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A 3-CONNECTED COMPONENT OF A CUBIC SUBGRAPH OF THE MIDDLE TWO LAYER'S GRAPH USING MODULAR MATCHINGS

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Abstract

Modular natchings were first studied by Duffus, Kierstead and Snevily in 1994 with the aim of using them to solve the middle two layers conjecture. The conjecture has now been proved by T. Mütze (2014). In 2005, Horák, Kaiser, Rosenfeld and Ryjáček used them to show that the middle two layers graph has a hamiltonian prism. They also showed that the graph formed by 3 consecutive modular matchings is connected. In this paper we show a stronger result that a component of two consecutive modular matchings and any other third modular matching is 3-connected.

1. Introduction

In 1971 Tutte made a conjecture that every 3-connected cubic bipartite graph is hamiltonian. Although this was refuted by Horton [3] in 1982 by constructing a 3-connected

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bicubic graph with 92 vertices having no hamiltonian cycle. In fact, earlier(in 1976) he had constructed such a graph with 96 vertices. In this paper 3-connected cubic bipartite graphs are formed by suitably chosing 3 modular matchings.

This provides with another class of such graphs, in which Tutte conjecture can be tested or refused. We will first introduce the necessary notation used to define modular matchings. They were introduced by Duffus, Kierstead and Snevily [1] in connection with the middle two layer's conjecture which says that the middle two layer's graph is hamiltonian. Since any hamilton cycle can be decomposed into two perfect matchings, if there is a large collection of matchings at hand, to work with, perhaps two of them can be joined appropriately to give a hamilton cycle. It was with this in mind, that the modular matchings were introduced and they provided an aid to understand the middle to layer's graph, which will also be defined below.

The n-dimensional discrete cube, $\mathbf{Q_n}$ is defined as the graph (V, E) where V consists of all the subsets of $[n]$, with $[n]$ being the n-element set $\{1, 2, \ldots, n\}$. And the edge set $E = \{(A, B) : \quad | A \triangle B |= 1\}$. The collection of all the j-element subsets of $[n]$ will be denoted by $\mathbf{R_j}$ and called the j'th layer of the discrete cube. When n is odd, say $n = 2k + 1$, the middle two layers have the same size since $\binom{n}{k} = \binom{n}{k+1}$. The graph induced by these middle two layers $\mathbf{R_k}$ and $\mathbf{R_{k+1}}$ of the discrete cube $\mathbf{Q_{2k+1}}$ is called the middle two layers graph and will be denoted by $\mathbf{B_k}$. Given any j-element subset A, we will write it as $A =< a_1, a_2, \ldots, a_j >$ where we use the convention $a_1 < a_2 < \cdots < a_j$, and we write the complement of the set A as $A^c =< \bar{a}_1, \bar{a}_2 \ldots \bar{a}_{n-j} >$, where it is understood that $\bar{a}_1 > \bar{a}_2 > \ldots > \bar{a}_{n-j}$. So, whenever there is no chance of confusion we will write a_r to mean that it is the r'th smallest element of A and $\bar{a}_r$ will denote the r'th largest element of A^c. S_{2k+1}, as usual represents the symmetric group on $[2k + 1]$. The subset $\{x - r, x - r + 1, \ldots x - 1\}$ which is a segment of r contiguous elements will be denoted by any of the following self-evident ways: $[x - r, x), [x - r, x - 1], (x - r - 1, x), (x - r, x - 1]$ depending on the inclusion/exclusion of the endpoints and where the addition and subtraction is done modulo $2k + 1$. A 1-*factor* of a graph G, is a spanning $1-$regular subgraph of G. It is a perfect matching of G. Decomposition of the edge-set $E(G)$ into a collection of 1-*factors* is called a 1-*factorization* of G. So a 1-*factorization* of $\mathbf{B_k}$ is a collection of $k+1$ disjoint perfect matchings. The modular matchings defined provides one such factorization of $\mathbf{B_k}$. For convenience, we will consider the modular matching in $\mathbf{B_k}$ as an injection $\mathbf{m} : \mathbf{R_k} \to \mathbf{R_{k+1}}$, such that A is adjacent to $\mathbf{m}(A)$.

