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COMMON FIXED POINT THEOREM IN COMPLEX VALUED METRIC SPACE UNDER WEAKER CONDITIONS

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Abstract

In this paper, a common fixed point result for two pairs of weakly compatible mappings in complex valued metric space has been proved using a weaker condition called property $(E.A)$. Our result improves the results of Ajam et al [3] in the sense that the completeness of the space has been relaxed and a weaker condition of property $(E.A)$ has been used.

1 Introduction

Azam et al. [3] established the existence of common fixed points of a pair of mappings satisfying a contractive condition by introducing the notion of complex-valued metric space. Then the complex-valued normed spaces and complex-valued Hilbert spaces were defined on complex-valued metric spaces. Azam et al [3] and Bhatt et al [4] used the rational inequality in a complex-valued metric space as contractive condition.

Key Words : *Common fixed point; coincidence point, contractive type mapping; complex valued metric space.* 2000 AMS Subject Classification : 47H10, 54H25.

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The concept of property $(E.A)$ in a complex-valued metric space was introduced by Verma and Pathak [7] to prove some common fixed point results for a quadruple of self-mappings satisfying a contractive condition of 'max' type. Their results generalize various theorems of ordinary metric spaces.

Ahmad et al [2] established some common fixed results for mappings fulfilling rational expressions on a closed ball in complex valued metric spaces, while Rafiq et al [6] proved some common fixed point theorems of weakly compatible mappings in complex valued metric spaces.

In this paper, we improve the results of Ajam et al [3] in the sense that the completeness of the space has been relaxed and a weaker condition of property $(E.A)$ has been used.

2 Preliminaries.

The following definitions are introduced by Azam et al [3]

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$.

Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$z_1 \precsim z_2$ if and only if

$$Re(z_1) \leq Re(z_2), Im(z_1) \leq Im(z_2)$$

Therefore one can infer $z_1 \precsim z_2$ if any one of the following conditions is satisfied:

(i) $Re(z_1) = Re(z_2), Im(z_1) < Im(z_2)$,

(ii) $Re(z_1) < Re(z_2), Im(z_1) = Im(z_2)$,

(iii) $Re(z_1) < Re(z_2), Im(z_1) < Im(z_2)$,

(iv) $Re(z_1) = Re(z_2), Im(z_1) = Im(z_2)$

From $(i), (ii)$ and (iii) we have $|z_1| < |z_2|$ and from (iv) we have $|z_1| = |z_2|$.

We write $z_1 \prec z_2$ only if (iii) is satisfied.

Also

$$0 \precsim z_1 \precsim z_2 \Rightarrow |z_1| \leq |z_2|$$
$$z_1 \precsim z_2, z_2 \prec z_3 \Rightarrow z_1 \prec z_3$$

Definition 2.1 *[3] Let X be a nonempty set whereas $\mathbb{C}$ be the set of complex numbers. Suppose that the mapping $d : X \times X \to \mathbb{C}$, satisfies:*

(1) $0 \precsim d(x, y)$*, for all* $x, y \in X$ *and* $d(x, y) = 0$ *if and only if* $x = y$;

(2) $d(x, y) = d(y, x)$ *for all* $x, y \in X$;

(3) $d(x, y) \precsim d(x, z) + d(z, y)$, *for all* $x, y, z \in X$.

Then d is called a complex valued metric on X, and (X, d) is called a complex valued metric space.

Example 2.1 *Let X be a nonempty set.*
Define the mapping $d : X \times X \to \mathbb{C}$ by

$$d(x, y) = \frac{|x - y|}{x^2 + y^2} e^{ik}, k \in \mathbb{R}, x, y \in X.$$

Then (X, d) is a complex valued metric space.

