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ON THE GENERAL SUM CONNECTIVITY INDEX OF GENERALIZED xyz-POINT-LINE-TRANSFORMATION GRAPHS

B. BASAVANAGOUD[1] AND CHETANA S. GALI[2]
[1,2] Department of Mathematics,
Karnatak University, Dharwad - 580 003, India

Abstract

The *general sum connectivity index* is a molecular descriptor defined as $\chi_\alpha(G) = \sum_{uv \in E(G)} (d_G(u) + d_G(v))^\alpha$, where $d_G(u)$ denotes the degree of vertex u in a graph G and α is a real number different from zero. In this paper, we compute the expressions for general sum connectivity index of *generalized xyz−Point-Line transformation graphs.*

1. Introduction

In this paper, we consider simple and undirected graphs.

Let G be a graph with the vertex set $V(G) = V$ and edge set $E(G) = E$ such that $|V| = n$ is said to be *order* and $|E| = m$ the *size* of G. The *degree of a vertex* $v \in V(G)$ is the number of vertices adjacent to v in G and is denoted by $d_G(v)$. If u and v are two adjacent vertices of G, then the edge connecting them will be denoted by uv. The

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degree of an edge $e = uv$ in G is denoted by $d_G(e)$, and is defined by $d_G(e) = d_G(u) + d_G(v) - 2$. The *maximum* and *minimum vertex degree* of G are denoted by $\Delta(G) = \Delta$ and $\delta(G) = \delta$ respectively. Here $u \sim v$ ($u \nsim v$) means that the vertices u and v are adjacent (resp., not adjacent) in G, $e \sim f$ ($e \nsim f$) means that the edges e and f are adjacent (resp., not adjacent) and also $u \sim e$ ($u \nsim e$) means that the vertex u and edge e are incident (resp., not incident) in G.

The *complement* $\overline{G}$ of a graph G is the graph with the vertex set V, in which two vertices are adjacent if and only if they are not adjacent in G.

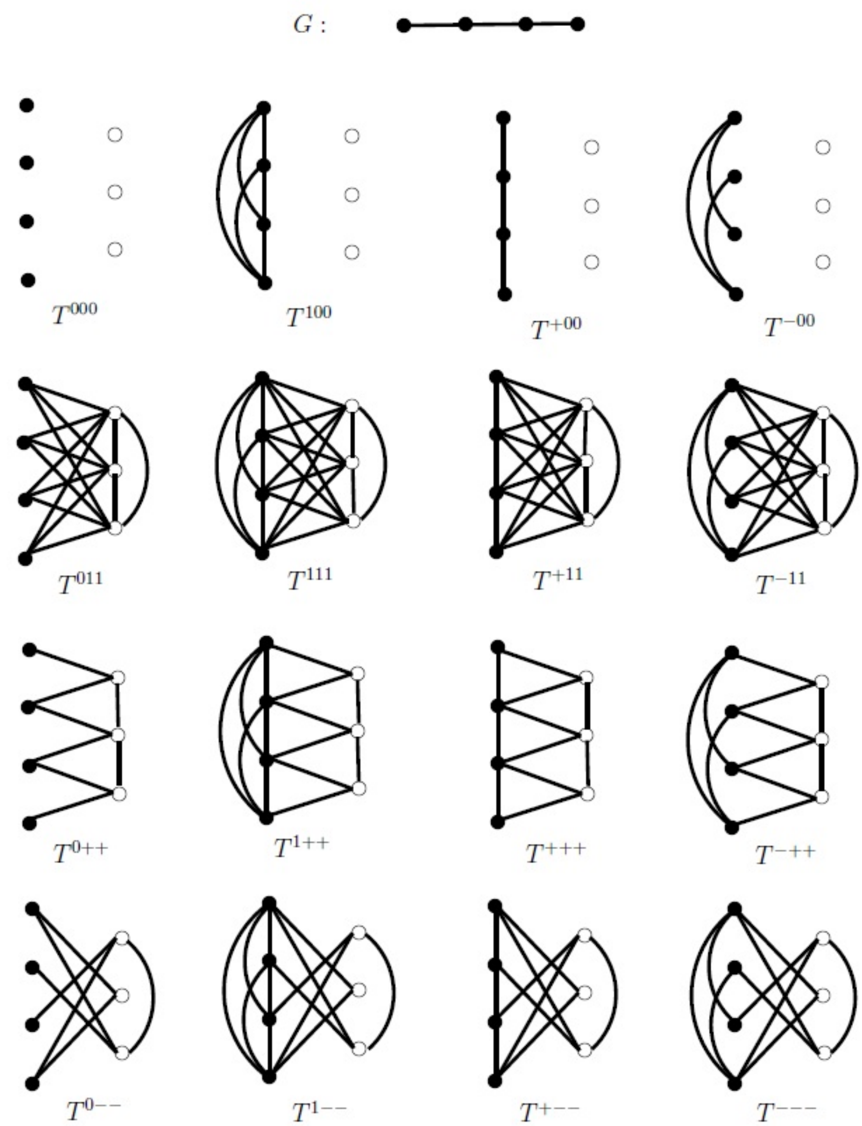
The *line graph* $L(G)$ of a graph G is the graph with vertex set E and two vertices are adjacent in $L(G)$ if and only if the corresponding edges in G are adjacent.

Let $S(G)$ ($S^*(G)$) be the graph with the vertex set $V \cup E$ such that two vertices of $S(G)$ ($S^*(G)$) are adjacent if and only if one corresponds to a vertex v of G and other to an edge e of G and v is incident (resp., not incident) to e in G. Here $S(G)$ is the *subdivision graph* of G and $S^*(G)$ is the *partial complement of subdivision graph.*

For a graph G, let G^0 be the graph with the vertex set $V(G^0) = V$ and with no edges, G^1 the complete graph with $V(G^1) = V$, $G^+ = G$, and $G^- = \overline{G}$.

Definition [7] : Given a graph $G = (V, E)$ and three variables $x, y, z \in \{0, 1, +, -\}$, the *xyz−transformation* $T^{xyz}(G)$(or T^{xyz}) of G is the graph with the vertex set $V(T^{xyz}) = V \cup E$ and the edge set $E(T^{xyz}) = E(G^x) \cup E(L(G)^y) \cup E(W)$, where $W = S(G)$ if $z = +$, $W = S^*(G)$ if $z = -$, W is graph with $V(W) = V \cup E$ and with no edges if $z = 0$, and W is the complete bipartite graph with parts V and E if $z = 1$.

Since there are sixty four distinct 3-permutations of $\{1, 0, +, -\}$, there are sixty four different $xyz-$transformations of a given graph G. Among them 16 graphs with $z = 0$ are always disconnected and 16 graphs with $z = 1$ are always connected. These *xyz−transformation graphs* are also termed as *generalized xyz−Point-Line transformation graphs* [1] in order to avoid confusion with other transformation graphs. In literature, it is interesting to see that T^{00+} is the subdivision graph, T^{+++} is the total graph $T(G)$, T^{---} is the complement of $T(G)$, T^{+0+} is the semitotal-point graph [12], T^{0++} is the semitotal-line graph [12], T^{-++} is the quasi-total graph [2] and T^{1++} is the quasivertex-total graph [11]. The vertex v of T^{xyz} corresponding to a vertex v of G is referred to as *point-vertex* and vertex e of T^{xyz} corresponding to an edge e of G is referred to as *line-vertex*. Some self-explanatory examples of T^{xyz} graphs are depicted in Figure 1, dark circles represents the point-vertices and light circles represents the line-vertices of T^{xyz}. Here we refer [10] for undefined terminology and notation.



The following lemmas are useful to prove our main results.

