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# ON AUTOMORPHISM GROUPS OF S-VALUED GRAPHS

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**Abstract**

Motivated by the study of isomorphism on S-valued graphs, in this paper we study the automorphism groups of S-valued graphs. In particular, we define the concepts of the group of vertex and the group of induced edge automorphism on S-valued graphs.

## 1. Introduction

Group theory and graph theory both provide interesting and meaningful ways to examine the relationship between the elements of a given set. In the field of abstract algebra one of the fundamental concept is that of combining particular sets with operations on their elements and studying the resulting behaviour. Another natural question to ask is how the elements of a given set is related to each other. For example, given a set of cities we are interested to know which of them are directly connected by roads. Similarly we are interested to know the friends of a given person in a given group of people. Both

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of these sets lead themselves to be represented in the form of a graph. Suppose that we are building a graph to represent the cities and roads between them to know the shortest route between any two cities. In this case, it would be important to know how long each of the roads are. We could then assign a weight to any given edge of our graph. This leads to the concept of semiring valued graphs in which the vertices and the edges are labelled with the values from a given semiring.

In his monograph, Jonathan Golan [1] has mentioned the notion of R-valued graphs, where R is a semiring. He has assigned values to the edges of the graph only. Nothing more theory has been dealt by him. Motivated by this, Chandramouleeswaran and others started investigating the notion of graph thoery in which, unlike the notion in Golan's book [1], values from semiring are given to the vertices of a given graph using the canonical preorder existing in the semiring weights can be assigned to the edges also. Thus it becomes natural to use some of our tools from our study of abstract algebra to understand the theory of graphs much better.

In this paper we start by refining the idea of a group automorphism to apply to the theory of graphs. In the case of a group automorphism we needed a permutation on the set of elements that had the additional property of preserving the structure of the group under the operation associated with it. In the case of a crisp graph, we have no operation to preserve, but we like to maintain the information provided by the graph through the isomorphism. This preserving the relationship between two vertices that are connected by an edge. In the case of S-valued graphs we need to maintain the information provided by the S-valued graph through the isomorphism which is defined as a pair of mappings $\phi = (\alpha, \beta)$,where $\alpha$ is a graph isomorphism and $\beta$ is a semiring isomorphism that provides the relationship between the two vertices that are not only connected by an edge but also that preserves the canonical preorder existing in the semiring [1].

## 2. Preliminaries

In this section we recall some basic definitions that are needed for our work.

**Definition 2.1 [2]** : A semiring $(S, +, \cdot)$ is an algebraic system with a non-empty set $S$ together with two binary operations $+$ and $\cdot$ such that

1. $(S, +, \cdot)$ is a monoid.

2. $(S, \cdot)$ is a semigroup.

3. For all $a, b, c \in S, a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$

4. $0 \cdot x = x \cdot 0 = 0 \ \ \forall x \in S$.

The element 0 in $S$ is called the additive identity as well as the zero of the semiring $S$.

**Definition 2.2 [2]** : Let $(S, +, \cdot)$ be a semiring. $\preceq$ is said to be a canonical pre-order if for $a, b \in S, a \preceq b$ if and only if there exists $c \in S$ such that $a + c = b$.

**Definition 2.3 [2]** : Let $G = (V, E \subset V \times V)$ be a given graph with $V, E \neq \phi$. For any semiring $(S, +, \cdot)$, a semiring valued graph (or a S-valued graph ) $G^S$ is defined to be the graph $G^S = (V, E, \sigma, \psi)$ where $\sigma : V \to S$ and $\psi : E \to S$ is defined by: For all $(x, y) \in E \subseteq V \times V$

$$\psi(x, y) = \begin{cases} min\{\sigma(x), \sigma(y)\} & if\ \sigma(x) \preceq \sigma(y) (or) \sigma(y) \preceq \sigma(x) \\ 0 & otherwise \end{cases}$$

We call $\sigma$, a $S-$vertex set and $\psi$ a $S-$edge set of the $S-$valued graph $G^S$.