Definition 1.1 : Given $n = 2k+1$, the i-th modular matching for $i = 1, 2, \cdots k+1$ is defined as the function $\mathbf{m} : \mathbf{R_k} \to \mathbf{R_{k+1}}$ where $\mathbf{m_i}(A) = A \cup \{\bar{a}_j\}$ where j is given by

$$j = i + \sum_{a \in A} a \pmod{k+1}$$

Thus the i'th modular matching $\mathbf{m_i}$ in $\mathbf{B_k}$ consists of edges of the form $(A, \mathbf{m_i}(A))$, where A is any k-element subset of $[n]$. Also for the set A, note that we have the following relation between the element $\bar{a}_j$ and j.

$$j = 2k + 1 - \bar{a}_j - \mid a \in A : a > \bar{a}_j \mid +1$$

We also give the following definition which gives a rule from $\mathbf{m} : \mathbf{R_{k+1}} \to \mathbf{R_k}$ which is used in [**?**] to show that the modular matchings are well-defined. It is, in fact the inverse of the modular matching function.

Definition 1.2 : Given $n = 2k+1$, for $i = 1, 2, \cdots k+1$ let $\mathbf{b_i}$ be the function $\mathbf{b_i} : \mathbf{R_{k+1}} \to \mathbf{R_k}$ where $\mathbf{b_i}(B) = B \setminus \{a_l\}$ where l is given by

$$l \equiv i + \sum_{b \in B} b \pmod{k+1}$$

Here, as before, note that b_l is the l-th smallest element of B.

Definition 1.3 : Let $H_{a,b,c}(k)$ denote the spanning subgraph of $\mathbf{B_k}$ whose edge set is $\{\mathbf{m_a}, \mathbf{m_b}, \mathbf{m_c}\}$. If $a = b-1$ and $c = b+1$, that is, the matchings are consecutive, we will use the notation $H_b(k)$ to denote the resulting subgraph. We will use $H_{a,b,c}$ or H_b, if the value of k is clear from the context.

The above two subgraphs are cubic. Also, let $H_{a,b}$ denote the 2-regular subgraph of $\mathbf{B_k}$ whose edge set is $\{\mathbf{m_a}, \mathbf{m_b}\}$. The *weight* of the set A is the sum of all the elements of A. For any permutation α of $[n]$ and a subset A of $[n]$, we let $\alpha(A)$ be the set obtained by permuting the elements of A with α. One such set obtained by using the rotation permutation $\sigma = (1, 2, \ldots n)$ will be of particular interest. The resulting set, $\sigma(A)$, also called the shift of A, is obtained by shifting each of the element by $1 \pmod{2k+1}$. One immediately sees that $\sigma^{2k+1}(A) = A$.

2. On Connectivity

Horák, Kaiser, Rosenfeld and Ryjáček [5] showed that the subgraph H_i is connected. They did this by proving that every set A, which is not equal to $\{1, 2, \cdots, k\}$ is always connected to another set of a smaller weight, using only the edges of H_i. This, thus implied that every set is in the middle two layers graph, $\mathbf{B_k}$ is connected to the set with the smallest weight $\{1, 2, \cdots, k\}$ in the subgraph H_i. Hence the entire graph is connected. The same result was proved using a different method by Kelkar and Maharshi [12]. The corollaries from [5] from their following theorem will be used to prove the extended result on 3-connectedness of a suitably chosen cubic subgraph of $\mathbf{B_k}$

Theorem 2.1 [**Horák, Kaiser, Rosenfeld and Ryjáček**] : Let A be a k-set on a cycle C of the subgraph $H_{i,i+1}$ consisting of t segments, then $\sigma(A)$ lies on the cycle C and $d_C(A, \sigma(A)) = 2t$ or $2t + 1$.

Corollary 2.1 [**Horák, Kaiser, Rosenfeld and Ryjáček**] : Let C be the cycle of $H_{i,i+1}$ containing the k-set A of $\mathbf{B_k}$, then $d_C(A, \sigma(A)) = d_C(\sigma^j(A), \sigma^{j+1}(A))$ where $j \in \{1, 2, \ldots, 2k\}$

Corollary 2.2 [**Horák, Kaiser, Rosenfeld and Ryjáček**] : Let C be the cycle of $H_{i,i+1}$ containing the k-set A of $\mathbf{B_k}$ and let (A, B) be an $\mathbf{m_l}$ edge where $l \neq i, i + 1$. Either B is not on the cycle C or $d_C(A, B) > d_C(A, \sigma(A))$.

We will now prove the following theorem which essentially says that any component of the cubic subgraph of $\mathbf{B_k}$ formed by two consecutive modular matchings and any other third matching is 3-connected.

