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FIXED POINT OF ASYMPTOTICALLY REGULAR MAPPINGS OF c-DISTANCE ON CONE METRIC SPACE

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Abstract

In this paper we introduce and establish fixed point theorems of asymptotically regular mappings of c-distance on Cone Metric Space. Our results generalise and extends some fixed some theorems exiting in the literature.

1. Introduction

Definition 1.1 [9] : Let E be a real Banach space and P be a subset of E. P is called a cone if

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(i) P is a closed, non-empty and $P = \{0\}$

(ii) $a, b \in R$, $a, b \geq 0$, $x, y \in P$ implies $ax + by \in P$.

(iii) $x \in P$ and $-x \in P$ imply $x = 0$.

Given a cone $P \subseteq E$, we define a partial ordering " $\leq''$ in E by $x \leq y$ if $y - x \in P$. We write $x < y$ to denote $x \leq y$ but $x = y$ and $x < y$ to denote $y - x \in P^0$, where P^0 stands for the interior of P.

Proposition 1.2 [1] : Let P be a cone in a real Banach space E. If $a \in P$ and $a \leq ka$, for some $k \in [0, 1)$ then $a = 0$.

Proposition 1.3 [1] : Let P be a cone in a real Banach space E. If for $a \in E$ and $a \ll c$, for all $c \in P^0$, then $a = 0$.

Remark 1.4 [10] : $\lambda P^0 \subseteq P^0$, for $\lambda > 0$ and $P^0 + P^0 \subseteq P^0$.

Definition 1.5 [9] : Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies

(a) $0 \leq d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$.

(b) $d(x, y) = d(y, x)$ for all $x, y \in X$.

(c) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Example 1.6 ([4],[13]) : Let $E = R^3$, $P = \{(x, y, z) \in B : x, y, z \geq 0\}$ and $X = R$. Define $d : X \in X \to E$ by $d(x, y) = (a|x-y|, \beta|x-y|, \gamma|x-y|)$ where α, β, γ are positive constants. Then (X, d) is a cone metric space.

Definition 1.7 [9] : Let (X, d) be a cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with $0 \ll c$ there is a positive integer Nc such that for all $n > Nc$, $d(x_n, x)c$, then the sequence $\{x_n\}$ is said to converges to x and x is called limit of $\{x_n\}$. We write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

Definition 1.8 [9] : Let (X, d) be a cone metric space. Let $\{x - n\}$be a sequence in X. If for any $c \in E$ with $0 \ll c$ there is a natural number N such that for all n, $m > N$, $d(x_n, x_m) \ll c$, then the sequence $\{x_n\}$ is said to be a Cauchy sequence in X.

Definition 1.9 [9] : Let (X, d) be a cone metric space. If every Cauchy sequence in X is convergent in X, then X is called a complete cone metric space.

Proposition 1.10 [12] : Let (X, d) be a cone metric space and P be a cone in a real Banach space E. If $u \leq v$, $v \ll w$ then $u \ll w$.

Lemma 1.11 [12] : Let (X, d) be a cone metric space and P be a cone in a real Banach space E and k_1, k_2, k_3, k_4, $k > 0$. If $x_n \to x$, $y_n \to y$, $z_n \to z$ and $p_n \to p$ in X and (i) $ka \leq k_1 d(x_n, x) + k_2 d(y_n, y) + k_3 d(z_n, z) + k_4 d(p_n, p)$ then $a = 0$.

Definition 1.11 : Let (X, d) be a cone metric space. A sequence $\{x_n\}$ in X is said to be asymptotically T-regular if $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Definition 1.12 : Let (X, d) be a cone metric space. Then a function $p : X \times X \to E$ is called a c-distance on X if the followings are satisfied.

(1) $0 \leq p(x, y)$ for all $x, y \in X$.

(2) $p(x, z) \leq p(x, y) + p(y, z)$ for all $x, y, z \in X$;

(3) for each $x \in X$ and $n \geq 1$, if $p(x, y_n) \leq u$ for some $u = u_n$, then $p(x, y) \leq u$ whenever $\{y_n\}$ is a sequence in X conversing to a point $y \in X$;

(4) for all $c \in E$ with $0 \ll c$, there exists $e \in E$ with $0 \ll e$ such that $p(z, x) \ll e$ and $p(z, y) \ll e$ imply $d(x, y) \ll c$.

Example 1.13 [16] : Let (X, d) be a cone metric space and P be a normal cone. Define a mapping $p : X \times X \to E$ by $p(x, y) = d(x, y)$ for all $x, y \in X$. Then p is c-distance.