**Definition 2.4 [4]** : Let $G_1^{S_1} = (V_1, E_1, \sigma_1, \psi_1)$ and $G_2^{S_2} = (V_2, E_2, \sigma_2, \psi_2)$be two given $S_1-$valued and $S_2-$valued graphs. A homomorphism $f = (\alpha, \beta) : G_1^{S_1} \to G_2^{S_2}$of S-valued graphs is a pair of homomorphisms $\alpha : V_1 \to V_2$ which is a bijection and $\beta : S_1 \to S_2$such that $\beta(\sigma_1(v_i)) \preceq \sigma_2(\alpha(v_i))$and $\beta(\psi_1(v_i v_j) \preceq \psi_2(\alpha(v_i)\alpha(v_j)) \ \forall v_i, v_j \in V_1$.

**Definition 2.5 [4]** : A weak isomorphism$f = (\alpha, \beta) : G_1^{S_1} \to G_2^{S_2}$ is a pair of homomorphisms $\alpha : V_1 \to V_2$ which is a bijection and $\beta : S_1 \to S_2$ such that $\beta(\sigma_1(v_i)) = \sigma_2(\alpha(v_i))\forall v_i \in V_1$. A Co-weak isomorphism $f = (\alpha, \beta) : G_1^{S_1} \to G_2^{S_2}$ is a pair of homomorphisms $\alpha : V_1 \to V_2$ which is a bijection and $\beta : S_1 \to S_2$such that $\beta(\psi_1(v_i v_j) = \psi_2(\alpha(v_i)\alpha(v_j))\forall v_i, v_j \in V_1$.

**Definition 2.6 [4]** : An isomorphism $f = (\alpha, \beta) : G_1^{S_1} \to G_2^{S_2}$ is a pair of isomomorphisms $\alpha : V_1 \to V_2, \beta : S_1 \to S_2$are such that $\beta(\sigma_1(v_i)) = \sigma_2(\alpha(v_i))\forall v_i \in V_1$and$\beta(\psi_1(v_i v_j) = \psi_2(\alpha(v_i)\alpha(v_j))\forall v_i, v_j \in V_1$.

If such an isomorphism from $G_1^{S_1}$ to $G_2^{S_2}$ exists and if both $\alpha$ and $\beta$ are onto, then $G_1^{S_1}$ is said to be $S-$ valued isomorphic to $G_2^{S_2}$ and we write it as $G_1^{S_1} \cong_S G_2^{S_2}$.

**Definition 2.7** : A $S-$valued graph $G^S = (V, E, \sigma, \psi)$ is said to be $S-$complete if its underlying graph $G$ is complete along with $S-$values.

## Automorphism Groups of $S-$valued Graphs

In this section we determine the automorphism groups

**Definition 3.1 :** An S-valued automorphism of $G^S$ is a pair of isomorphisms $(\alpha, \beta)$ that satisfies the property that $\{(v_i, a), (v_j, b)\} \in E(G^S)$ if and only if $\{(\alpha(v_i), \beta(a)), (\alpha(v_j), \beta(b))\} \in E(G^S)$.

**Theorem 3.2 :** The set $Aut(G^S)$ of all $S-$valued graph automorphisms of a $S-$valued graph $G^S$ forms a group under function composition.

**Proof :** We will show that the elements of $Aut(G^S)$ satisfy the group axioms.

1. Let $f, g \in Aut(G^S)$.
   Since $f, g$ are automorphisms they permute the vertices of $G^S$.
   It follows that the combination of two will still permute the vertices of $G^S$.
   Thus the resulting permutation is an automorphism. This yields that the function composition is closed.

2. Since composition is associative this point follows by assumption.

3. The identity automorphism $(\alpha_i, \beta_i)$ is the permutation defined by $(\alpha_i, \beta_i)(v, a) = (v, a), \forall\ (v, a) \in G^S$ act as the identity element of $Aut(G^S)$. For, Given any automorphism $(\gamma, \delta) \in Aut(G^S)$, we find that $((\gamma, \delta) \circ (\alpha_i, \beta_i))(v_i, a) = (\gamma(v_i), \delta(a))$ and that $((\alpha_i, \beta_i) \circ (\gamma, \delta)(v_i, a)) = (\gamma(v_i), \delta(a)), \forall\ (v_i, a) \in G^S$.

4. Let us consider $(f, g) \in Aut(G^S)$. The size of $Aut(G^S)$ is finite, we cannot have almost $n!$ permutations on n vertices.
   Thus $(f, g) \circ (f, g) \circ ...(f, g) = (f, g)^j$ must eventually equal the identity permutation for some positive integer j, otherwise $Aut(G^S)$ is infinite.
   Let $(\gamma, \delta) = (f, g)^{(j-1)}$. Then $(f, g) \circ (\gamma, \delta) = (\gamma, \delta) \circ (f, g) = (f, g)^j = (\alpha_i, \beta_i)$.
   This completes the proof that $Aut(G^S)$ is a group under function composotion.