Lemma 1.1 [5] : Let G be a graph of order n, size m and let v be the point-vertex of T^{xyz} corresponding to a vertex v of G. Then

(i) $$d_{T^{xy0}}(v) = \begin{cases} 0 & if\ x = 0, y \in \{0, 1, +, -\}. \\ n - 1 & if\ x = 1, y \in \{0, 1, +, -\}. \\ d_G(v) & if\ x = +, y \in \{0, 1, +, -\}. \\ n - 1 - d_G(v) & if\ x = -, y \in \{0, 1, +, -\}. \end{cases}$$

(ii) $$d_{T^{xy1}}(v) = \begin{cases} m & if\ x = 0, y \in \{0, 1, +, -\}. \\ n + m - 1 & if\ x = 1, y \in \{0, 1, +, -\}. \\ m + d_G(v) & if\ x = +, y \in \{0, 1, +, -\}. \\ n + m - 1 - d_G(v) & if\ x = -, y \in \{0, 1, +, -\}. \end{cases}$$

(iii) $$d_{T^{xy+}}(v) = \begin{cases} d_G(v) & if\ x = 0, y \in \{0, 1, +, -\}. \\ n - 1 + d_G(v) & if\ x = 1, y \in \{0, 1, +, -\}. \\ 2d_G(v) & if\ x = +, y \in \{0, 1, +, -\}. \\ n - 1 & if\ x = -, y \in \{0, 1, +, -\}. \end{cases}$$

(iv) $$d_{T^{xy-}}(v) = \begin{cases} m - d_G(v) & if\ x = 0, y \in \{0, 1, +, -\}. \\ n + m - 1 - d_G(v) & if\ x = 1, y \in \{0, 1, +, -\}. \\ m & if\ x = +, y \in \{0, 1, +, -\}. \\ n + m - 1 - 2d_G(v) & if\ x = -, y \in \{0, 1, +, -\}. \end{cases}$$

Lemma 1.2 **[5]** : Let G be a graph of order n, size m and let e be the line-vertex of T^{xyz} corresponding to an edge $e = uv$ of G. Then

(i) $$d_{T^{xy0}}(e) = \begin{cases} 0 & if\ y = 0, x \in \{0, 1, +, -\}. \\ m - 1 & if\ y = 1, x \in \{0, 1, +, -\}. \\ d_G(u) + d_G(v) - 2 & if\ y = +, x \in \{0, 1, +, -\}. \\ m + 1 - d_G(u) - d_G(v) & if\ y = -, x \in \{0, 1, +, -\}. \end{cases}$$

(ii) $$d_{T^{xy1}}(e) = \begin{cases} n & if\ y = 0, x \in \{0, 1, +, -\}. \\ n + m - 1 & if\ y = 1, x \in \{0, 1, +, -\}. \\ n - 2 + d_G(u) + d_G(v) & if\ y = +, x \in \{0, 1, +, -\}. \\ n + m + 1 - d_G(u) - d_G(v) & if\ y = -, x \in \{0, 1, +, -\}. \end{cases}$$

(iii) $$d_{T^{xy+}}(e) = \begin{cases} 2 & if\ y = 0, x \in \{0, 1, +, -\}. \\ m + 1 & if\ y = 1, x \in \{0, 1, +, -\}. \\ d_G(u) + d_G(v) & if\ y = +, x \in \{0, 1, +, -\}. \\ m + 3 - d_G(u) - d_G(v) & if\ y = -, x \in \{0, 1, +, -\}. \end{cases}$$

(iv) $$d_{T^{xy-}}(e) = \begin{cases} n - 2 & if\ y = 0, x \in \{0, 1, +, -\}. \\ n + m - 3 & if\ y = 1, x \in \{0, 1, +, -\}. \\ n - 4 + d_G(u) + d_G(v) & if\ y = +, x \in \{0, 1, +, -\}. \\ n + m - 1 - d_G(u) - d_G(v) & if\ y = -, x \in \{0, 1, +, -\}. \end{cases}$$

The *first and second Zagreb indices* are defined as [8].

$$M_1 = M_1(G) = \sum_{uv \in E(G)} [d_G(u) + d_G(v)] \text{ and } M_2 = M_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v).$$

The *sum-connectivity index* of a graph G is defined as [14]

$$\chi(G) = \sum_{uv \in E(G)} [d_G(u) + d_G(v)]^{-\frac{1}{2}}.$$

It has been extended to the *general sum-connectivity index* defined as [15]

$$\chi_\alpha(G) = \sum_{uv \in E(G)} [d_G(u) + d_G(v)]^\alpha, \text{ where } \alpha \text{ is any real number.}$$

The Zagreb indices of semitotal-point graph, semitotal-line graph and generalized $xyz-$ Point-Line transformation graphs can be found in [3, 4, 5, 9]. The bounds for general sum-connectivity index of semitotal-point graph, semitotal-line graph, and total transformation graphs obtained in [13]. However we gave equality expressions for general sum-connectivity index of semitotal-point graph, semitotal-line graph and total graph and corrected some errors of [13] in [6]. This motivated us to compute expressions for general sum connectivity index of generalized $xyz-$ Point-Line transformation graphs.

2. General Sum Connectivity Index of T^{xyz} Graphs

Theorem 2.1 : Let G be a graph of order n and size m. Then

1. $\chi_\alpha(T^{000}) = 0$
2. $\chi_\alpha(T^{100}) = 2^{\alpha-1} n(n-1)^{\alpha+1}$
3. $\chi_\alpha(T^{+00}) = \chi_\alpha(G)$
4. $\chi_\alpha(T^{-00}) = \sum_{u \nsim v;\, u,v \in V(G)} [2(n-1) - d_G(u) - d_G(v)]^\alpha$
5. $\chi_\alpha(T^{010}) = 2^{\alpha-1} m(m-1)^{\alpha+1}$
6. $\chi_\alpha(T^{110}) = 2^{\alpha-1} \left[n(n-1)^{\alpha+1} + m(m-1)^{\alpha+1}\right]$
7. $\chi_\alpha(T^{+10}) = \chi_\alpha(G) + 2^{\alpha-1} m(m-1)^{\alpha+1}$
8. $\chi_\alpha(T^{-10}) = 2^{\alpha-1} m(m-1)^{\alpha+1} + \sum_{u \nsim v;\, u,v \in V(G)} [2(n-1) - d_G(u) - d_G(v)]^\alpha$
9. $\chi_\alpha(T^{001}) = nm(n+m)^\alpha$
10. $\chi_\alpha(T^{101}) = \binom{n}{2} [2(n+m-1)]^\alpha + nm\,(2n+m-1)^\alpha$
11. $\chi_\alpha(T^{011}) = \binom{m}{2} [2(n+m-1)]^\alpha + nm\,(n+2m-1)^\alpha$

12. $\chi_\alpha(T^{111}) = [2(n+m-1)]^\alpha \left[\binom{n}{2} + \binom{m}{2} + nm\right]$

Proof : 1. $\chi_\alpha(T^{000}) = \sum_{xy\in E(T^{000})} [d_{T^{000}}(x) + d_{T^{000}}(y)]^\alpha = \sum_{xy\in E(T^{000})} [0+0]^\alpha = 0.$

2. $\chi_\alpha(T^{100}) = \sum_{xy\in E(T^{100})} [d_{T^{100}}(x) + d_{T^{100}}(y)]^\alpha$

$$= \sum_{uv\in E(T^{100})\cap E(K_n)} [d_{T^{100}}(u) + d_{T^{100}}(v)]^\alpha$$

$$= \sum_{uv\in E(K_n)} [(n-1) + (n-1)]^\alpha$$

$$= [2(n-1)]^\alpha \binom{n}{2}$$

$$\chi_\alpha(T^{100}) = 2^{\alpha-1} n(n-1)^{\alpha+1} = \chi_\alpha(K_n).$$

3. $\chi_\alpha(T^{+00}) = \sum_{uv\in E(T^{+00})} [d_{T^{+00}}(u) + d_{T^{+00}}(v)]^\alpha = \sum_{uv\in E(G)} [d_G(u) + d_G(v)]^\alpha = \chi_\alpha(G).$

