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LOCALLY STANDARD $(\mathbb{Z}_2)^m$-MANIFLODS OVER POLYGONS AND PRODUCTS OF TWO SIMPLICES

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Abstract

In this paper, we calculate the number of equivariant homeomorphism classes of locally standard $(\mathbb{Z}_2)^m$-manifolds over polygons for $m \geq 2$ and the number of Davis-Januszkiewicz equivalence classes of locally standard $(\mathbb{Z}_2)^m$-manifolds over $\Delta^{n_1} \times \Delta^{n_2}$ for $m \geq n_1 + n_2$, where Δ^{n_i} is an n_i-simplex for $i = 1, 2$.

1. Introduction Let M^n be an n-dimensional closed manifold with a locally standard $(\mathbb{Z}_2)^n$-action (see [7]) and $\pi : M^n \to X^n = M^n/(\mathbb{Z}_2)^n$ be the orbit map. Then X^n is a nice n-manifold with corners and the $(\mathbb{Z}_2)^n$-action determines a characteristic function ν_π (also called $(\mathbb{Z}_2)^n$-coloring) on the facets of X^n. In particular, when X^n is a simple convex polytope, M^n is a small cover over X^n and there is a standard construction to recover M^n from the characteristic function ν_π on X^n (see [7]). Generally, we need an additional data to recover M^n. In [10], Yu defined a general notion of locally standard $(\mathbb{Z}_2)^m$-actions on n-dimensional closed manifolds for all $m \geq 1$, which is actually a

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generalization of the notion of locally standard 2-torus manifold defined in [8] where m is required to be equal to n.

This paper is motivated by the works [2], [3], [4], [5] and [9], which enumerate the number of Davis-Januszkiewicz equivalence classes and equivariant homeomorphism classes of small covers over a specific polytope. By using the ideas in the above papers, we determine the number of equivariant homeomorphism classes of locally standard $(\mathbb{Z}_2)^m$-manifolds over polygons for $m \geq 2$ (see Theorem 3.2). Moreover, we calculate the number of Davis-Januszkiewicz equivalence classes of locally standard $(\mathbb{Z}_2)^m$-manifolds over $\Delta^{n_1} \times \Delta^{n_2}$ for $m \geq n_1 + n_2$, where Δ^{n_i} is an n_i-simplex for $i = 1, 2$ (see Theorem 4.1).

The paper is organized as follows. In Section 2, we review the notion of locally standard $(\mathbb{Z}_2)^m$-actions on n-dimensional manifolds and basic results about locally standard $(\mathbb{Z}_2)^m$-manifolds over n-dimensional simple convex polytopes. In Section 3, we determine the number of locally standard $(\mathbb{Z}_2)^m$-manifolds over polygons up to equivariant homeomorphism. In Section 4, we calculate the number of locally standard $(\mathbb{Z}_2)^m$-manifolds over $\Delta^{n_1} \times \Delta^{n_2}$ up to Davis-Januszkiewicz equivalence.

2. Preliminaries

First, let us give the definition of locally standard $(\mathbb{Z}_2)^m$-actions on n-dimensional manifolds for any $m \geq 1$ (see [10]). Let $g = (g_1, \cdots, g_m)$ be an arbitrary element of $(\mathbb{Z}_2)^m$.

(1) If $m \leq n$, the standard $(\mathbb{Z}_2)^m$-action on $\mathbb{R}^n$ is:

$$(x_1, \cdots, x_n) \longmapsto ((-1)^{g_1} x_1, \cdots, (-1)^{g_m} x_m, x_{m+1}, \cdots, x_n),$$

whose orbit space is $\mathbb{R}_+^{n,m} := \{(x_1, \cdots, x_n) | x_i \geq 0 \text{ for } 1 \leq i \leq m\}$.

(2) For $m > n$, the standard $(\mathbb{Z}_2)^m$-action on $\mathbb{R}^n \times (\mathbb{Z}_2)^{m-n}$ is:

$$((x_1, \cdots, x_n), (h_1, \cdots, h_{m-n})) \longmapsto$$
$$(((-1)^{g_1} x_1, \cdots, (-1)^{g_n} x_n), (g_{n+1} + h_1, \cdots, g_m + h_{m-n})),$$

whose orbit space is $\mathbb{R}_+^{n,n}$.