**Theorem 3.3 :** $Aut(G^S) = Aut(\bar{G^S})$

**proof :** We will prove by showing set inclusion on both sides.

First, let $(f, g) \in Aut(G^S)$ and an edge $(v_i, v_j) \notin E_{G^S}$.

By the definition of complement of $S-$valued graph, $(v_i, v_j) \in E_{\bar{G^S}}$.

Since by the definition of $S-$valued graph automorphism that $(f, g)(v_i, v_j) \notin E_{G^S}$ and hence we find that $(f, g)(v_i, v_j) \in E_{\bar{G^S}}$.

Thus we have shown that $Aut(G^S) \subseteq Aut(\bar{G^S})$

Similarly we can prove that $Aut(\bar{G^S}) \subseteq Aut(G^S)$

Thus we have shown that the two automorphism groups are equal.

## 4. Edge Automorphism Groups of $S-$valued Graphs

**Definition 4.1** : An edge $e_i^j$ is said to incident at $v_i$ iff $\psi(e_i^j) \preceq \sigma(v_i)$.

**Definition 4.2** : An edge automorphism is a pair of mappings $\phi = (\alpha, \beta)$ on the set of edges $E_S$ that satisfies the property that $e_i^j, e_i^k$ are adjacent iff $\alpha(e_i^j)$ and $\alpha(e_i^k)$ are adjacent and $\beta(\psi(e_i^j)) = \psi(e_{\alpha(v_i)}^{\alpha(v_j)})$ and $\beta(\psi(e_i^k)) = \psi(e_{\alpha(v_i)}^{\alpha(v_k)})$.

The set of all edge automorphisms of $S-$valued graphs is denoted by $Aut_E(G^S)$.
The induced edge automorphism is a particular case of vertex automorphism that preserves the adjacency of edges of $S-$valued graphs and the set of all such automorphisms will be represented by $Aut_I(G^S)$.

It is clear that $Aut_I(G^S) \subseteq Aut_E(G^S)$
Analogous to theorem 3.2, we can prove the following theorem.

**Theorem 4.3** : The set $Aut_E(G^S)$ of all $S-$edge automorphisms of a graph $G^S$ forms a group under function composition.

**Example 4.4** : The following example illustrates the existence of isomorphism between two edge automorphism groups corresponding to two non-isomorphic $S$-valued graphs. Consider the semiring $S = (\{0, a, b, c\}, +, \cdot)$ with the binary operations '$+$' and '$\cdot$' defined by the following Cayley tables.

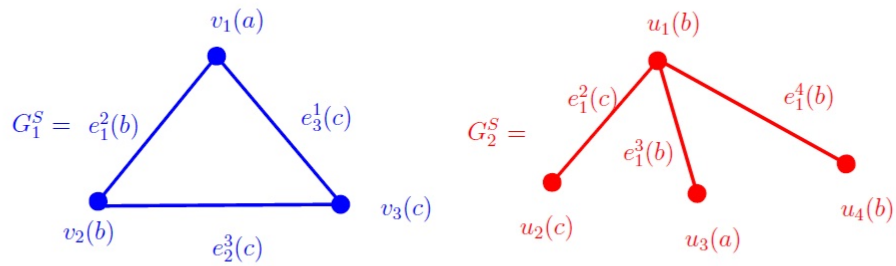
| $+$ | $0$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $0$ | $0$ | $a$ | $b$ | $c$ |
| $a$ | $a$ | $a$ | $a$ | $a$ |
| $b$ | $b$ | $a$ | $b$ | $b$ |
| $c$ | $c$ | $a$ | $b$ | $c$ |

| $\cdot$ | $0$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ |
| $a$ | $0$ | $a$ | $a$ | $a$ |
| $b$ | $0$ | $b$ | $b$ | $b$ |
| $c$ | $0$ | $b$ | $b$ | $b$ |

In $S$ we define a canonical pre-order $\preceq$as follows:

$$0 \preceq 0,\ 0 \preceq a,\ 0 \preceq b,\ 0 \preceq c,\ a \preceq a,\ b \preceq b,\ c \preceq c,\ b \preceq a,\ c \preceq a,\ c \preceq b,$$