4. $\chi_\alpha(T^{-00}) = \sum_{uv\in E(T^{-00})\cap E(\overline{G})} [d_{T^{-00}}(u) + d_{T^{-00}}(v)]^\alpha$

$$= \sum_{uv\in E(\overline{G})} [n-1-d_G(u) + n-1-d_G(v)]^\alpha$$

$$\chi_\alpha(T^{-00}) = \sum_{u\nsim v;\ u,v\in V(G)} [2(n-1) - d_G(u) - d_G(v)]^\alpha = \chi_\alpha(\overline{G}).$$

5. $\chi_\alpha(T^{010}) = \sum_{e_ie_j\in E(T^{010})\cap E(K_m)} [d_{T^{010}}(e_i) + d_{T^{010}}(e_j)]^\alpha$

$$= \sum_{e_ie_j\in E(K_m)} [(m-1) + (m-1)]^\alpha$$

$$= [2(m-1)]^\alpha \binom{m}{2}$$

$$\chi_\alpha(T^{010}) = 2^{\alpha-1} m(m-1)^{\alpha+1} = \chi_\alpha(K_m).$$

6. Since $T^{110} \cong K_n \cup K_m$, we have $\chi_\alpha(T^{110}) = \chi_\alpha(K_n) + \chi_\alpha(K_m)$.

7. Since $T^{+10} \cong G \cup K_m$, we have $\chi_\alpha(T^{+10}) = \chi_\alpha(G) + \chi_\alpha(K_m)$.

8. Since $T^{-10} \cong \overline{G} \cup K_m$, we have $\chi_\alpha(T^{-10}) = \chi_\alpha(\overline{G}) + \chi_\alpha(K_m)$.

9. $\chi_\alpha(T^{001}) = \sum_{ue\in E(T^{001})\cap E(K_{n,m})} [d_{T^{001}}(u) + d_{T^{001}}(e)]^\alpha = \sum_{ue\in E(K_{n,m})} [m+n]^\alpha = mn(n+m)^\alpha.$

10. $\chi_\alpha(T^{101}) = \sum\limits_{uv\in E(K_n)} [(n+m-1)+(n+m-1)]^\alpha + \sum\limits_{ue\in E(K_{n,m})} [(n+m-1)+n]^\alpha.$

11. $\chi_\alpha(T^{011}) = \sum\limits_{e_ie_j\in E(K_m)} [(n+m-1)+(n+m-1)]^\alpha + \sum\limits_{ue\in E(K_{n,m})} [m+(n+m-1)]^\alpha.$

12. $\chi_\alpha(T^{111}) = \sum\limits_{uv\in E(K_n)} [2(n+m-1)]^\alpha + \sum\limits_{e_ie_j\in E(K_m)} [2(n+m-1)]^\alpha$

$$+ \sum_{ue\in E(K_{n,m})} [2(n+m-1)]^\alpha. \qquad \square$$

Theorem 2.2 : Let G be a $(n,\, m)$ graph and $\alpha < 0$. Then

1. $4^\alpha(\delta-1)^\alpha\left[\frac{1}{2}M_1 - m\right] \geq \chi_\alpha(T^{0+0}) \geq 4^\alpha(\Delta-1)^\alpha\left[\frac{1}{2}M_1 - m\right]$

2. $\chi_\alpha(T^{1+0}) \leq 2^{\alpha-1}n(n-1)^{\alpha+1} + 4^\alpha(\delta-1)^\alpha\left[\frac{1}{2}M_1 - m\right]$
   $\chi_\alpha(T^{1+0}) \geq 2^{\alpha-1}n(n-1)^{\alpha+1} + 4^\alpha(\Delta-1)^\alpha\left[\frac{1}{2}M_1 - m\right]$

3. $\chi_\alpha(T^{++0}) \leq \chi_\alpha(G) + 4^\alpha(\delta-1)^\alpha\left[\frac{1}{2}M_1 - m\right]$
   $\chi_\alpha(T^{++0}) \geq \chi_\alpha(G) + 4^\alpha(\Delta-1)^\alpha\left[\frac{1}{2}M_1 - m\right]$

4. $\chi_\alpha(T^{-+0}) \leq 2^\alpha(n-1-\delta)^\alpha\left[\binom{n}{2} - m\right] + 4^\alpha(\delta-1)^\alpha\left[\frac{1}{2}M_1 - m\right]$
   $\chi_\alpha(T^{-+0}) \geq 2^\alpha(n-1-\Delta)^\alpha\left[\binom{n}{2} - m\right] + 4^\alpha(\Delta-1)^\alpha\left[\frac{1}{2}M_1 - m\right];$
   the equalities hold if and only if G is a regular graph.

Proof : 1. $\chi_\alpha(T^{0+0}) = \sum\limits_{e_ie_j\in E(T^{0+0})\cap E(L(G))} [d_{T^{0+0}}(e_i) + d_{T^{0+0}}(e_j)]^\alpha$

$$= \sum_{e_ie_j\in E(L(G)),e_i=uv,e_j=vw} [d_G(u)+d_G(v)-2+d_G(v)+d_G(w)-2]^\alpha$$

Note that $d_G(u) \leq \Delta$ and $d_G(u) \geq \delta$ for any vertex $u \in V(G)$. The equalities hold if and only if G is a regular graph. Also given that $\alpha < 0$.

Thus $\chi_\alpha(T^{0+0}) \leq \sum\limits_{e_i\sim e_j} [4\delta - 4]^\alpha$

$$\chi_\alpha(T^{0+0}) \leq 4^\alpha(\delta-1)^\alpha\left[\frac{1}{2}M_1 - m\right],$$

Similarly, we can show that $\chi_\alpha(T^{0+0}) \geq 4^\alpha(\Delta-1)^\alpha\left[\frac{1}{2}M_1 - m\right]$.

Also note that $\chi_\alpha(T^{0+0}) = \chi_\alpha(L(G))$ but $T^{0+0} \ncong L(G)$.

2. Since $T^{1+0} \cong K_n \cup L(G)$, we have $\chi_\alpha(T^{1+0}) = \chi_\alpha(K_n) + \chi_\alpha(L(G))$.
3. Since $T^{++0} \cong G \cup L(G)$, we have $\chi_\alpha(T^{++0}) = \chi_\alpha(G) + \chi_\alpha(L(G))$.
4. Since $T^{-+0} \cong \overline{G} \cup L(G)$, we have $\chi_\alpha(T^{-+0}) = \chi_\alpha(\overline{G}) + \chi_\alpha(L(G))$. □

Theorem 2.3 : Let G be a (n, m) graph and $\alpha < 0$. Then

1. $\chi_\alpha(T^{0-0}) \leq 2^\alpha(m+1-2\delta)^\alpha \left[\binom{m}{2} + m - \frac{1}{2}M_1\right]$
   $\chi_\alpha(T^{0-0}) \geq 2^\alpha(m+1-2\Delta)^\alpha \left[\binom{m}{2} + m - \frac{1}{2}M_1\right]$
2. $\chi_\alpha(T^{1-0}) \leq 2^{\alpha-1}n(n-1)^{\alpha+1} + 2^\alpha(m+1-2\delta)^\alpha \left[\binom{m}{2} + m - \frac{1}{2}M_1\right]$
   $\chi_\alpha(T^{1-0}) \geq 2^{\alpha-1}n(n-1)^{\alpha+1} + 2^\alpha(m+1-2\Delta)^\alpha \left[\binom{m}{2} + m - \frac{1}{2}M_1\right]$
3. $\chi_\alpha(T^{+-0}) \leq \chi_\alpha(G) + 2^\alpha(m+1-2\delta)^\alpha \left[\binom{m}{2} + m - \frac{1}{2}M_1\right]$
   $\chi_\alpha(T^{+-0}) \geq \chi_\alpha(G) + 2^\alpha(m+1-2\Delta)^\alpha \left[\binom{m}{2} + m - \frac{1}{2}M_1\right]$
4. $\chi_\alpha(T^{--0}) \leq \sum_{u \nsim v;\, u,v \in V(G)} [2(n-1) - d_G(u) - d_G(v)]^\alpha + 2^\alpha(m+1-2\delta)^\alpha \left[\binom{m}{2} + m - \frac{1}{2}M_1\right]$
   $\chi_\alpha(T^{--0}) \geq \sum_{u \nsim v;\, u,v \in V(G)} [2(n-1) - d_G(u) - d_G(v)]^\alpha + 2^\alpha(m+1-2\Delta)^\alpha \left[\binom{m}{2} + m - \frac{1}{2}M_1\right]$;
   the equalities hold if and only if G is a regular graph.