Suppose that $(\mathbb{Z}_2)^m$ acts effectively on an n-dimensional closed manifold M^n. A local isomorphism of M^n with the standard action consists of:

(1) a group automorphism $\sigma : (\mathbb{Z}_2)^m \to (\mathbb{Z}_2)^m$;

(2) $(\mathbb{Z}_2)^m$-stable open sets V in M^n and U in $\mathbb{R}^n$ (if $m \leq n$) or $\mathbb{R}^n \times (\mathbb{Z}_2)^{m-n}$ (if $m \geq n$);

(3) a σ-equivariant homeomorphism $f : V \to U$, i.e. $f(g \cdot v) = \sigma(g) \cdot f(v)$ for

any $g \in (\mathbb{Z}_2)^m$ and $v \in V$.

M^n is locally isomorphic to the standard action if each point of M^n is in the domain of some local isomorphism. Under this condition, we say the $(\mathbb{Z}_2)^m$-action on M^n is locally standard and say that M^n is a locally standard $(\mathbb{Z}_2)^m$-manifold over $M^n/(\mathbb{Z}_2)^m$.
Now, suppose that M^n is a locally standard $(\mathbb{Z}_2)^m$-manifold over $M^n/(\mathbb{Z}_2)^m$. Then the orbit space $X^n = M^n/(\mathbb{Z}_2)^m$ is a nice manifold with corners (see [6] for the details of a nice manifold with corners). Suppose $\pi : M^n \to X^n$ is the orbit map. Let the set of facets (faces of codimension one) of X^n be $\mathcal{F}(X^n) = \{F_1, \cdots, F_\ell\}$. The characteristic function ($(\mathbb{Z}_2)^m$-coloring) $\nu_\pi : \mathcal{F}(X^n) \to (\mathbb{Z}_2)^m$ is defined as follows:

$\nu_\pi(F_j) =$ the element of $(\mathbb{Z}_2)^m$ that fixes $\pi^{-1}(F_j)$ pointwise.

We may find that whenever $F_{j_1} \cap \cdots \cap F_{j_s} \neq \emptyset, \{\nu_\pi(F_{j_1}), \cdots, \nu_\pi(F_{j_s})\}$ must be linearly independent vectors in $(\mathbb{Z}_2)^m$ over $\mathbb{Z}_2$.

An n-dimensional convex polytope is said to be simple, if exactly n facets meet at each of its vertices (see [11]). An n-dimensional simple convex polytope is obviously a nice n-manifold with corners. If $m < n$, the dimension of any face of the orbit space X^n is at least $n - m$. So X^n must not be a simple convex polytope when $m < n$. In the rest of the paper, suppose $m \geq n$ and that X^n is an n-dimensional simple convex polytope. In fact, a locally standard $(\mathbb{Z}_2)^n$-manifold over an n-dimensional simple convex polytope X^n is just a small cover over X^n.

In [10], Yu gave a reconstruction process of M^n by using the characteristic function ν_π and the product bundle $X^n \times (\mathbb{Z}_2)^m$ over X^n up to equivariant homeomorphism. Following Davis and Januszkiewicz [7], we say that two locally standard $(\mathbb{Z}_2)^m$-manifolds M^n and N^n over X^n are Davis-Januszkiewicz equivalent (or simply D-J equivalent) if there is a homeomorphism $f : M^n \to N^n$ together with an element $\sigma \in \mathrm{GL}(m, \mathbb{Z}_2)$ such that

(1) $f(g \cdot x) = \sigma(g) \cdot f(x)$ for all $g \in (\mathbb{Z}_2)^m$ and $x \in M^n$, and

(2) f induces the identity on the orbit space X^n.

Two locally standard $(\mathbb{Z}_2)^m$-manifolds M^n and N^n over X^n are equivariantly homeomorphic if there is a homeomorphism $f: M^n \to N^n$ such that $f(g \cdot x) = g \cdot f(x)$ for all $g \in (\mathbb{Z}_2)^m$ and $x \in M^n$. Let $\nu(X^n, (\mathbb{Z}_2)^m) := \{\nu : \mathcal{F}(X^n) \to (\mathbb{Z}_2)^m \mid \nu(F_{j_1}), \cdots, \nu(F_{j_s})$ are linearly independent vectors in $(\mathbb{Z}_2)^m$ whenever $F_{j_1} \cap \cdots \cap F_{j_s} \neq \emptyset\}$.
Then we have

Theorem 2.1 ([10]) : The set of D-J equivalence classes of locally standard $(\mathbb{Z}_2)^m$-manifolds over X^n bijectively corresponds to the coset $\nu(X^n, (\mathbb{Z}_2)^m)/GL(m, \mathbb{Z}_2)$, where $\mathrm{GL}(m, \mathbb{Z}_2)$ acts on $\nu(X^n, (\mathbb{Z}_2)^m)$ via automorphisms of the coefficient $(\mathbb{Z}_2)^m$.