Consider the two $S-$valued graphs $G_1^S$ and $G_2^S$




Here $Aut_E(G_1^S) = \{(\alpha, \beta) \mid \alpha : E(G_1(S)) \to E(G_1(S))$ is an automorphism and $\beta : S \to S$ such that $\beta(a) = a, \forall a \in S\}$
and $Aut_E(G_2^S) = \{(f, \beta) \mid f : E(G_2(S)) \to E(G_2(S))$ is an automorphism and $\beta : S \to S$ such that $\beta(a) = a, \forall a \in S\}$
Then

$$Aut_E(G_1^S) = \{((1), \beta), ((e_3^1\ e_2^3), \beta), ((e_3^1\ e_1^2), \beta), ((e_2^3\ e_1^2), \beta), (e_2^3\ e_3^1\ e_1^2), \beta), ((e_3^1\ e_1^2\ e_2^3), \beta)\}.$$

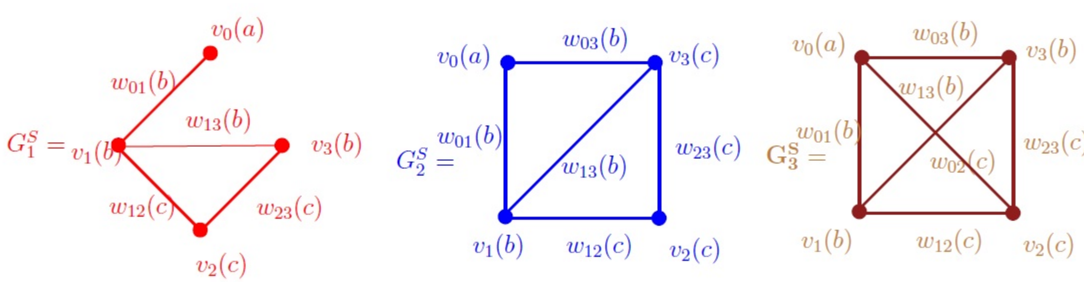
and

$$Aut_E(G_2^S) = \{((1), \beta), ((e_1^3\ e_1^4), \beta), ((e_1^2\ e_1^4), \beta), ((e_1^2\ e_1^3), \beta), (e_1^2\ e_1^3\ e_1^4), \beta), ((e_1^2\ e_1^4\ e_1^3), \beta)\}.$$

We can easily prove that $Aut_E(G_1^S) \cong_S Aut_E(G_2^S)$
But $G_1^S \ncong_S G_2^S$, because $G_1^S$ and $G_2^S$ has different number of vertices, therefore there is no bijection between $V(G_1^S)$ and $V(G_2^S)$.

The following examples of $S-$valued graphs has an edge automorphism that are not induced by any $S-$valued graph automorphism.

**Example 4.5** : Let $(S = \{0, a, b, c\}, +, \cdot)$ be a semiring with canonical preorder given in the example 4.4
Consider the three $S-$valued graphs $G_1^S, G_2^S$ and $G_3^S$



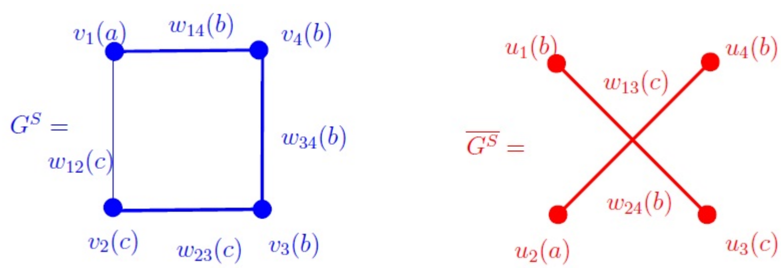
Then $\phi_1 = (\alpha_1, \beta)$, where $\alpha_1 = (w_{01} \;\; w_{23})$ and $\beta(\psi(w_{ij})) = w_{ij}$,
$\phi_2 = (\alpha_2, \beta)$, where $\alpha_2 = (w_{01} \;\; w_{12} \;\; w_{23} \;\; w_{03})$ and $\beta(\psi(w_{ij})) = w_{ij}$
and $\phi_3 = (\alpha_3, \beta)$, where $\alpha_3 = (w_{01} \;\; w_{23})$ and $\beta(\psi(w_{ij})) = w_{ij}$ are the $S-$valued edge automorphisms corresponding to the $S-$valued graphs $G_1^S, G_2^S$ and $G_3^S$ that are not induced by any $S-$valued graph automorphism.

