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# THEOREMS ON INERT SUBGROUPS AND STABLE IMAGES OF FREE GROUPS

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**Abstract**

Theory of fixed subgroups of free groups created the vast literature and a major line of research in past. In this paper we are going to see some interesting results on fixed subgroups, inert subgroups and the stable images of free groups.

## 1. Introduction

Free groups, its fixed point subgroups and inert subgroups are interesting domain of research since previous century. A wide body of results on fixed subgroups of free groups by their automorphisms as well as endomorphisms and inert subgroups of free groups have been studied earlier.

Let $F(A)$ and $F(B)$ are free groups generated by the sets $A$ and $B$ respectively. Then $End(F)$ and $Aut(F)$ represent the set of endomorphisms and the set of automorphisms on $F$ respectively. Fixed subgroup Fix $\phi = \{u \in F | \phi(u) = u\}$, where $\phi \in End(F)$. The equaliser of two free group homomorphisms $g, h : F(A) \to F(B)$ is the set of points

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Key Words : *Free group, Fixed subgroups, Endomorphism, Inert subgroup, Stable image, Generalized stable image.*

where they have same images, so $Eq(g,h) = \{x \in F(A) | g(x) = h(x)\}$. If one of the homomorphism $g$ or $h$ is injective then $Eq(g,h)$ has finite rank, $rk(Eq(g,h)) < \infty$, it was proved by Goldstein and Turner in [11]. Fixed subgroups of free groups have generated a lot of literature from the 1970s onwards by Dyer and Scott in [6], Jaco and Shalen in [14], Gersten in [8], Ventura in [17], Bogopolski and Maslakova in [3], Feighn and Handel in [7]. Bestvina and Handel in [2] used Thurston's train-track maps to prove that $rk(Fix(\varphi)) \leq |A|$ for $\varphi \in Aut(F(A))$ and Imrich and Turner extended this bound to all endomorphisms in [12]. Bergman further extended this bound to all sets of endomorphisms in [1]. Similarly equalisers have been studied other than fixed subgroups in the papers of Goldstein and Turner in [9], [10] and [11], Ciobanu, Martino, Ventura in [4] and Myasnikov, Nikolaev, and Ushakov in [15].

## 2. Inert Subgroups

A subgroup $H$ of a free group $F(A)$ is inert if for all $K \leq F(A)$ we have $rk(H \cap K) \leq rk(K)$. Examples of inert subgroups include free factors, and more generally fixed subgroups of sets of monomorphisms are discussed by Dicks and Ventura in [5], and there are inert subgroups which are not fixed subgroups have been studied by Rosenmann in [16]. Recent development of the work on inert subgroups has been focused on trying to determine inertness(or sometimes inertia) algorithmically by quantifying it by Ivanov in [13].

## 3. Equalisers as Fixed Subgroups

In this section we view equalisers as fixed subgroups, as explained below. We begin with a theorem which, under very specific conditions, allows us to view equalisers as fixed subgroups.

If $g, h : F(A) \to F(B)$ are homomorphisms with $h$ injective and $im(g) \leq im(h)$ then we may define: $\psi(g,h) : h^{-1}(im(g)) \to h^{-1}(im(g))$

$$x \to h^{-1}(g(x)).$$

Here we can apply $h^{-1}$ to $g(x)$ as $im(g) \leq im(h)$ and the map $\psi(g,h)$ is a function as $h$ is injective.

**Theorem 3.1** : Let $g, h : F(A) \to F(B)$ be homomorphisms with $h$ injective and

$im(g) \leq im(h)$. Then $Eq(g,h) = Fix(\psi(g,h))$.

**Proof** : Clearly $Fix(\psi(g,h)) \leq Eq(g,h)$, while $Eq(g,h) \leq Fix(\psi(g,h))$ since if $g(x) = h(x)$ then $h^{-1}(g(x)) = x$, as $h$ is injective, and clearly $x \in h^{-1}(im(g))$.

## 4. Generalized Stable Image

For endomorphisms : $F \to F$ the stable image of $\varphi$ is $\varphi^{\infty}(F) = \bigcap_{i=0}^{\infty} \varphi^i(F)$. Imrich and Turner used this tool to prove that $rk(Fix(\varphi)) \leq rk(F)$ in [16]. We now generalize this construction to equalisers. Let $g, h : F(A) \to F(B)$ be homomorphisms. Then we define $H_0 = F(A)$ and by mathematical induction define $H_{i+1} = h^{-1}(g(H_i) \cap (H_i))$. Then finally define $GSI(g,h) = \bigcap_{i=0}^{\infty} H_i$.

We call $GSI(g,h)$ the generalized stable image of $g$ with $h$. The name ggeneralized stable imageh is because we can use the restrictions $g|GSI(g,h)$ and $h|GSI(g,h)$ to understand $Eq(g,h)$ (in theorem 6.1). By taking $A = B$ and $g$ to be the identity map, we see that the stable image is a special case of the generalized stable image.

**Theorem 4.1** : Let $h$ be injective. Then $GSI(g,h)$ is the maximal subgroup $K$ of $F(A)$ such that $g(K) \leq h(K)$.