Proof : 1. $\chi_\alpha(T^{0-0}) = \sum_{e_ie_j \in E(T^{0-0}) \cap E(\overline{L(G)})} [d_{T^{0-0}}(e_i) + d_{T^{0-0}}(e_j)]^\alpha$

$$= \sum_{e_i \nsim e_j, e_i = uv, e_j = wx} [m+1-d_G(u)-d_G(v)+m+1-d_G(w)-d_G(x)]^\alpha$$

Note that $d_G(u) \leq \Delta$ and $d_G(u) \geq \delta$ for any vertex $u \in V(G)$. The equalities hold if and only if G is a regular graph. Also given that $\alpha < 0$.

Thus $\chi_\alpha(T^{0-0}) \geq \sum_{e_i \nsim e_j} [2(m+1) - 4\Delta]^\alpha = 2^\alpha(m+1-2\Delta)^\alpha \left[\binom{m}{2} + m - \frac{1}{2}M_1\right]$.

Similarly, we can compute $\chi_\alpha(T^{0-0}) \leq 2^\alpha[m+1-2\delta]^\alpha \left[\binom{m}{2} + m - \frac{1}{2}M_1\right]$.

Also note that $\chi_\alpha(T^{0-0}) = \chi_\alpha(\overline{L(G)})$ but $T^{0-0} \ncong \overline{L(G)}$

2. Since $T^{1-0} \cong K_n \cup \overline{L(G)}$, we have $\chi_\alpha(T^{1-0}) = \chi_\alpha(K_n) + \chi_\alpha(\overline{L(G)})$.
3. Since $T^{+-0} \cong G \cup \overline{L(G)}$, we have $\chi_\alpha(T^{+-0}) = \chi_\alpha(G) + \chi_\alpha(\overline{L(G)})$.
4. Since $T^{--0} \cong \overline{G} \cup \overline{L(G)}$, we have $\chi_\alpha(T^{--0}) = \chi_\alpha(\overline{G}) + \chi_\alpha(\overline{L(G)})$. □

Theorem 2.4 : Let G be a graph of order n and size m. Then

1. $\chi_\alpha(T^{+01}) = \sum_{uv \in E(G)} [2m + d_G(u) + d_G(v)]^\alpha + m \sum_{u \in V(G)} [n + m + d_G(u)]^\alpha$
2. $\chi_\alpha(T^{-01}) = \sum_{u \nsim v;\, u,v \in V(G)} [2(n+m-1) - d_G(u) - d_G(v)]^\alpha + m \sum_{u \in V(G)} [2n + m - 1 - d_G(u)]^\alpha$
3. $\chi_\alpha(T^{+11}) = \sum_{uv \in E(G)} [2m + d_G(u) + d_G(v)]^\alpha + m \sum_{u \in V(G)} [n + 2m - 1 + d_G(u)]^\alpha + \binom{m}{2}[2(n+m-1)]^\alpha$
4. $\chi_\alpha(T^{-11}) = \sum_{u \nsim v;\, u,v \in V(G)} [2(n+m-1) - d_G(u) - d_G(v)]^\alpha + \binom{m}{2}[2(n+m-1)]^\alpha + m \sum_{u \in V(G)} [2(n+m-1) - d_G(u)]^\alpha$.

Proof : 1.
$$\chi_\alpha(T^{+01}) = \sum_{uv \in E(G)} [m + d_G(u) + m + d_G(v)]^\alpha + \sum_{ue \in E(K_{n,m})} [m + d_G(u) + n]^\alpha$$
$$= \sum_{uv \in E(G)} [2m + d_G(u) + d_G(v)]^\alpha + \sum_{u \in V(G)} m[n + m + d_G(u)]^\alpha.$$

2.
$$\chi_\alpha(T^{-01}) = \sum_{uv \in E(\overline{G})} [n + m - 1 - d_G(u) + n + m - 1 - d_G(v)]^\alpha$$
$$+ \sum_{ue \in E(K_{n,m})} [n + m - 1 - d_G(u) + n]^\alpha$$
$$= \sum_{u \nsim v;\, u,v \in V(G)} [2(n+m-1) - d_G(u) - d_G(v)]^\alpha + \sum_{u \in V(G)} m[2n + m - 1 - d_G(u)]^\alpha.$$

3.
$$\chi_\alpha(T^{+11}) = \sum_{uv \in E(G)} [m + d_G(u) + m + d_G(v)]^\alpha + \sum_{ue \in E(K_{n,m})} [m + d_G(u) + n + m - 1]^\alpha$$
$$+ \sum_{e_i e_j \in E(K_m)} [n + m - 1 + n + m - 1]^\alpha$$
$$= \sum_{uv \in E(G)} [2m + d_G(u) + d_G(v)]^\alpha + \sum_{u \in V(G)} m[n + 2m - 1 + d_G(u)]^\alpha$$

$$+\binom{m}{2}[2(n+m-1)]^{\alpha}.$$

4. $\chi_\alpha(T^{-11}) = \sum\limits_{uv\in E(\overline{G})}[2(n+m-1)-d_G(u)-d_G(v)]^{\alpha} + \sum\limits_{ue\in E(K_{n,m})}[2(n+m-1)-d_G(u)]^{\alpha}$

$$+\sum_{e_ie_j\in E(K_m)}[n+m-1+n+m-1]^{\alpha}$$

$$=\sum_{u\nsim v;\ u,v\in V(G)}[2(n+m-1)-d_G(u)-d_G(v)]^{\alpha}$$

$$+m\sum_{u\in V(G)}[2(n+m-1)-d_G(u)]^{\alpha}+\binom{m}{2}[2(n+m-1)]^{\alpha}.$$ □

Theorem 2.5 : Let G be a graph of order n and size m. Then

1. $\chi_\alpha(T^{00+}) = \sum\limits_{u\in V(G)} d_G(u)[d_G(u)+2]^{\alpha}$

2. $\chi_\alpha(T^{10+}) = \left(\sum\limits_{u\sim v}+\sum\limits_{u\nsim v}\right)[2(n-1)+d_G(u)+d_G(v)]^{\alpha} + \sum\limits_{u\in V(G)} d_G(u)[n+1+d_G(u)]^{\alpha}$

3. $\chi_\alpha(T^{-0+}) = [2(n-1)]^{\alpha}\left[\binom{n}{2}-m\right]+2m(n+1)^{\alpha}$

4. $\chi_\alpha(T^{01+}) = [2(m+1)]^{\alpha}\binom{m}{2}+\sum\limits_{u\in V(G)} d_G(u)[m+1+d_G(u)]^{\alpha}$

5. $\chi_\alpha(T^{11+}) = \left(\sum\limits_{u\sim v}+\sum\limits_{u\nsim v}\right)[2(n-1)+d_G(u)+d_G(v)]^{\alpha}+[2(m+1)]^{\alpha}\binom{m}{2}$
   $+\sum\limits_{u\in V(G)} d_G(u)[n+m+d_G(u)]^{\alpha}$

6. $\chi_\alpha(T^{+1+}) = 2^{\alpha}\chi_\alpha(G)+[2(m+1)]^{\alpha}\binom{m}{2}+\sum\limits_{u\in V(G)} d_G(u)[m+1+2d_G(u)]^{\alpha}$

7. $\chi_\alpha(T^{-1+}) = [2(n-1)]^{\alpha}\left[\binom{n}{2}-m\right]+[2(m+1)]^{\alpha}\binom{m}{2}+2m(n+m)^{\alpha}$

8. $\chi_\alpha(T^{1++}) = \left(\sum\limits_{u\sim v}+\sum\limits_{u\nsim v}\right)[2(n-1)+d_G(u)+d_G(v)]^{\alpha}+\sum\limits_{uvw\in E_2(G)}[d_G(u)+2d_G(v)+$
   $d_G(w)]^{\alpha}+\sum\limits_{uv\in E(G)}\{[n-1+2d_G(u)+d_G(v)]^{\alpha}+[n-1+d_G(u)+2d_G(v)]^{\alpha}\},$
   where $E_2(G)$ is the set of all pairs of adjacent edges in G.