Remark 1 : Without loss of generality, we assume that $F_1, \cdots, F_n$ of $\mathcal{F}(X^n)$ meet at one vertex p of X^n. Let $e_1, \cdots, e_m$ be the standard basis of $(\mathbb{Z}_2)^m$. Write $A(X^n) = \{\nu \in \nu(X^n, (\mathbb{Z}_2)^m) | \nu(F_i) = e_i, i = 1, \cdots, n\}$. In fact, $A(X^n)$ is the orbit space of $\nu(X^n, (\mathbb{Z}_2)^m)$ under the action of $\mathrm{GL}(m, \mathbb{Z}_2)$. By Theorem 2.1, the order $|A(X^n)|$ of $A(X^n)$ is the number of D-J equivalence classes in locally standard $(\mathbb{Z}_2)^m$-manifolds over X^n.

Let X^n be a simple convex polytope of dimension n. All faces of X^n form a poset (i.e., a partially ordered set by inclusion). An automorphism of $\mathcal{F}(X^n)$ is a bijection from $\mathcal{F}(X^n)$ to itself which preserves the poset structure of all faces of X^n, and by $Aut(\mathcal{F}(X^n))$ we denote the group of automorphisms of $\mathcal{F}(X^n)$. One can define the right action of $Aut(\mathcal{F}(X^n))$ on $\nu(X^n, (\mathbb{Z}_2)^m)$ by $\nu \times h \longmapsto \nu \circ h$, where $\nu \in \nu(X^n, (\mathbb{Z}_2)^m)$ and $h \in Aut(\mathcal{F}(X^n))$.

Theorem 2.2 : The set of equivariant homeomorphism classes of all n-dimensional locally standard $(\mathbb{Z}_2)^m$-manifolds over X^n bijectively corresponds to the coset
$\nu(X^n, (\mathbb{Z}_2)^m)/Aut(\mathcal{F}(X^n))$.

3. Locally Standard $(\mathbb{Z}_2)^m$-manifolds Over Polygons

From [7], we know that the connected sum Q^2 of $k-2$ $\mathbb{R}P(2)'s$ is a small cover over k-gon P_k, where $\mathbb{R}P(2)$ is the 2-dimensional real projective space. So Q^2 is a locally standard $(\mathbb{Z}_2)^2$-manifold over P_k. We can easily extend locally standard $(\mathbb{Z}_2)^2$-action on Q^2 to locally standard $(\mathbb{Z}_2)^m$-action on $Q^2 \times (\mathbb{Z}_2)^{m-2}$ with the orbit space unchanged for $m > 2$. Thus, $Q^2 \times (\mathbb{Z}_2)^{m-2}$ is a locally standard $(\mathbb{Z}_2)^m$-manifold over P_k for $m > 2$. A coloring on k-gon P_k (with $2^m - 1$ colors) means to color edges of P_k in such a way that any adjacent edges have different colors.

Lemma 3.1 : $|\nu(P_k, (\mathbb{Z}_2)^m)| = (2^m - 2)^k + (-1)^k(2^m - 2)$.

Proof : Let $S(k)$ be a segment with $k+1$ vertices including the endpoints, so $S(k)$ has k segments. The number of coloring segments of $S(k)$ with $2^m - 1$ colors in such a way that any adjacent edges have different colors is $(2^m - 1) \cdot (2^m - 2)^{k-1}$. If the two end segments have different colors, then it produces a coloring on P_k by gluing the end points of $S(k)$. If the two end segments have the same color, then it produces a coloring on P_{k-1} by gluing the end segments of $S(k)$. Thus, we have that

(a) $|\nu(P_k, (\mathbb{Z}_2)^m)| + |\nu(P_{k-1}, (\mathbb{Z}_2)^m)| = (2^m - 1) \cdot (2^m - 2)^{k-1}$.