**Proof** : First of all we prove that $g(GSI(g,h)) \leq h(GSI(g,h))$. For this let $x \in GSI(g,h)$. Then $x \in H_i$ for all $i \geq 0$. Hence, for all $j \geq 1$ there exists some $y_j \in g(H_j) \cap h(H_j)$ such that $x \in g^{-1}(y_i)$. Then $g(x) = y_j \in h(H_j)$ and so $g(x) \in h(H_j)$ for all $j \geq 0$. Hence, $g(x) \in \cap h(H_j)$..

By injectivity of $h$, we have $\cap h(H_j) = h(\cap H_j) = h(GSI(g,h))$ and so $g(x) \in h(GSI(g,h))$ as required. For maximality, suppose $K \leq F(A)$ is such that $g(K) \leq h(K)$. Clearly $K7leH_0 = F(A)$. If $K \leq H_i$ then $g(K) \leq g(H_i) \cap h(H_i)$ and so $K \leq g^{-1}(g(H_i) \cap h(H_i)) = H_{i+1}$.

Hence, by induction $K \leq H_i$ for all $i \geq 0$ and so $K \leq \cap H_i = GSI(g,h)$ as required.

## 5. Examples Of Generalized Stable Image

We now give two examples of generalized stable image. Our first example shows that $GSI(g,h) \neq GSI(h,g)$ in general, even if both maps are injective.

**Example 5.1** : Define $g : F(x,y) \to F(a,b)$ by $g : x \to^2, y \to b$ and $h : x \to a, y \to b^2$. As $g(\langle x \rangle) \leq h(\langle x \rangle)$, we have that $x \in GSI(g,h)$ by Theorem 4.1, and similarly $y \notin GSI(h,g)$. On the other hand, $y \notin GSI(g,h)$ and $x \notin GSI(h,g)$, as $im(g) \cap im(h)$ is a proper subgroup of both $im(g)$ and $im(h)$ and so neither generalized stable image is the whole of $F(x, y)$. Hence $GSI(g, h) \neq GSI(h, g)$.

Our next example shows that generalized stable images are not necessarily finitely generated.

**Example 5.2** : Define $g : x \to ab$, $y \to 1$ and $h : x \to a^2$ ,$y \to b^2$. Then $im(g) \cap im(h)$ is trivial, and so the subgroup $H_1$ in the definition of the generalized stable image is $ker(g)$. We then see inductively that .$H_i = ker(g)$ for all $i \geq 0$ and so $GSI(g, h) = ker(g)$. As $ker(g)$ is a normal subgroup of infinite index in $F(A)$, we have that $GSI(g, h) = ker(g)$ is not finitely generated.

## 6. More On Generalized Stable Image

**Theorem 6.1** : $Eq(g, h) = Eq(g|GSI(g, h), h|GSI(g, h))$.

**Proof** : Clearly $Eq(g|GSI(g, h), h|GSI(g, h)) \leq Eq(g, h)$. For the other direction we prove that $Eq(g, h) \leq GSI(g, h)$, which is sufficient. So, let $x \in Eq(g, h)$. Then $x \in H_0$, while if $x \in H_i$ then $g(x) = h(x) \in g(H_i) \cap h(H_i)$ and so $x \in g^{-1}(g(h_i) \cap h(H_i)) = H_{i+1}$. Therefore, by induction we have that $x \in H_i$ for all $i \geq 0$ and so $x \in \cap H_i = GSI(g, h)$ as required.

If $h$ is injective then we can define $\psi(g|GSI(g, h), h|GSI(g, h)) \in End(GSI(g, h))$ as in Theorem 3.1. Crucially, $Eq(g, h)$ is the set of fixed points of this map.

**Theorem 6.2** : Let $g, h : F(A) \to F(B)$ be homomorphisms with $h$ injective. Then the endomorphism $\varphi(g, h) = \psi(g|GSI(g, h), h|GSI(g, h)) \in End(GSI(g, h))$ satisfies $Eq(g, h) = Fix(\varphi(g, h))$.

**Proof** : By Theorem 4.1, the maps $g|GSI(g, h)$ and $h|GSI(g, h)$ satisfy the conditions of Theorem 3.1, and so $Eq(g|GSI(g, h), h|GSI(g, h)) = Fix((g, h))$. The result then follows using Theorem 6.1.

**Theorem 6.3** : Let $g, h : F(A) \to F(B)$ be homomorphisms.

(a) If $h$ is injective then $rk(Eq(g, h)) \leq rk(GSI(g, h))$.

(b) If both $g$ and $h$ are injective then $Eq(g, h)$ is inert in $GSI(g, h)$.

**Proof** : Suppose $h$ is injective. Consider the map $\varphi(g, h) \in End(GSI(g, h))$ from Theorem 6.2, with $Fix(\varphi(g, h)) = Eq(g, h)$. Then $rk(Eq(g, h)) = rk(Fix(\varphi(g, h))) \leq rk(g(GSI(g, h)))$ by [16], as required.

Suppose both $g$ and $h$ are injective. Recalling that $\varphi(g,h) \in End(GSI(g,h))$ is defined by $x \to h^{-1}(g(x))$, as $g$ is injective the map $\varphi(g,h)$ is also injective. Hence, $Fix(\varphi(g,h))$ is inert in $g(GSI(g,h))$ by [6]. Finally the result follows as $Eq(g,h) = Fix(\varphi)$, by Theorem 6.2.

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