Proof : 1. $\chi_\alpha(T^{00+}) = \sum_{ue\in E(T^{00+})\cap E(S(G))} [d_{T^{00+}}(u) + d_{T^{00+}}(e)]^\alpha$

$$= \sum_{ue\in E(S(G))} [d_G(u) + 2]^\alpha$$

$$\chi_\alpha(T^{00+}) = \sum_{u\in V(G)} d_G(u)[d_G(u) + 2]^\alpha.$$

2. $\chi_\alpha(T^{10+}) = \sum_{uv\in E(K_n)} [2(n-1) + d_G(u) + d_G(v)]^\alpha + \sum_{ue\in E(S(G))} [n-1+d_G(u)+2]^\alpha$

$$= \left(\sum_{u\sim v} + \sum_{u\nsim v}\right) [2(n-1) + d_G(u) + d_G(v)]^\alpha + \sum_{u\in V(G)} d_G(u)[n+1+d_G(u)]^\alpha.$$

3. $\chi_\alpha(T^{-0+}) = \sum_{uv\in E(\overline{G})} [2(n-1)]^\alpha + \sum_{ue\in E(S(G))} [(n-1)+2]^\alpha$

$$= [2(n-1)]^\alpha \left[\binom{n}{2} - m\right] + 2m(n+1)^\alpha.$$

4. $\chi_\alpha(T^{01+}) = \sum_{e_ie_j\in E(K_m)} [(m+1)+(m+1)]^\alpha + \sum_{ue\in E(S(G))} [d_G(u)+m+1]^\alpha$

$$= [2(m+1)]^\alpha \binom{m}{2} + \sum_{u\in V(G)} d_G(u)[m+1+d_G(u)]^\alpha.$$

5. $\chi_\alpha(T^{11+}) = \sum_{uv\in E(K_n)} [(n-1+d_G(u))+(n-1+d_G(v))]^\alpha + \sum_{e_ie_j\in E(K_m)} [(m+1)+(m+1)]^\alpha$

$$+ \sum_{u\sim e} [n-1+d_G(u)+(m+1)]^\alpha$$

$$\chi_\alpha(T^{11+}) = \left(\sum_{u\sim v} + \sum_{u\nsim v}\right) [2(n-1) + d_G(u) + d_G(v)]^\alpha + [2(m+1)]^\alpha \binom{m}{2}$$

$$+ \sum_{u\in V(G)} d_G(u)[n+m+d_G(u)]^\alpha.$$

6. $\chi_\alpha(T^{+1+}) = \sum_{uv\in E(G)} [2d_G(u) + 2d_G(v)]^\alpha + \sum_{e_ie_j\in E(K_m)} [(m+1)+(m+1)]^\alpha$
$+ \sum_{u\sim e} [2d_G(u) + (m+1)]^\alpha$

$$= 2^\alpha \chi_\alpha(G) + [2(m+1)]^\alpha \binom{m}{2} + \sum_{u\in V(G)} d_G(u)[m+1+2d_G(u)]^\alpha.$$

7. $\chi_\alpha(T^{-1+}) = \sum_{uv\in E(\overline{G})} [2(n-1)]^\alpha + \sum_{e_ie_j\in E(K_m)} [(m+1)+(m+1)]^\alpha$

$$+\sum_{ue\in E(S(G))}[(n-1)+(m+1)]^{\alpha}$$

$$=[2(n-1)]^{\alpha}\left[\binom{n}{2}-m\right]+[2(m+1)]^{\alpha}\binom{m}{2}+2m(n+m)^{\alpha}.$$

8. $\chi_{\alpha}(T^{1++})=\sum_{uv\in E(K_n)}[2(n-1)+d_G(u)+d_G(v)]^{\alpha}$

$$+\sum_{e_i\sim e_j,e_i=uv,e_j=vw}[d_G(u)+d_G(v)+d_G(v)+d_G(w)]^{\alpha}$$

$$+\sum_{u\sim e,e=uv}[n-1+d_G(u)+d_G(u)+d_G(v)]^{\alpha}.$$ □

Theorem 2.6 : Let G be a graph of order n and size m. Then

1. $\chi_{\alpha}(T^{00-})=\sum_{u\in V(G)}[m-d_G(u)]\,[n+m-2-d_G(u)]^{\alpha}$
2. $\chi_{\alpha}(T^{10-})=\left(\sum_{u\sim v}+\sum_{u\nsim v}\right)[2(n+m-1)-d_G(u)-d_G(v)]^{\alpha}+\sum_{u\in V(G)}[m-d_G(u)]\,[2n+m-3-d_G(u)]^{\alpha}$
3. $\chi_{\alpha}(T^{+0-})=m\cdot(2m)^{\alpha}+m(n-2)[n+m-2]^{\alpha}$
4. $\chi_{\alpha}(T^{-0-})=2^{\alpha}\sum_{u\nsim v}[n+m-1-d_G(u)-d_G(v)]^{\alpha}+\sum_{u\in V(G)}[m-d_G(u)]\,[2n+m-3-2d_G(u)]^{\alpha}$
5. $\chi_{\alpha}(T^{01-})=[2(n+m-3)]^{\alpha}\binom{m}{2}+\sum_{u\in V(G)}[m-d_G(u)]\,[n+2m-3-d_G(u)]^{\alpha}$
6. $\chi_{\alpha}(T^{11-})=\left(\sum_{u\sim v}+\sum_{u\nsim v}\right)[2(n+m-1)-d_G(u)-d_G(v)]^{\alpha}+[2(n+m-3)]^{\alpha}\binom{m}{2}+\sum_{u\in V(G)}[m-d_G(u)]\,[2(n+m-2)-d_G(u)]^{\alpha}$
7. $\chi_{\alpha}(T^{+1-})=2^{\alpha}\cdot m^{\alpha+1}+[2(n+m-3)]^{\alpha}\binom{m}{2}+m(n-2)[n+2m-3]^{\alpha}$
8. $\chi_{\alpha}(T^{-1-})=2^{\alpha}\sum_{u\nsim v;\,u,v\in V(G)}[n+m-1-d_G(u)-d_G(v)]^{\alpha}+[2(n+m-3)]^{\alpha}\binom{m}{2}+2^{\alpha}\sum_{u\in V(G)}[m-d_G(u)]\,[n+m-2-d_G(u)]^{\alpha}$

Proof: 1. $\chi_{\alpha}(T^{00-})=\sum_{ue\in E(T^{00-})\cap E(S^*(G))}[d_{T^{00-}}(u)+d_{T^{00-}}(e)]^{\alpha}$

$$=\sum_{u\nsim e}[m-d_G(u)+n-2]^{\alpha}$$

$$\chi_\alpha(T^{00-}) = \sum_{u\in V(G)} [m - d_G(u)][n+m-2-d_G(u)]^\alpha.$$

2. $$\chi_\alpha(T^{10-}) = \sum_{uv\in E(K_n)} [(n+m-1-d_G(u)) + (n+m-1-d_G(v))]^\alpha + \sum_{u\nsim e} [n+m-1-d_G(u)+(n-2)]^\alpha$$
$$= \left(\sum_{u\sim v} + \sum_{u\nsim v}\right) [2(n+m-1) - d_G(u) - d_G(v)]^\alpha + \sum_{u\in V(G)} [m-d_G(u)][2n+m-3-d_G(u)]^\alpha.$$