A point $x \in X$ is called interior point of a set $A \subseteq X$ whenever there exists $0 \prec r \in \mathbb{C}$ such that

$$B(x, r) = \{y \in X : d(x, y) \prec r\} \subseteq A$$

A point $x \in X$ is called a limit point of A whenever for every $0 \prec r \in \mathbb{C}$,

$$B(x, r) \cap (A - X) \neq \phi$$

A is called open whenever each element of A is an interior point of A. A subset $B \subseteq X$ is called closed whenever each limit point of B belongs to B.
The family

$$F = \{B(x, r) : x \in X, 0 \prec r\}$$

is a sub-basis for a Hausdorff topology τ on X.
Aamri and Moutawakil [1] developed several common fixed point theorems for mappings having the property $(E.A)$ on a metric space under rigorous contractive conditions.

Definition 2.2 *[1] A pair of self-mappings f, g defined on a metric space (X, d) is said to satisfy the property $(E.A.)$ if there exists a sequence $\{x_n\}$ in X such that*

$$\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} gx_n = t \in X.$$

Definition 2.3 *[4] A pair of self-mappings $A, S : X \to X$ is called weakly-compatible if they commute at their coincidence points. That is, if there be a point $u \in X$ such that $Au = Su$, then $ASu = SAu$, for each $u \in X$.*

In complex valued metric space, Verma and Pathak [7] defined property $(E.A)$ as follows:

Definition 2.4 *[7] Let $A, S : X \to X$ be two self-maps of a complex-valued metric space (X, d). The pair (A, S) is said to satisfy property $(E.A)$, if there exists a sequence $\{x_n\}$ in X such that*

$$\lim_{n \to +\infty} Ax_n = \lim_{n \to +\infty} Sx_n = t$$

for some $t \in X$.

Pathak et al [5] has that weakly compatibility and property $(E.A)$ are independent to each other (see Ex.2.5, Ex.2.6, Ex.2.7 of [5]).

3 Main Result.

Theorem 3.1 *Let (X, d) be a complex valued metric space and mappings $f, g, S, T : X \to X$ satisfying:*
(3.1.1) *$S(X) \subset g(X)$ and $T(X) \subset f(X)$;*
(3.1.2) $\forall x, y \in X$,

$$d(Sx, Ty) \precsim \lambda d(fx, gy) + \frac{\mu d(fx, Sx) d(gy, Ty)}{1 + d(fx, gy)}$$

where λ and μ are non-negative real numbers such that $\lambda + \mu < 1$,
(3.1.3) *the pairs (S, f) and (T, g) are weakly compatible,*
(3.1.4) *one of the pairs (S, f) and (T, g) satisfies the property $(E.A)$.*
If the range of one of the mappings f or g is a complete subspace of X, then mappings f, g, S and T have unique common fixed point in X.

Proof. If the pair (S, f) satisfies property $(E.A)$, then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n \to +\infty} Sx_n = \lim_{n \to +\infty} fx_n = z,$$

Proof. If the pair (S, f) satisfies property $(E.A)$, then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n \to +\infty} Sx_n = \lim_{n \to +\infty} fx_n = z,$$

for some $z \in X$.

Since $S(X) \subset g(X)$, there exists a sequence $\{y_n\}$ in X such that $gy_n = Sx_n$ for all

$n \in N$.
Which implies

$$\lim_{n\to+\infty} gy_n = \lim_{n\to+\infty} Sx_n = z$$

Now to show that

$$\lim_{n\to+\infty} Ty_n = z$$

From (3.1.2)

$$d(Sx_n, Ty_n) \precsim \lambda d(fx_n, gy_n) + \frac{\mu d(fx_n, Sx_n) d(gy_n, Ty_n)}{1 + d(fx_n, gy_n)}$$

Taking $n \to \infty$, we get

$$d(z, Ty_n) \precsim \lambda d(z, z) + \frac{\mu d(z, z) d(z, Ty_n)}{1 + d(z, z)}$$