Theorem 2.2 : Any component H of $H_{i,i+1,l}$ is 3-connected where $l \notin \{i - 1, i\}$.

Proof : First, note that since H is connected and a cubic graph, it will be 3-connected if it is 3-edge connected. Now, let F be an edge cut of H and let $x \in \mathbf{m_a}$ be an edge in F where $a \in \{i - 1, i, l\}$. $|F| \geq 2$ since the cycle through x in the 2-factor of $\mathbf{m_a} \cup \mathbf{m_b}$ where $b \neq a$ and $b \in \{i - 1, i, l\}$ will contain another edge $y \neq x$ such that $y \in F$. Suppose for contradiction that $|F| = 2$. If $y \in \mathbf{m_b}$ where $b \neq a$, then we consider a cycle through x in the 2-factor $\mathbf{m_a} \cup \mathbf{m_c}$ where $c \in \{i - i, i, l\}$ and $c \notin \{a, b\}$. Then, F contains two edges from this cycle and along with $y \in \mathbf{m_b}$, we have $|F| \geq 3$. Hence both x and y are from the same matching $\mathbf{m_a}$.

First, suppose that $a \in \{i - 1, i\}$, so both the edges $x, y \in \mathbf{m_a}$. Consider the cycle C in $\mathbf{m_{i-1}} \cup \mathbf{m_i}$ passing through x (and also y, otherwise we will be done!). Let P_1 and

P_2 be the two paths of $C \setminus \{x, y\}$ with P_1 having the smaller length. Let P_1 be in the component R of $H \setminus F$ and $S = H \setminus R$. Let A_x be the vertex in R that is incident with the edge x of the edge-cut F. If $\sigma(A_x) \notin P_2$, then $\sigma^{2k}(A_x)$ will be on P_2. This is true since by corollary 3.1 we have $d_C(A_x, \sigma(A_x)) = d_C(A_x, \sigma^{2k}(A_x))$ and since $|P_1| \leq |P_2|$, if $\sigma^{2k}(A)$ also lies on P_1, the two distances will not be equal. Hence $\sigma^t(A_x) \in P_2$ either for $t = 1$ or for $t = 2k$. Let $B = \mathbf{m_l}(A_x)$. If B is not on the cycle C, consider the cycle C^1 of $\mathbf{m_{i-1}} \cup \mathbf{m_i}$ passing through the set B. The two cycles do not intersect, so C^1 lies completely in R. All the shifts of B are in C^1 so also in R. But then, $\sigma^t(B) \in R$ and we have $\sigma^t(A_x) \in S$, so the edge $(\sigma^t(B), \sigma^t(A_x)) \in \mathbf{m_l}$ will be in the edge-cut F making $|F| \geq 3$. So, B is on the cycle C. If it is on P_2, then the edge (A_x, B) is in the edge-cut and we have $|F| \geq 3$. So B is on P_1. This forces $\sigma^t(B)$ on P_1 and hence the edge $(\sigma^t(B), \sigma^t(A_x)) \in \mathbf{m_l}$ will again be in F, making $|F| \geq 3$

Now consider the case when both the edges in F are in $\mathbf{m_l}$. Let $x = (A_x, B_x) \in \mathbf{m_l}$ be an edge of F. Let C_1 be the cycle in $\mathbf{m_{i-1}} \cup \mathbf{m_i}$ passing through A_x and C_2 be the cycle in $\mathbf{m_{i-1}} \cup \mathbf{m_i}$ passing through B_x. If any of these two edges lie in F, its cardinality will be greater than 3. Hence C_1 and C_2 lie in different components of $V \setminus F$ and do not intersect. But, $\sigma^j(A_x) \in C_1$ and $\sigma^j(B_x) \in C_2$ for all $j = 0 \ldots 2k$. And hence $(\sigma^j(A_x), \sigma^j(B_x)) \in \mathbf{m_l}$ are all edges between C_1 and C_2 for $j = 0 \ldots 2k$. This gives $|F| \geq 2k + 1 \geq 3$. So, we conclude that G is 3-edge connected and hence 3-connected.

3. Conclusion

Since we already know that the subgraph H_i is connected. By the above result, it will be 3-connected. If we can use a similar technique to show that the subgraph formed by two consecutive modular matchings and any other third matching is connected, we will obtain a stronger result. In fact, the main motivation to give an alternate proof regarding connectivity of H_i was to show this stronger result. Efforts are continued in this direction.

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