3. $$\chi_\alpha(T^{+0-}) = \sum_{uv\in E(G)} [m+m]^\alpha + \sum_{u\nsim e} [m+(n-2)]^\alpha$$
$$= m\cdot(2m)^\alpha + m(n-2)[n+m-2]^\alpha.$$

4. $$\chi_\alpha(T^{-0-}) = \sum_{uv\in E(\overline{G})} [(n+m-1-2d_G(u)) + (n+m-1-2d_G(v))]^\alpha + \sum_{u\nsim e} [n+m-1-2d_G(u)+(n-2)]^\alpha$$
$$= 2^\alpha \sum_{u\nsim v} [n+m-1-d_G(u)-d_G(v)]^\alpha + \sum_{u\in V(G)} [m-d_G(u)]\ [2n+m-3-2d_G(u)]^\alpha.$$

5. $$\chi_\alpha(T^{01-}) = \sum_{e_ie_j\in E(K_m)} [(n+m-3)+(n+m-3)]^\alpha + \sum_{u\nsim e} [m-d_G(u)+(n+m-3)]^\alpha$$
$$= [2(n+m-3)]^\alpha \binom{m}{2} + \sum_{u\in V(G)} [m-d_G(u)]\ [n+2m-3-d_G(u)]^\alpha.$$

6. $$\chi_\alpha(T^{11-}) = \sum_{uv\in E(K_n)} [(n+m-1-d_G(u)) + (n+m-1-d_G(v))]^\alpha + \sum_{e_ie_j\in E(K_m)} [(n+m-3)+(n+m-3)]^\alpha + \sum_{u\nsim e} [n+m-1-d_G(u)+(n+m-3)]^\alpha$$
$$= \left(\sum_{u\sim v} + \sum_{u\nsim v}\right) [2(n+m-1) - d_G(u) - d_G(v)]^\alpha + [2(n+m-3)]^\alpha \binom{m}{2} + \sum_{u\in V(G)} [m-d_G(u)]\ [2(n+m-2)-d_G(u)]^\alpha.$$

7. $$\chi_\alpha(T^{+1-}) = \sum_{uv\in E(G)} [m+m]^\alpha + \sum_{e_ie_j\in E(K_m)} [(n+m-3)+(n+m-3)]^\alpha$$

$$+\sum_{u\nsim e}[m+(n+m-3)]^{\alpha}$$

$$\chi_{\alpha}(T^{+1-}) = 2^{\alpha}\cdot m^{\alpha+1} + [2(n+m-3)]^{\alpha}\binom{m}{2} + m(n-2)[n+2m-3]^{\alpha}.$$

8. $\chi_{\alpha}(T^{-1-}) = \sum_{uv\in E(\overline{G})}[(n+m-1-2d_G(u))+(n+m-1-2d_G(v))]^{\alpha}$

$$+\sum_{e_ie_j\in E(K_m)}[(n+m-3)+(n+m-3)]^{\alpha} + \sum_{u\nsim e}[n+m-1-2d_G(u)+(n+m-3)]^{\alpha}$$

$$= 2^{\alpha}\sum_{u\nsim v;\ u,v\in V(G)}[n+m-1-d_G(u)-d_G(v)]^{\alpha} + [2(n+m-3)]^{\alpha}\binom{m}{2}$$
$$+2^{\alpha}\sum_{u\in V(G)}[m-d_G(u)]\,[n+m-2-d_G(u)]^{\alpha}.$$ □

Theorem 2.7 : Let G be a $(n,\, m)$ graph and $\alpha < 0$. Then

1. $\chi_{\alpha}(T^{0+1}) \le 2^{\alpha}(n-2+2\delta)^{\alpha}\left[\frac{1}{2}M_1 - m\right] + n\sum_{uv\in E(G)}[n+m-2+d_G(u)+d_G(v)]^{\alpha}$
   $\chi_{\alpha}(T^{0+1}) \ge 2^{\alpha}(n-2+2\Delta)^{\alpha}\left[\frac{1}{2}M_1 - m\right] + n\sum_{uv\in E(G)}[n+m-2+d_G(u)+d_G(v)]^{\alpha}$

2. $\chi_{\alpha}(T^{1+1}) \le \binom{n}{2}[2(n+m-1)]^{\alpha} + 2^{\alpha}(n-2+2\delta)^{\alpha}\left[\frac{1}{2}M_1 - m\right]$
   $+\, n\sum_{uv\in E(G)}[2n+m-3+d_G(u)+d_G(v)]^{\alpha}$
   $\chi_{\alpha}(T^{1+1}) \ge \binom{n}{2}[2(n+m-1)]^{\alpha} + 2^{\alpha}(n-2+2\Delta)^{\alpha}\left[\frac{1}{2}M_1 - m\right]$
   $+\, n\sum_{uv\in E(G)}[2n+m-3+d_G(u)+d_G(v)]^{\alpha}$

3. $\chi_{\alpha}(T^{++1}) \le \sum_{uv\in E(G)}[2m+d_G(u)+d_G(v)]^{\alpha} + 2^{\alpha}(n-2+2\delta)^{\alpha}\left[\frac{1}{2}M_1 - m\right] + nm\,[n+m-2+3\delta]^{\alpha}$
   $\chi_{\alpha}(T^{++1}) \ge \sum_{uv\in E(G)}[2m+d_G(u)+d_G(v)]^{\alpha} + 2^{\alpha}(n-2+2\Delta)^{\alpha}\left[\frac{1}{2}M_1 - m\right] + nm\,[n+m-2+3\Delta]^{\alpha}$

4. $\chi_{\alpha}(T^{-+1}) \le 2^{\alpha}(n+m-1-\delta)^{\alpha}\left[\binom{n}{2} - m\right] + 2^{\alpha}(n-2+2\delta)^{\alpha}\left[\frac{1}{2}M_1 - m\right]$
   $+\, mn(2n+m-3+\delta)^{\alpha}$
   $\chi_{\alpha}(T^{-+1}) \ge 2^{\alpha}(n+m-1-\Delta)^{\alpha}\left[\binom{n}{2} - m\right] + 2^{\alpha}(n-2+2\Delta)^{\alpha}\left[\frac{1}{2}M_1 - m\right]$
   $+\, mn(2n+m-3+\Delta)^{\alpha}$

5. $\chi_{\alpha}(T^{0-1}) \le 2^{\alpha}[n+m+1-2\delta]^{\alpha}\left[\binom{m}{2} + m - \frac{1}{2}M_1\right] + n\sum_{uv\in E(G)}[n+2m+1-d_G(u)-d_G(v)]^{\alpha}$

$\chi_\alpha(T^{0-1}) \geq 2^\alpha[n+m+1-2\Delta]^\alpha \left[\binom{m}{2}+m-\frac{1}{2}M_1\right]+n\sum_{uv\in E(G)}[n+2m+1-d_G(u)-d_G(v)]^\alpha$

6. $\chi_\alpha(T^{1-1}) \leq [2(n+m-1)]^\alpha\binom{n}{2}+2^\alpha(n+m-1-2\delta)^\alpha\left[\binom{m}{2}+m-\frac{1}{2}M_1\right] + n\sum_{uv\in E(G)}[2(n+m)-d_G(u)-d_G(v)]^\alpha$

