cdots, m\}$ and $E_S = \left\{(e_i^j, \psi(e_i^j))/1 \leq i, j \leq m\right\}$.
Then a $S-$ valued graph corresponding to the given crisp graph G is given by $G^S = (V_S, E_S)$.
The line graph $L(G^S)$ of G^S is given by
$V_S(L(G^S)) = \left\{e_i^J \in E_S\right\}$ and$E_S(L(G^S)) = \left\{(e_i^j, e_i^k)/e_i^j, e_i^k \in V_S(L(G^S))\right\}$.
This gives a bijection between the edge set of G^S and the vertex set of $L(G^S)$ and also a bijection between the vertex set of G^S and the edge set of $L(G^S)$. These bijections complete the proof of the theorem.

Theorem 3.7 : The line graph of the star$K_{1,n}^S$ is the complete graph K_n^S

Proof : Consider the star $K_{1,n}^S = (V_S, E_S)$ where $V_S = \{(v_0, \sigma(v_0)), (v_i, \sigma(v_i)) \mid i = 1, 2, ..., n\}$ with $(v_0, \sigma(v_0))$ as its pole.
Then the edge set of $k_{1,n}$ is $E_S = \{(e_0^i, \psi(e_0^i)) \mid i = 1, 2, ..., n\}$
For $L(K_{1,n}^S)$ the $V(L(K_{1,n}^S)) = \{(e_0^i, \psi(e_0^i)) \mid 1 = 1, 2, ..., n\}$ so that $|V(L(K_{1,n}^S))| = n$
In $E(K_{1,n}^S), e, f$ are adjacent if e, f are adjacent in $K_{1,n}^S$. Which is obvious as the common point shared by them in the pole $(v_0, \sigma(v_0)$.
Thus in $L(K_{1,n}^S)$ every pair of vertices $(e_0^i, e_0^j), 1 \leq i, j \leq n$ are adjacent, proving that $L(K_{1,n}^S)$ is a complete graph on n vertices.

Theorem 3.8 : The line graph of $S-$cycle C_n^S is a $S-$cycle.

Proof : Consider the cycle $C_n^S = (V_S, E_S)$ where $V_S = \{(v_i, \sigma(v_i)) \mid i = 1, 2, \cdots, n\}$ with $(v_1, \sigma(v_1))$ as its initial and end point. Then the edge set of C_n^S is $E_S = \{(e_i^{i+1}, \psi(e_i^{i+1})) \mid i = 1, 2, \cdots, n\}$.
For $L(C_n^S)$ the $V(L(C_n^S)) = \left\{(e_i^{i+1}, \psi(e_i^{i+1})) \mid 1 = 1, 2, \cdots, n\right\}$ so that $\mid V(L(C_n^S)) \mid = n$.
In $E(L(C_n^S))$, two edges e, f are adjacent if the edges e, f are adjacent in C_n^S.. Consider $e_{i-1}^i, e_i^{i+1}, e_{i+1}^{i+2}, \ 2 \leq i \leq (n-1)$. Then the edge $e_i^{i+1}, \ \ 2 \leq i \leq (n-1)$ is adjacent to the edges e_{i+1}^{i+2} and e_{i-1}^i. Further, e_1^2 is adjacent to both the edges e_2^3 and e_n^1. Thus, e_i^{i+1} will be adjacent to only two edges only, for $i = 1, 2, \cdots, n$, proving that the line graph $L(C_n^S)$ of a cycle C_n^S is a cycle.

Theorem 3.9 : The line graph of a $S-$path P_n^S is a $S-$path.

Proof : Consider the path $P_n^S = (V_S, E_S)$ where $V_S = \{(v_0, \sigma(v_0)), (v_i, \sigma(v_i)) \mid i = 1, 2, \cdots, n\}$. Then the edge set of P_n^S is $E_S = \{(e_i^{i+1}, \psi(e_i^{i+1})) \mid i = 0, 1, 2, \cdots, n\}$.

For $L(P_n^S)$ the $V(L(P_n^S)) = \left\{(e_i^{i+1}, \psi(e_i^{i+1})) \mid 1 = 1, 2, \cdots, n\right\}$ so that $|V(L(P_n^S))| = n$. As for the case of a cycle, in a $S-$path also, two edges in $L(P_n^S)$ will be adjacent if and only if they are adjacent in P_n^S. Consider the edges $e_{i-1}^i, e_i^{i+1}, e_{i+1}^{i+2},\ 1 \le i \le (n-2)$. Then the edge $e_i^{i+1},\ \ 1 \le i \le (n-2)$ is adjacent to the edges e_{i-1}^i and e_i^{i+1}. Further, e_{n-1}^n is adjacent to the edge e_{n-2}^{n-1} only and the edge e_1^2 is adjacent to the edge e_2^3 only. This shows that the line graph $L(P_n^S)$ of a path P_n^S is a $S-$path from the vertex e_1^2 of $L(P_n^S)$ (the edge e_1^2 of P_n^S connecting v_1 to v_2) to the vertex e_{n-1}^n of $L(P_n^S)$ (the edge e_{n-1}^n of P_n^S connecting v_{n-1} to v_n), thus completing the proof of the theorem.

A line graph can be used to convert the edge colouring problem into a vertex colouring problem. Thus we have the following theorem.

Theorem 3.10 : For any graph G^S, $\chi'_S(G^S) = \chi_S(L(G^S)$.

Proof : The vertices of a line graph $L(G)$ correspond to the edges of graph G, it follows directly from the definitions that the edge colorings of G are in one-to-one correspondence with the vertex colorings of $L(G)$. This bijection preserves lots of properties (the coloring to be proper, equitable, having at least a given distance between any two elements of the same color or at most a given diameter in each component of every colour class, excluding alternately bi-colored cycles, etc.). Hence the corresponding versions of the chromatic index of G are equal to those of the chromatic number of $L(G)$, thus proving the theorem.

Alternatively, we shall prove the same result, in detail, as follows:

Consider the $S-$valued graph $G^S = (V_S, E_S)$, where $V_S = \{(v_i, \sigma(v_i))/i = 1, 2, \cdots, m\}$ and $E_S = \left\{(e_i^j, \psi(e_i^j))/1 \le i, j \le m\right\}$.

Consider the $S-$valued graph $G^S = \{V, E, \sigma, \psi\}$ and a colouring class $\mathcal{C} = \{c_1, c_2, \cdots, c_k\}$of G^S.

We shall introduce the following notation. For an edge $e_i^j = (v_i, v_j) \in E_S$ the corresponding vertex in $V(L(G^S))$ will be denoted by $(e_i^j)'$ and vice-versa.

Let a vertex-colouring of G^S be given by a function $\phi : V_S \to \mathcal{C}$ such that for all $v_i \in V_S, \phi(v_i) = c(v_i)\ c(v_i) \in \mathcal{C}$. To this vertex-colouring of G^S, we associate an edge-colouring $\pi : E(L(G^S)) \to \mathcal{C}$ of the line graph $L(G^S)$ by

$$\pi((e_i^j)', (e_i^k)') = \phi(v_i)\ where\ \{v_i\} = e_i^j \cap e_i^k \quad \cdots\cdots (1)$$

Due to the degree condition, every vertex of G^S is the intersection of atleast two edges.

Hence every colour used in ϕ occurs in the colouring π also. Thus, it is easy to prove that the definition given in (1) that gives the correspondence $\pi \mapsto \phi$ is a bijection, completing the proof of the theorem.

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