**Definition 4.6** : A graph $G^S$ is said to be self complementary if $G^S$ and $\bar{G^S}$ are isomorphic.

Analogous to theorem 3.3, we cannot have $Aut_E(G^S) \neq Aut_E(\bar{G^S})$ in general. This is illustrated in the following example.

**Example 4.7** : Let $(S = \{0, a, b, c\}, +, \cdot)$ be a semiring with canonical preorder given in the example 4.4

Consider the $S-$valued graphs $G^S$ and its complement $\bar{G^S}$




$$Aut_E(G^S) = \{(1), (w_{12} \;\; w_{34}), (w_{23} \;\; w_{34}), (w_{12} \;\; w_{23})(w_{34} \;\; w_{14}), (w_{12} \;\; w_{34})(w_{23} \;\; w_{14}),$$

$$(w_{12} \;\; w_{14})(w_{23} \;\; w_{34}), (w_{12} \;\; w_{23} \;\; w_{34} \;\; w_{14}), (w_{12} \;\; w_{14} \;\; w_{34} \;\; w_{23})\}$$

$Aut_E(\bar{G^S}) = \{\}$

It is clear from the above two sets that $Aut_E(G^S) \neq Aut_E(\bar{G^S})$.

**Theorem 4.8** : If $G^S$ is self complementary then $Aut_E(G^S) = Aut_E(\bar{G^S})$.

**Proof** : We will prove by showing set inclusion on both sides.

First, let $(f, g) \in Aut_E(G^S)$ and an edge $(w_{ij}) \notin E_{G^S}$.

By the definition of complement of $S-$valued graph, $(w_{ij}) \in E_{\bar{G^S}}$.

Since by the definition of $S-$valued edge automorphism that $\alpha(v_i)$ and $\alpha(v_j)$ are not adjacent in$G^S$.

Hence we find that there is an edge between $\alpha(v_i)$ and $\alpha(v_j)$ in $\bar{G^S}$.
Thus we have shown that $Aut_E(G^S) \subseteq Aut_E(\bar{G^S})$
Similarly we can prove that $Aut_E(\bar{G^S}) \subseteq Aut_E(G^S)$
Thus we have shown that the two automorphism groups are equal.

**Theorem 4.9** : Let $G^S$ be any connected $S-$valued graph. Then $Aut(G^S) \cong Aut_I(G^S)$ if and only if $G^S \ncong K_2^S$.

**Proof** : First we assume that $G^S$ is a connected $S-$valued graph on atleast three vertices. That is $G^S$ must have at least two edges.
Define $\tau : Aut(G^S) \to Aut_I(G^S)$ such that $\tau(\phi) = \tau(\alpha, \beta) = (\alpha_I, \beta_I)$, where $(\alpha_I, \beta_I)$ is the edge automorphism induced by $(\alpha, \beta)$.
We need to prove that this $\tau$ is an isomorphism.

1. Let $\phi_1 = (\alpha_1, \beta), \phi_2 = (\alpha_2, \beta) \in Aut(G^S)$ such that $\phi_1 \neq \phi_2$. and let $a, b, c \in S$ and without loss of generality assume that $a \preceq b \preceq c$.
   Therefore, there must be a vertex $(v_i, a) \in V_S$ such that $\phi_1(v_i, a) \neq \phi_2(v_i, a)$ and let $(v_j, b)$ be a vertex adjacent to $(v_i, a)$.
   If $\phi_1(v_i, a) \neq \phi_2(v_j, b)$ or $\phi_2(v_j, b) \neq \phi_1(v_i, a)$, then we found that for the edge $(w_{ij}, a)$, the induced automorphisms $\phi_{1I} \neq \phi_{2I}$.
   Now assume that $\phi_1(v_i, a) = \phi_2(v_j, b)$ and $\phi_2(v_j, b) = \phi_1(v_i, a)$.
   Since $G^S$ has alleast three vertices, there exists another vertex $(v_k, c)$ such that $(v_k, c)$ is adjacent to either $(v_i, a)$ or $(v_j, b)$ or both.
   We suppose that $(w_{jk}, b) = (v_j, b)(v_k, c) \in E_S$, we arrive at $\phi_{1I}(w_{jk}, b) \neq \phi_{2I}(w_{jk}, b)$.
   Thus in all the cases, we have shown that $\tau$ is one-one.