Hence $\lim_{n\to+\infty} Ty_n = z$.
Similarly, If the pair (T, g) satisfies property $(E.A)$, then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to+\infty} Tx_n = \lim_{n\to+\infty} gx_n = u,$$

for some $u \in X$.
Since $T(X) \subset f(X)$, there exists a sequence $\{y_n\}$ in X such that $fy_n = Tx_n$ for all $n \in N$.
Which implies

$$\lim_{n\to+\infty} fy_n = \lim_{n\to+\infty} Tx_n = u$$

Now to show that

$$\lim_{n\to+\infty} Sy_n = u$$

From (3.1.2)

$$d(Sy_n, Tx_n) \precsim \lambda d(fy_n, gx_n) + \frac{\mu d(fy_n, Sy_n) d(gx_n, Tx_n)}{1 + d(fy_n, gx_n)}$$

Taking $n \to \infty$, we get

$$d(Sy_n, u) \precsim \lambda d(u, u) + \frac{\mu d(u, Sy_n) d(u, u)}{1 + d(u, u)}$$

Hence $\lim_{n\to+\infty} Sy_n = u$.

Now suppose $f(X)$ is complete, then $fu = z$ for some $u \in X$.

Therefore

$$\lim_{n\to+\infty} Sx_n = \lim_{n\to+\infty} Ty_n = \lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} gy_n = z = fu$$

Now we show that

$$Su = fu = z.$$

Using (3.1.2)

$$d(Su, Ty_n) \lesssim \lambda d(fu, gy_n) + \frac{\mu d(fu, Su) d(gy_n, Ty_n)}{1 + d(fu, gy_n)}$$

Let $n \to \infty$, then

$$d(Su, fu) \lesssim \lambda d(fu, fu) + \frac{\mu d(fu, Su) d(fu, fu)}{1 + d(fu, fu)}$$

Therefore

$$Su = fu = z$$

Hence u is a coincidence point of S and f.

Using the weak compatibility of the pair (S, f), we have

$$Sfu = fSu$$

or

$$Sz = fz.$$

Since $S(X) \subset g(X)$, there exists w in X such that

$$Su = gw.$$

Therefore

$$Su = fu = gw = z$$

To show that w is the coincidence point of (T, g) i.e. $Tw = gw = z$. Using (3.1.2)

$$d(Su, Tw) \lesssim \lambda d(fu, gw) + \frac{\mu d(fu, Su) d(gw, Tw)}{1 + d(fu, gw)}$$

Which gives

$$d(z, Tw) \lesssim \lambda d(z, z) + \frac{\mu d(z, z) d(z, Tw)}{1 + d(z, z)}$$

i.e. $Tw = z$. Hence

$$Tw = gw = z.$$

Therefore w is a coincidence point of (T, g).
Using weak compatibility of (T, g)

$$Tgw = gTw$$

or

$$Tz = gz$$

Therefore z is the common coincidence point of S, T, f and g.
To show z is the common fixed point of S, T, f and g.
Using (3.1.2)

$$d(Su, Tz) \precsim \lambda d(fu, gz) + \frac{\mu d(fu, Su) d(gz, Tz)}{1 + d(fu, gz)}$$

Which gives

$$d(z, Tz) \precsim \lambda d(z, Tz) + \frac{\mu d(z, z) d(gz, Tz)}{1 + d(z, Tz)}$$

i.e.

$$d(z, Tz) \precsim \lambda d(z, Tz)$$

which is a contradiction as $0 < \lambda < 1$. Therefore $Tz = z$.
Hence

$$Sz = Tz = fz = gz.$$

i.e. z is the common fixed point of S, T, f and g.
Similar result follows when we assume that $g(X)$ is complete.
Now to show the uniqueness, let $v \in X$ be another common fixed point of S, T, f and g i.e.

$$Sv = Tv = fv = gv = v.$$

Using (3.1.2),

$$d(Sz, Tv) \precsim \lambda d(fz, gv) + \frac{\mu d(fz, Sz) d(gv, Tv)}{1 + d(fz, gv)}$$

which gives

$$d(z, v) \precsim \lambda d(z, v) + \frac{\mu d(z, z) d(v, v)}{1 + d(z, v)}$$

i.e.