It follows that

$$\begin{aligned}
&|\nu(P_k, (\mathbb{Z}_2)^m)| - (2^m - 2)|\nu(P_{k-1}, (\mathbb{Z}_2)^m)| \\
&= -(|\nu(P_{k-1}, (\mathbb{Z}_2)^m)| - (2^m - 2)|\nu(P_{k-2}, (\mathbb{Z}_2)^m)|) \\
&= \cdots \\
&= (-1)^{k-3}(|\nu(P_3, (\mathbb{Z}_2)^m)| - (2^m - 2)|\nu(P_2, (\mathbb{Z}_2)^m)|).
\end{aligned}$$

and a observation shows that $|\nu(P_3, (\mathbb{Z}_2)^m)| = (2^m - 1) \cdot (2^m - 2) \cdot (2^m - 3)$ and $|\nu(P_2, (\mathbb{Z}_2)^m)| = (2^m - 1) \cdot (2^m - 2)$, so

$$|\nu(P_k, (\mathbb{Z}_2)^m)| - (2^m - 2)|\nu(P_{k-1}, (\mathbb{Z}_2)^m)| = (-1)^k(2^m - 1) \cdot (2^m - 2).$$

The lemma then follows from (a) and (b). □

By $a_1, \cdots, a_k$ we denote all edges of k-gon P_k in their general order. Let x, y be two automorphisms of $Aut(\mathcal{F}(P_k))$ with the following properties respectively:

(1) $x(a_i) = a_{i+1} (i = 1, 2, \cdots, k-1), x(a_k) = a_1$;

(2) $y(a_i) = a_{k+1-i} (i = 1, 2, \cdots, k)$.

Then, all automorphisms of $Aut(\mathcal{F}(P_k))$ can be written in a simple form as follows:

x^j or $x^j y$, where $j \in \mathbb{Z}_k$.

Theorem 3.2 : Let φ denote the Euler's totient function, that is, $\varphi(1) = 1$ and $\varphi(N)$ for a positive integer $N (N \geq 2)$ is the number of positive integers both less than N and coprime to N. Let $E(P_k)$ denote the number of equivariant homeomorphism classes of locally standard $(\mathbb{Z}_2)^m$-manifolds over P_k. Then

$$E(P_k) = \frac{1}{2k}\{\sum_{d>1,d|k}\varphi(\tfrac{k}{d})|\nu(P_d,(\mathbb{Z}_2)^m)| + \tfrac{1+(-1)^k}{2}\cdot(2^m-1)\cdot(2^m-2)^{\frac{k}{2}}\cdot\tfrac{k}{2}\}.$$

Proof : The famous Burnside Lemma (see [1]) says that if G is a finite group acting on a set X, then

$$|X/G| = \frac{1}{|G|}\sum_{g\in G}|X^g|$$

where $X^g = \{x \in X | gx = x\}$. From Theorem 2.2 and Burnside Lemma, we have that

$$\text{(a)}\quad E(P_k) = \frac{1}{2k}\sum_{j=0}^{k-1}(|\nu(P_k,(\mathbb{Z}_2)^m)^{x^j}| + |\nu(P_k,(\mathbb{Z}_2)^m)^{x^jy}|)$$

Let d be the greatest common divisor of j and k. Then all edges of P_k are divided into d orbits under the action of $g = x^j$, and each orbit contains $\frac{k}{d}$ edges. Thus, each $(\mathbb{Z}_2)^m$-coloring of $\nu(P_k,(\mathbb{Z}_2)^m)^{x^j}$ gives the same coloring on all $\frac{k}{d}$ edges of each orbit. This means that if $d \neq 1, |\nu(P_k,(\mathbb{Z}_2)^m)^{x^j}| = |\nu(P_d,(\mathbb{Z}_2)^m)|$. If $d = 1$, then all edges of P_k have the same coloring, which is impossible by the linear independence condition of $(\mathbb{Z}_2)^m$-colorings in $\nu(P_k,(\mathbb{Z}_2)^m)$. On the other hand, for every $d > 1$, there are exactly $\varphi(\frac{k}{d})$ automorphisms of the form x^j, each of which divides all edges of P_k into d orbits. Thus

$$\text{(b)}\quad \sum_{j=0}^{k-1}|\nu(P_k,(\mathbb{Z}_2)^m)^{x^j}| = \sum_{d>1,d|k}\varphi(\frac{k}{d})|\nu(P_d,(\mathbb{Z}_2)^m)|$$

At the same time, since x^jy is a reflection obtained by x and y, we have

$$\text{(c)}\quad |\nu(P_k,(\mathbb{Z}_2)^m)^{x^jy}| = \begin{cases} (2^m-1)\cdot(2^m-2)^{\frac{k}{2}}, & \text{when } k \text{ is even and } j \text{ is odd},\\ 0, & \text{otherwise.}\end{cases}$$