2. By the construction of induced automorphism, for each $\phi_I \in Aut_I(G^S)$, there is $\phi \in Aut(G^S)$ such that $\tau(\phi) = \phi_I$. This proves $\tau$ is onto.

3. Let $(w_{ij}, a) = \{(v_i, a)(v_j, b)\}$ Take $u = (v_i, a)$ and $v = (v_j, b)$.
   Define $\phi_2(u) = u',\ \phi_2(v) = v',\ \phi_1(u') = u'',\ \phi_1(v') = v''$.

$$Then\ \tau(\phi_1\ \phi_2)(w_{ij}, a) = \tau(\phi_1\ \phi_2)(\{u, v\}) = \phi_{1I}\ \phi_{2I}(\{u, v\})$$

$$= \{(\phi_1\ \phi_2)(u), (\phi_1\ \phi_2)(v)\} = \{\phi_1(u'), \phi_2(v')\} = \{u'', v''\}.$$

On the other hand,

$$\tau(\phi_1)\ \tau(\phi_2)(w_{ij}, a) = \tau(\phi_1)\ \tau(\phi_2)(\{u, v\}) = \tau(\phi_1)(\{\phi_2(u), \phi_2(v)\})$$

$$= \tau(\phi_1)(\{u', v'\}) = \{\phi_1(u'), \phi_1(v')\} = \{u'', v''\}.$$

Thus we shown that $\tau(\alpha_1\ \alpha_2) = \tau(\alpha_1)\tau(\alpha_2)$

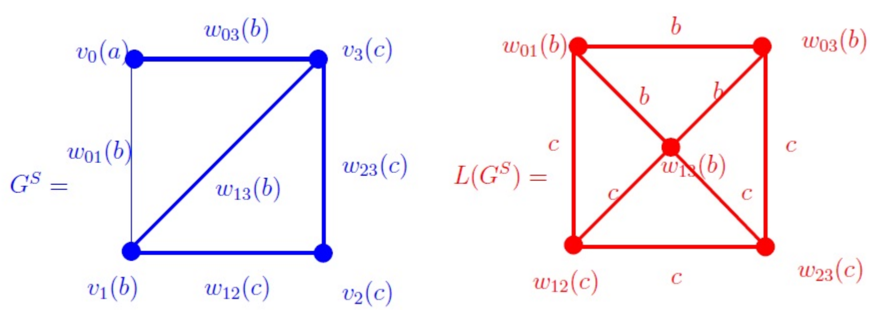
Hence $\tau$ is an isomorphism.

**Remark 4.10** : From the example 4.5, we have noted that the $S-$valued graphs $G_1^S, G_2^S$ and $G_3^S$ are the only connected graphs on three or more vertices that have edge automorphisms which are not included in the group of induced edge automorphisms. This remark along with the above theorem leads to the following corollary.

**Corollary 4.11** : If $G^S$ is a connected graph on 3 or more vertices which has no subgraph isomorphic to a triangle or a cycle, then $Aut(G^S) = Aut_I(G^S) = Aut_E(G^S)$.

**Definition 4.12 [3]** : Let $G^S$ be any $S-$valued graph. The line graph of $G^S$, denoted by $L(G^S)$, is a graph whose vertices are the edges of $G^S$ such that the edges with a common end point is adjacent in $L(G^S)$. That is, $e_i^j, e_k^l \in E(L(G^S))$ if and only if $e_i^j, e_k^l$ are the edges in $G^S$ having a common end point.

**Example 4.13** :




**Theorem 4.14** : $Aut(L(G^S))$ forms a group under composition of functions.

**Theorem 4.15** : Let $G^S$ be a connected graph on 3 or more vertices. Then $Aut(G^S) \cong Aut(L(G^S))$ if $G^S$ has no subgraph isomorphic to a triangle or a cycle.

**Proof** : From the construction of the line graph $G^S$, there is an isomorphism between the sets $E_S$ and $V_{L(G^S)}$.

Hence there is an isomorphism between $Aut_E(G^S)$ and $Aut(L(G^S))$.

By theorem 4.9 and corollary 4.11, we have $Aut(G^S) \cong Aut_E(G^S) \cong Aut(L(G^S))$.

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