$$d(z,v) \lesssim \lambda d(z,v)$$

which is a contradiction as $0 < \lambda < 1$. Therefore $z = v$. for some $z \in X$.
Since $S(X) \subset g(X)$, there exists a sequence $\{y_n\}$ in X such that $gy_n = Sx_n$ for all $n \in N$.
Which implies

$$\lim_{n\to+\infty} gy_n = \lim_{n\to+\infty} Sx_n = z$$

Now to show that

$$\lim_{n\to+\infty} Ty_n = z$$

From (3.1.2)

$$d(Sx_n, Ty_n) \lesssim \lambda d(fx_n, gy_n) + \frac{\mu d(fx_n, Sx_n) d(gy_n, Ty_n)}{1 + d(fx_n, gy_n)}$$

Taking $n \to \infty$, we get

$$d(z, Ty_n) \lesssim \lambda d(z,z) + \frac{\mu d(z,z) d(z, Ty_n)}{1 + d(z,z)}$$

Hence $\lim_{n\to+\infty} Ty_n = z$.
Similarly, If the pair (T, g) satisfies property $(E.A)$, then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to+\infty} Tx_n = \lim_{n\to+\infty} gx_n = u,$$

for some $u \in X$.
Since $T(X) \subset f(X)$, there exists a sequence $\{y_n\}$ in X such that $fy_n = Tx_n$ for all $n \in N$.
Which implies

$$\lim_{n\to+\infty} fy_n = \lim_{n\to+\infty} Tx_n = u$$

Now to show that

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From (3.1.2)

$$d(Sy_n, Tx_n) \lesssim \lambda d(fy_n, gx_n) + \frac{\mu d(fy_n, Sy_n) d(gx_n, Tx_n)}{1 + d(fy_n, gx_n)}$$

Taking $n \to \infty$, we get

$$d(Sy_n, u) \precsim \lambda d(u,u) + \frac{\mu d(u, Sy_n) d(u,u)}{1 + d(u,u)}$$

Hence $\lim_{n \to +\infty} Sy_n = u$.

Now suppose $f(X)$ is complete, then $fu = z$ for some $u \in X$.

Therefore

$$\lim_{n \to +\infty} Sx_n = \lim_{n \to +\infty} Ty_n = \lim_{n \to +\infty} fx_n = \lim_{n \to +\infty} gy_n = z = fu$$

Now we show that

$$Su = fu = z.$$

Using (3.1.2)

$$d(Su, Ty_n) \precsim \lambda d(fu, gy_n) + \frac{\mu d(fu, Su) d(gy_n, Ty_n)}{1 + d(fu, gy_n)}$$

Let $n \to \infty$, then

$$d(Su, fu) \precsim \lambda d(fu, fu) + \frac{\mu d(fu, Su) d(fu, fu)}{1 + d(fu, fu)}$$

Therefore

$$Su = fu = z$$

Hence u is a coincidence point of S and f.

Using the weak compatibility of the pair (S, f), we have

$$Sfu = fSu$$

or

$$Sz = fz.$$

Since $S(X) \subset g(X)$, there exists w in X such that

$$Su = gw.$$

Therefore

$$Su = fu = gw = z$$

To show that w is the coincidence point of (T, g) i.e. $Tw = gw = z$.

Using (3.1.2)

$$d(Su, Tw) \precsim \lambda d(fu, gw) + \frac{\mu d(fu, Su) d(gw, Tw)}{1 + d(fu, gw)}$$

Which gives

$$d(z,Tw) \lesssim \lambda d(z,z) + \frac{\mu d(z,z)d(z,Tw)}{1+d(z,z)}$$

i.e. $Tw = z$. Hence

$$Tw = gw = z.$$

Therefore w is a coincidence point of (T,g). Using weak compatibility of (T,g)

$$Tgw = gTw$$

or

$$Tz = gz$$

Therefore z is the common coincidence point of S,T,f and g.

To show z is the common fixed point of S,T,f and g.