   $\chi_\alpha(T^{1-1}) \geq [2(n+m-1)]^\alpha\binom{n}{2}+2^\alpha(n+m-1-2\Delta)^\alpha\left[\binom{m}{2}+m-\frac{1}{2}M_1\right] + n\sum_{uv\in E(G)}[2(n+m)-d_G(u)-d_G(v)]^\alpha$

7. $\chi_\alpha(T^{+-1}) \leq \sum_{uv\in E(G)}[2m+d_G(u)+d_G(v)]^\alpha+2^\alpha[n+m-1-2\delta]^\alpha\left[\binom{m}{2}+m-\frac{1}{2}M_1\right] + mn[n+2m+1-\delta]^\alpha$

   $\chi_\alpha(T^{+-1}) \geq \sum_{uv\in E(G)}[2m+d_G(u)+d_G(v)]^\alpha+2^\alpha[n+m-1-2\Delta]^\alpha\left[\binom{m}{2}+m-\frac{1}{2}M_1\right] + mn[n+2m+1-\Delta]^\alpha$

8. $\chi_\alpha(T^{--1}) \leq \sum_{u\nsim v;\ u,v\in V(G)}[2(n+m-1)-d_G(u)-d_G(v)]^\alpha+2^\alpha[n+m+1-2\delta]^\alpha\left[\binom{m}{2}+m-\frac{1}{2}M_1\right] + mn[2(n+m)-3\delta]^\alpha$

   $\chi_\alpha(T^{--1}) \geq \sum_{u\nsim v;\ u,v\in V(G)}[2(n+m-1)-d_G(u)-d_G(v)]^\alpha+2^\alpha[n+m+1-2\Delta]^\alpha\left[\binom{m}{2}+m-\frac{1}{2}M_1\right] + mn[2(n+m)-3\Delta]^\alpha$;

9. $\chi_\alpha(T^{0-+}) \leq 2^\alpha[m+3-2\delta]^\alpha\left[\binom{m}{2}+m-\frac{1}{2}M_1\right]+\sum_{u\in V(G)}d_G(u)[m+3-d_G(u)]^\alpha$

   $\chi_\alpha(T^{0-+}) \geq 2^\alpha[m+3-2\Delta]^\alpha\left[\binom{m}{2}+m-\frac{1}{2}M_1\right]+\sum_{u\in V(G)}d_G(u)[m+3-d_G(u)]^\alpha$

10. $\chi_\alpha(T^{1-+}) \leq \left(\sum_{u\sim v}+\sum_{u\nsim v}\right)[2(n-1)+d_G(u)+d_G(v)]^\alpha+2^\alpha[m+3-2\delta]^\alpha\left[\binom{m}{2}+m-\frac{1}{2}M_1\right] + \sum_{u\in V(G)}d_G(u)[n+m+2-d_G(u)]^\alpha$

    $\chi_\alpha(T^{1-+}) \geq \left(\sum_{u\sim v}+\sum_{u\nsim v}\right)[2(n-1)+d_G(u)+d_G(v)]^\alpha+2^\alpha[m+3-2\Delta]^\alpha\left[\binom{m}{2}+m-\frac{1}{2}M_1\right] + \sum_{u\in V(G)}d_G(u)[n+m+2-d_G(u)]^\alpha$; the equalities hold if and only if G is a regular graph.

Proof : 1.
$$\chi_\alpha(T^{0+1}) = \sum_{e_ie_j\in E(L(G)),e_i=uv,e_j=vw}[n-2+d_G(u)+d_G(v)+n-2+d_G(v)+d_G(w)]^\alpha + \sum_{ue\in E(K_{n,m}),e=vw}[m+n-2+d_G(v)+d_G(w)]^\alpha$$
$$= \sum_{e_i\sim e_j,e_i=uv,e_j=vw}[n-2+d_G(u)+d_G(v)+n-2+d_G(v)+d_G(w)]^\alpha$$

$$+n \sum_{uv\in E(G)} [n+m-2+d_G(u)+d_G(v)]^{\alpha}.$$

Since $d_G(u) \geq \delta$ for any vertex $u \in V(G)$ and $\alpha < 0$.

$$\chi_\alpha(T^{0+1}) \leq \sum_{e_i\sim e_j} [2n-4+4\delta]^{\alpha} + n \sum_{uv\in E(G)} [n+m-2+d_G(u)+d_G(v)]^{\alpha}$$

Thus $\chi_\alpha(T^{0+1}) \leq 2^{\alpha}(n-2+2\delta)^{\alpha} \left[\frac{1}{2}M_1 - m\right] + n \sum_{uv\in E(G)} [n+m-2+d_G(u)+d_G(v)]^{\alpha}$.

Similarly, we calculate

$$\chi_\alpha(T^{0+1}) \geq 2^{\alpha}(n-2+2\Delta)^{\alpha} \left[\frac{1}{2}M_1 - m\right] + n \sum_{uv\in E(G)} [n+m-2+d_G(u)+d_G(v)]^{\alpha}.$$

2. $\chi_\alpha(T^{1+1}) = \sum_{uv\in E(K_n)} [2(n+m-1)]^{\alpha} + \sum_{ue\in E(K_{n,m}),e=vw} [n+m-1+n-2+d_G(v)+d_G(w)]^{\alpha}$
$+ \sum_{e_ie_j\in E(L(G)),e_i=uv,e_j=vw} [n-2+d_G(u)+d_G(v)+n-2+d_G(v)+d_G(w)]^{\alpha}$

3. $\chi_\alpha(T^{++1}) = \sum_{uv\in E(G)} [m+d_G(u)+m+d_G(v)]^{\alpha}$
$+ \sum_{ue\in E(K_{n,m}),e=vw} [m+d_G(u)+n-2+d_G(v)+d_G(w)]^{\alpha}$
$+ \sum_{e_ie_j\in E(L(G)),e_i=uv,e_j=vw} [n-2+d_G(u)+d_G(v)+n-2+d_G(v)+d_G(w)]^{\alpha}$

4. $\chi_\alpha(T^{-+1}) = \sum_{uv\in E(\overline{G})} [n+m-1-d_G(u)+n+m-1-d_G(v)]^{\alpha}$
$+ \sum_{ue\in E(K_{n,m}),e=vw} [n+m-1-d_G(u)+n-2+d_G(v)+d_G(w)]^{\alpha}$
$+ \sum_{e_ie_j\in E(L(G)),e_i=uv,e_j=vw} [n-2+d_G(u)+d_G(v)+n-2+d_G(v)+d_G(w)]^{\alpha}$

5. $\chi_\alpha(T^{0-1}) = \sum_{e_i\nsim e_j,e_i=uv,e_j=wx} [n+m+1-d_G(u)-d_G(v)+n+m+1-d_G(w)-d_G(x)]^{\alpha}$
$+ \sum_{ue\in E(K_{n,m}),e=vw} [m+n+m+1-d_G(v)-d_G(w)]^{\alpha}$

6. $\chi_\alpha(T^{1-1}) = \sum_{uv\in E(K_n)} [(n+m-1)+(n+m-1)]^{\alpha}$
$+ \sum_{e_i\nsim e_j,e_i=uv,e_j=wx} [n+m+1-d_G(u)-d_G(v)+n+m+1-d_G(w)-d_G(x)]^{\alpha}$
$+ \sum_{ue\in E(K_{n,m}),e=vw} [n+m-1+n+m+1-d_G(v)-d_G(w)]^{\alpha}$

7. $\chi_\alpha(T^{+-1}) = \sum_{uv\in E(G)} [m+d_G(u)+m+d_G(v)]^{\alpha}$
$+ \sum_{e_i\nsim e_j,e_i=uv,e_j=wx} [n+m+1-d_G(u)-d_G(v)+n+m+1-d_G(w)-d_G(x)]^{\alpha}$
$+ \sum_{ue\in E(K_{n,m}),e=vw} [m+d_G(u)+n+m+1-d_G(v)-d_G(w)]^{\alpha}$