Putting (b) and (c) into (a), we obtain the formula in the theorem. □

4. Locally Standard $(\mathbb{Z}_2)^m$-manifolds Over $\Delta^{n_1} \times \Delta^{n_2}$

From [7], we have that $\mathbb{R}P(n_1) \times \mathbb{R}P(n_2)$ is a small cover over $\Delta^{n_1} \times \Delta^{n_2}$. Thus, $\mathbb{R}P(n_1)\times\mathbb{R}P(n_2)$ is a locally standard $(\mathbb{Z}_2)^n$-manifold over $\Delta^{n_1}\times\Delta^{n_2}$ with $n_1+n_2 = n$. Similarly, we have that $\mathbb{R}P(n_1)\times\mathbb{R}P(n_2)\times(\mathbb{Z}_2)^{m-n}$ is a locally standard $(\mathbb{Z}_2)^m$-manifold over $\Delta^{n_1} \times \Delta^{n_2}$ for $m > n$.

To be convenient, we introduce the following marks. By $F'_1, \cdots, F'_{n_1+1}$ we denote all facets of n_1-simplex Δ^{n_1}, and by $F'_{n_1+2}, \cdots, F'_{n_1+n_2+2}$ we denote all facets of n_2-simplex

Δ^{n_2}. Set $\mathcal{F}' = \{F_i = F_i' \times \Delta^{n_2} | 1 \leq i \leq n_1+1\}$ and $\mathcal{F}'' = \{F_i = \Delta^{n_1} \times F_i' | n_1+2 \leq i \leq n_1+n_2+2\}$. Then $\mathcal{F}(\Delta^{n_1} \times \Delta^{n_2}) = \mathcal{F}' \bigcup \mathcal{F}''$.

Next, we determine the number of locally standard $(\mathbb{Z}_2)^m$-manifolds over $\Delta^{n_1} \times \Delta^{n_2}$ up to D-J equivalence.

Theorem 4.1 : Let DJ$(\Delta^{n_1} \times \Delta^{n_2})$ denote the number of D-J equivalence classes of locally standard $(\mathbb{Z}_2)^m$-manifolds over $\Delta^{n_1} \times \Delta^{n_2}$ with $n_1+n_2 = n$. Then

$$\begin{aligned} DJ(\Delta^{n_1} \times \Delta^{n_2}) &= 2^{2m} - 3\cdot 2^{m+n} + 2^{2n+1} - 2^m + 2^n + 2^{m+n_1+1} - 2^{n+n_1+1} \\ &\quad + 2^{m+n_2+1} - 2^{n+n_2+1} + 2^{n_1} + 2^{n_2} - 1. \end{aligned}$$

Proof : Let $e_1, e_2, \cdots, e_m$ be the standard basis of $(\mathbb{Z}_2)^m$, then $(\mathbb{Z}_2)^m$ contains 2^m-1 nonzero elements (or 2^m-1 colors). We choose $F_1, \cdots, F_{n_1}$ from $\mathcal{F}'$ and $F_{n_1+2}, \cdots, F_{n_1+n_2+1}$ from $\mathcal{F}''$ such that $F_1, \cdots, F_{n_1}, F_{n_1+2}, \cdots, F_{n_1+n_2+1}$ meet at one vertex of $\Delta^{n_1} \times \Delta^{n_2}$. Then

$$\begin{aligned} A(\Delta^{n_1} \times \Delta^{n_2}) &= \{\nu \in \nu(\Delta^{n_1} \times \Delta^{n_2}, (\mathbb{Z}_2)^m) | \nu(F_i) \\ &= e_i, 1 \leq i \leq n_1; \nu(F_i) \\ &= e_{i-1}, n_1+2 \leq i \leq n_1+n_2+1. \end{aligned}$$

Write

$$A_0(\Delta^{n_1} \times \Delta^{n_2}) = \{\nu \in A(\Delta^{n_1} \times \Delta^{n_2}) | \nu(F_{n_1+1}) = e_1 + e_2 + \cdots + e_{n_1}\},$$

$$A_1(\Delta^{n_1} \times \Delta^{n_2}) = \{\nu \in A(\Delta^{n_1} \times \Delta^{n_2}) | \nu(F_{n_1+1}) = e_1 + e_2 + \cdots + e_{n_1} + e_{k_1} + \cdots + e_{k_i},$$

where $n_1+1 \leq k_1 < \cdots < k_i \leq n_1+n_2$ and $1 \leq i \leq n_2\}$,

$$A_2(\Delta^{n_1} \times \Delta^{n_2}) = \{\nu \in A(\Delta^{n_1} \times \Delta^{n_2}) | \nu(F_{n_1+1}) = e_{t_1} + \cdots + e_{t_j} + e_{g_1} + \cdots + e_{g_h},$$

where $n+1 \leq t_1 < \cdots < t_j \leq m, 1 \leq g_1 < \cdots < g_h \leq n, 1 \leq j \leq m-n$ and $0 \leq h \leq n\}$.