Using (3.1.2)

$$d(Su,Tz) \lesssim \lambda d(fu,gz) + \frac{\mu d(fu,Su)d(gz,Tz)}{1+d(fu,gz)}$$

Which gives

$$d(z,Tz) \lesssim \lambda d(z,Tz) + \frac{\mu d(z,z)d(gz,Tz)}{1+d(z,Tz)}$$

i.e.

$$d(z,Tz) \lesssim \lambda d(z,Tz)$$

which is a contradiction as $0 < \lambda < 1$. Therefore $Tz = z$.

Hence

$$Sz = Tz = fz = gz.$$

i.e. z is the common fixed point of S,T,f and g.

Similar result follows when we assume that $g(X)$ is complete.

Now to show the uniqueness, let $v \in X$ be another common fixed point of S,T,f and g i.e.

$$Sv = Tv = fv = gv = v.$$

Using (3.1.2),

$$d(Sz,Tv) \lesssim \lambda d(fz,gv) + \frac{\mu d(fz,Sz)d(gv,Tv)}{1+d(fz,gv)}$$

which gives

$$d(z,v) \lesssim \lambda d(z,v) + \frac{\mu d(z,z)d(v,v)}{1+d(z,v)}$$

i.e.

$$d(z,v) \lesssim \lambda d(z,v)$$

which is a contradiction as $0 < \lambda < 1$. Therefore $z = v$.
If we put $f = g = I$ in the Theorem 3.1, we get

Corollary 3.2 *Let (X,d) be a complex valued metric space and mappings $S,T : X \to X$ satisfying:*
(3.2.1) *$S(X) \subset T(X)$ and $T(X) \subset f(X)$;*
(3.2.2) $\forall x, y \in X$,

$$d(Sx,Ty) \lesssim \lambda d(x,y) + \frac{\mu d(x,Sx)d(y,Ty)}{1+d(x,y)}$$

where λ and μ are non-negative real numbers such that $\lambda + \mu < 1$,
(3.2.3) *the pair (S,T) is weakly compatible,*
(3.2.4) *the pairs (S,T) satisfies the property $(E.A)$.*
If the range of one of the mappings $T(X)$ or $S(X)$ is a complete subspace of X, then mappings S and T have unique common fixed point in X.

If we put $f = g = I$ and $S = T$ in the Theorem 3.1, we get

Corollary 3.3 *Let (X,d) be a complex valued metric space and mapping $T : X \to X$ satisfying:*

$$d(Tx,Ty) \lesssim \lambda d(x,y) + \frac{\mu d(x,Tx)d(y,Ty)}{1+d(x,y)}$$

$\forall x, y \in X$, where λ and μ are non-negative real numbers such that $\lambda + \mu < 1$.
If the range of the mapping $T(X)$ is a complete subspace of X, then the mapping T has a unique common fixed point in X.

Reference

[1] Aamri A., Moutawakil D.El, Some new common fixed point theorems under strict contractive conditions. J Math Anal Appl. 270 (2002) 181-188.

[2] Ahmad J., Azam A., Saejung S. , Common fixed point results for contractive mappings in complex valued metric spaces, Fixed Point Theory Appl., 2014 (2014) 11.

[3] Azam A., Fisher B., Khan M., Common fixed point theorems in complex valued metric spaces, Numer. Funct. Anal. Optim. 32 (3)(2011) 243-253.

[4] Bhatt S., Chaukiyal S., Dimri R.C., A common fixed point theorem for weakly compatible maps in complex-valued metric spaces, Int. J. Math. Sci. Appl. 1 (3) (2011) 1385-1389.

[5] Pathak H.K., Lop R.R., Verma R.K., A common fixed point theorem of integral type using implicit relation, Nonlinear Funct. Anal. Appl. 15 (1)(2009) 1-12.

[6] Rafiq A., Rouzkard F., Imdad M., Shin Min Kang, Some common fixed point theorem of weakly compatible mappings in complex valued metric spaces, Mitteilungen Klosterneuburg 65 (2015) 1 422-432.

[7] Verma R.K., Pathak H.K., Common fixed point theorems using property (E.A) in complex valued metric spaces, Thai Journal of Mathematics, 11(2) (2013) 275-283.