8. $\chi_\alpha(T^{--1}) = \sum_{uv\in E(\overline{G})} [n+m-1-d_G(u)+n+m-1-d_G(v)]^{\alpha}$
$+ \sum_{e_i\nsim e_j,e_i=uv,e_j=wx} [2(n+m+1)-d_G(u)-d_G(v)-d_G(w)-d_G(x)]^{\alpha}$

$$+\sum_{ue\in E(K_{n,m}),e=vw}[n+m-1-d_G(u)+n+m+1-d_G(v)-d_G(w)]^{\alpha}$$

9. $\chi_\alpha(T^{0-+})=\sum_{e_i\nsim e_j,e_i=uv,e_j=wx}[m+3-d_G(u)-d_G(v)+m+3-d_G(w)-d_G(x)]^{\alpha}$

$$+\sum_{ue\in E(S(G)),e=uv}[d_G(u)+m+3-d_G(u)-d_G(v)]^{\alpha}$$

$$=\sum_{e_i\nsim e_j,e_i=uv,e_j=wx}[2(m+3)-d_G(u)-d_G(v)-d_G(w)-d_G(x)]^{\alpha}$$

$$+\sum_{u\in V(G)}d_G(u)[m+3-d_G(u)]^{\alpha}.$$

10. $\chi_\alpha(T^{1-+})=\sum_{uv\in E(K_n)}[2(n-1)+d_G(u)+d_G(v)]^{\alpha}$

$$+\sum_{e_i\nsim e_j,e_i=uv,e_j=wx}[m+3-d_G(u)-d_G(v)+m+3-d_G(w)-d_G(x)]^{\alpha}$$

$$+\sum_{ue\in E(S(G)),e=uv}[n-1+d_G(u)+m+3-d_G(u)-d_G(v)]^{\alpha}.$$

Since $d_G(u)\geq\delta$ for any vertex $u\in V(G)$ and $\alpha<0$. On solving above expressions, we obtain the upper bound with the equality if and only if G is a regular graph. Similarly, we can compute the lower bound for above graphs. □

Theorem 2.8 : Let G be a $(n,\, m)$ graph and $\alpha<0$. Then

1. $\chi_\alpha(T^{0+-})\leq 2^{\alpha}[n-4+2\delta]^{\alpha}\left[\frac{1}{2}M_1-m\right]+m(n-2)[n+m-4+\delta]^{\alpha}$

   $\chi_\alpha(T^{0+-})\geq 2^{\alpha}[n-4+2\Delta]^{\alpha}\left[\frac{1}{2}M_1-m\right]+m(n-2)[n+m-4+\Delta]^{\alpha}$

2. $\chi_\alpha(T^{1+-})\leq\left(\sum_{u\sim v}+\sum_{u\nsim v}\right)[2(n+m-1)-d_G(u)-d_G(v)]^{\alpha}+2^{\alpha}[n-4+2\delta]^{\alpha}\left[\frac{1}{2}M_1-m\right]$
   $+m(n-2)[2n+m-5+\delta]^{\alpha}$

   $\chi_\alpha(T^{1+-})\geq\left(\sum_{u\sim v}+\sum_{u\nsim v}\right)[2(n+m-1)-d_G(u)-d_G(v)]^{\alpha}+2^{\alpha}[n-4+2\Delta]^{\alpha}\left[\frac{1}{2}M_1-m\right]$
   $+m(n-2)[2n+m-5+\Delta]^{\alpha}$

3. $\chi_\alpha(T^{0--})\leq 2^{\alpha}[n+m-1-2\delta]^{\alpha}\left[\binom{m}{2}-\frac{1}{2}M_1+m\right]+m(n-2)[n+2m-1-3\delta]^{\alpha}$

   $\chi_\alpha(T^{0--})\geq 2^{\alpha}[n+m-1-2\Delta]^{\alpha}\left[\binom{m}{2}-\frac{1}{2}M_1+m\right]+m(n-2)[n+2m-1-3\Delta]^{\alpha}$

4. $\chi_\alpha(T^{1--})\leq\left(\sum_{u\sim v}+\sum_{u\nsim v}\right)[2(n+m-1)-d_G(u)-d_G(v)]^{\alpha}$

   $+m(n-2)[2(n+m-1)-3\delta]^{\alpha}+2^{\alpha}[n+m-1-2\delta]^{\alpha}\left[\binom{m}{2}-\frac{1}{2}M_1+m\right]$

   $\chi_\alpha(T^{1--})\geq\left(\sum_{u\sim v}+\sum_{u\nsim v}\right)[2(n+m-1)-d_G(u)-d_G(v)]^{\alpha}$

$$+ m(n-2)[2(n+m-1)-3\Delta]^\alpha + 2^\alpha[n+m-1-2\Delta]^\alpha \left[\binom{m}{2} - \frac{1}{2}M_1 + m\right];$$

the equalities hold if and only if G is a regular graph.

Proof : 1. $\chi_\alpha(T^{0+-}) = \sum_{e_i\sim e_j, e_i=uv, e_j=vw} [n-4+d_G(u)+d_G(v)+n-4+d_G(v)+d_G(w)]^\alpha$

$$+ \sum_{u\nsim e, e=vw} [m-d_G(u)+n-4+d_G(v)+d_G(w)]^\alpha.$$

Since $d_G(u) \leq \Delta$ for any vertex $u \in V(G)$ and $\alpha < 0$.

$$\chi_\alpha(T^{0+-}) \geq \sum_{e_i\sim e_j} [2n-8+4\Delta]^\alpha + \sum_{u\nsim e} [n+m-4+\Delta]^\alpha$$

$$\chi_\alpha(T^{0+-}) \geq 2^\alpha[n-4+2\Delta]^\alpha \left[\frac{1}{2}M_1 - m\right] + m(n-2)[n+m-4+\Delta]^\alpha.$$

Similarly, we calculate the other side inequality.

2. $\chi_\alpha(T^{1+-}) = \sum_{uv\in E(K_n)} [2(n+m-1)-d_G(u)-d_G(v)]^\alpha$

$$\sum_{e_i\sim e_j, e_i=uv, e_j=vw} [n-4+d_G(u)+d_G(v)+n-4+d_G(v)+d_G(w)]^\alpha$$

$$+ \sum_{u\nsim e, e=vw} [n+m-1-d_G(u)+n-4+d_G(v)+d_G(w)]^\alpha.$$

3. $\chi_\alpha(T^{0--}) = \sum_{e_i\nsim e_j, e_i=uv, e_j=wx} [2(n+m-1)-d_G(u)-d_G(v)-d_G(w)-d_G(x)]^\alpha$

$$+ \sum_{u\nsim e, e=vw} [m-d_G(u)+n+m-1-d_G(v)-d_G(w)]^\alpha.$$

4. $\chi_\alpha(T^{1--}) = \sum_{uv\in E(K_n)} [(n+m-1)-d_G(u)+(n+m-1)-d_G(v)]^\alpha$

$$+ \sum_{e_i\nsim e_j, e_i=uv, e_j=wx} [2(n+m-1)-d_G(u)-d_G(v)-d_G(w)-d_G(x)]^\alpha$$

$$+ \sum_{u\nsim e, e=vw} [2(n+m-1)-d_G(u)-d_G(v)-d_G(w)]^\alpha.$$

Since $d_G(u) \leq \Delta$ for any vertex $u \in V(G)$ and $\alpha < 0$. On solving above expressions, we obtain the lower bound with the equality if and only if G is a regular graph. Similarly, we can compute the upper bound for above graphs. □

3. Conclusion

In this paper, we have obtained the bounds for general sum connectivity index of some *generalized xyz–Point-Line transformation graphs*. In order to obtain sharp bounds, we compute the expressions in terms of order, size, maximum vertex degree, minimum vertex degree and summation over vertices (or edges) of G. Also note that, if $\alpha > 0$, the opposite inequality is valid.

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