By the linear independence condition of $(\mathbb{Z}_2)^m$-colorings in $\nu(\Delta_{n_1} \times \Delta_{n_2}, (\mathbb{Z}_2)^m)$, we have $|A(\Delta^{n_1} \times \Delta^{n_2})| = \sum_{i=0}^{2} |A_i(\Delta^{n_1} \times \Delta^{n_2})|$. Then, our argument is divided into the following cases.

Case 1. Calculation of $|A_0(\Delta^{n_1} \times \Delta^{n_2})|$.

By the linear independence condition of $(\mathbb{Z}_2)^m$-colorings, we have that $\nu(F_{n_1+n_2+2}) = e_{n_1+1} + \cdots + e_n + e_{f_1} + \cdots + e_{f_l}$ with $1 \leq f_1 < \cdots < f_l \leq n_1$ and $0 \leq l \leq n_1$, or

$e_{t_1} + \cdots + e_{t_j} + e_{g_1} + \cdots + e_{g_h}$ with $n+1 \le t_1 < \cdots < t_j \le m, 1 \le g_1 < \cdots < g_h \le n, 1 \le j \le m-n$ and $0 \le h \le n$.

Thus, $|A_0(\Delta^{n_1} \times \Delta^{n_2})| = 2^m - 2^n + 2^{n_1}$.

Case 2. Calculation of $|A_1(\Delta^{n_1} \times \Delta^{n_2})|$.

No matter which value of $\nu(F_{n_1+1})$ is chosen, by the linear independence condition of $(\mathbb{Z}_2)^m$-colorings, we have $\nu(F_{n_1+n_2+2}) = e_{n_1+1} + \cdots + e_n$ or $e_{t_1} + \cdots + e_{t_j} + e_{g_1} + \cdots + e_{g_h}$ with $n+1 \le t_1 < \cdots < t_j \le m, 1 \le g_1 < \cdots < g_h \le n, 1 \le j \le m-n$ and $0 \le h \le n$.

Thus, $|A_1(\Delta^{n_1} \times \Delta^{n_2})| = (2^{n_2} - 1) \cdot (2^m - 2^n + 1)$.

Case 3. Calculation of $|A_2(\Delta^{n_1} \times \Delta^{n_2})|$.

Without loss of generality, suppose $\nu(F_{n_1+1}) = e_{n+1}$ because other cases are similar. By the linear independence condition of $(\mathbb{Z}_2)^m$-colorings, we have that $\nu(F_{n_1+n_2+2}) = e_1 + \cdots + e_{n_1} + e_{n+1} + e_{f_1} + \cdots + e_{f_l}$ with $n_1 + 1 \le f_1 < \cdots < f_l \le n$ and $0 \le l \le n_2 - 2$, $e_{n_1+1} + \cdots + e_n + e_{l_1} + \cdots + e_{l_k}$ with $1 \le l_1 < \cdots < l_k \le n+1, l_i \neq n_1+1, \cdots, n$ for $1 \le i \le k$ and $0 \le k \le n_1 - 1$, $e_{h_1} + \cdots + e_{h_i}$ with $1 \le h_1 < \cdots < h_i \le n+1$ and $n \le i \le n+1$, or $e_{u_1} + \cdots + e_{u_l} + e_{v_1} + \cdots + e_{v_k}$ with $n+2 \le u_1 < \cdots < u_l \le m, 1 \le v_1 < \cdots < v_k \le n+1, 1 \le l \le m-n-1$ and $0 \le k \le n+1$.

Thus, $|A_2(\Delta^{n_1} \times \Delta^{n_2})| = (2^m - 2^n) \cdot (2^m - 2^{n+1} + 2^{n_1+1} + 2^{n_2} - 1)$.

Combining Cases 1-3, we complete the proof. $\square$

Remark 2 : In a similar way, we have that there are $2^m - 2^n + 1$ locally standard $(\mathbb{Z}_2)^m$-manifolds over Δ^n up to D-J equivalence for $m \ge n$.

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