Example 1.14 [16] : Let $E = R$ and $P = \{x \in E : x \geq 0\}$. Let $X = [0, \infty)$ and define a mapping $d : X \times X \to E$ by $d(x, y) = |x - y|$for all $x, y \in E$. Then (X, d) is a cone metric space. Define a mapping $p : X \times X \to E$ by $p(x, y) = y$ for all $x, y \in X$. Then p is c-distance.

Remark 1.15 : On c-distance $p(x, y) = p(y, x)$ does not necessarily hold and $p(x, y) = 0$ is not necessarily equivalent to $x = y$ for all $x, y \in X$.

Definition 1.16 : Let (X, d) be a cone metric space. A sequence $\{x_n\}$ in X is said to be asymptotically T-regular of c-distance if $\lim_{n\to\infty} p(x_n, Tx_n) = 0$.

Lemma 1.17 : Let (X, d) be cone metric space, p be a c-distance on X and $\{x_n\}$ be sequence in X. If there exists the sequence $\{x_n\}$ in P conversing to 0 and $p(x_n, x_m) \leq x_n$ for $m > n$, then $\{x_n\}$ is a Cauchy sequence in X.

2. Main Result

Theorem 2.1 : Let (X, d) be a complete cone metric space. Let p be a c-distance on X and T be a self-mapping of X satisfying the inequality

$$p(Tx, Ty) \leq a_1 p(x, Tx) + a_2 p(y, Ty) + a_3 p(x, Ty) + a_4 p(y, Tx) + a_5 p(x, y)$$

for all $x, y \in X$ where $a_1, a_2, a_3, a_4, a_5 \geq 0$ and $\max(a_1 + a_4), (a_3 + a_4 + a_5) < 1$.
If there exists an asymptotically T-regular sequence in X, then T has a unique fixed point.

Proof : Let $\{x_n\}$ be an asymptotically T-regular sequence of c-distance in X. Then

$$\begin{aligned}
p(x_n, x_m) &\leq p(x_n, Tx_n) + p(Tx_n, x_m) \\
&\leq p(x_n, Tx_n) + p(Tx_n, Tx_m) + p(Tx_m, x_m) \\
&\leq p(x_n, Tx_n) + p(Tx_m, x_m) + a_1 p(x_n, Tx_n) + a_2 p(x_m, Tx_m) \\
&\quad + a_3 p(x_n, Tx_m) + a_4 p(x_m, Tx_n) + a_5 p(x_n, x_m) \\
&\leq p(x_n, Tx_n) + p(Tx_m, x_m) + a_1 p(x_n, Tx_n) + a_2 p(x_m, Tx_m) \\
&\quad + a_3 p(x_n, x_m) + a_3 p(x_m, Tx_m) + a_4 p(x_m, x_n) + a_4 p(x_n, Tx_n) + a_5 p(x_n, x_m) \\
&= (1 + a_1 + a_4) p(x_n, Tx_n) + (1 + a_2 + a_3) p(x_m, Tx_m) + (a_3 + a_4 + a_5) p(x_n, x_m) \\
&\Rightarrow [1.(a3 + a4 + a5)] p(xn, xm).(1 + a1 + a4) p(xn, Txn) + (1 + a2 + a3) p(xm, Txm) \\
&\Rightarrow p(x_n, x_m) \leq \frac{(1 + a_1 + a_4)}{[1 - (a_3 + a_4 + a_5)]} p(x_n, Tx_n) + \frac{(1 + a_2 + a_3)}{[1 - (a_3 + a_4 + a_5)]} \\
&\Rightarrow p(x_n, x_m) \leq M_1 p(x_n, Tx_n) + M_2 p(x_m, Tx_m)
\end{aligned}$$

where $M_1 = \frac{(1+a_1+a_4)}{[1+(a_3+a_4+a_5)]}$ and $M_2 = \frac{(a_1+a_2+a_3)}{[1-(a_3+a_4+a_5)]}$.
Since $\{x_n\}$ is an asymptotically T-regular sequence of c-distance and $m > n$. Therefore, $p(x_n, Tx_n) = 0$ and $p(x_m, Tx_m) = 0$ when $n \to \infty$.
Choose a natural number N, such that $[M_1 d(x_n, Tx_n) + M_2 d(x_m, Tx_m)] \ll u_n$ for all $m, n \geq N$. Thus $p(x_n, x_m) \ll< u_n$ for $m > n$. Therefore $\{x_n\}$ is a Cauchy sequence in X which is a complete. So $\{x_n\} \to x \in X$.

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